Demonstration of the Carnot Relation

Derivation of $\left(\frac{Q_C}{Q_H}\right)_{\text{rev cycle}} = \frac{T_C}{T_H}$

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Introduction: The Carnot Relation

- We aim to demonstrate a fundamental relationship for any reversible power cycle operating between two constant temperatures.
- The equation is:

$$\left(\frac{Q_C}{Q_H}\right)_{\text{rev cycle}} = \frac{T_C}{T_H}$$

- Where:
  - Q_C is the heat rejected to the cold reservoir.
  - Q_H is the heat absorbed from the hot reservoir.
  - T_C is the absolute temperature of the cold reservoir.
  - T_H is the absolute temperature of the hot reservoir.
- This relation is a direct consequence of the Second Law of Thermodynamics and is key to understanding the maximum possible efficiency of heat engines.

The Carnot Cycle: A Reversible Benchmark

- The Carnot cycle is a theoretical reversible power cycle.
- Analyzing this cycle provides the basis for the fundamental relation between heat transfer and temperature for reversible processes.
- The cycle consists of four sequential reversible processes:
  1. Isothermal Expansion
  2. Adiabatic Expansion
  3. Isothermal Compression
  4. Adiabatic Compression
- We will analyze the heat transfer and entropy change during each process.

Process 1: Isothermal Expansion

- The working substance (e.g., an ideal gas) expands reversibly while in thermal contact with the hot reservoir at constant temperature T_H.
- Heat Q_H is absorbed from the hot reservoir.
- For an isothermal process, the change in internal energy $\Delta U = 0$.
- By the First Law of Thermodynamics ($\Delta U = Q - W$), the work done by the system is $W_{12} = Q_H$.
- The change in entropy for this reversible isothermal process is:

$$\Delta S_{12} = \frac{Q_H}{T_H}$$

Process 2: Adiabatic Expansion

- The working substance continues to expand reversibly, but it is now thermally insulated (adiabatic, $Q = 0$).
- The temperature of the working substance decreases from T_H to T_C.
- Work is done by the system ($W_{23} > 0$).
- For any reversible adiabatic process, there is no heat transfer, so the change in entropy is zero:

$$\Delta S_{23} = 0$$

Process 3: Isothermal Compression

- The working substance is compressed reversibly while in thermal contact with the cold reservoir at constant temperature T_C.
- Heat Q_C is rejected to the cold reservoir. Note that in our convention where Q entering is positive, Q_C here represents a positive magnitude of heat rejected, so the heat transfer value is $-Q_C$.
- For an isothermal process, $\Delta U = 0$.
- Work is done on the system ($W_{34} < 0$).
- The change in entropy for this reversible isothermal process is:

$$\Delta S_{34} = \frac{-Q_C}{T_C} = -\frac{Q_C}{T_C}$$

Process 4: Adiabatic Compression

- The working substance is compressed reversibly and adiabatically, returning to its initial state.
- No heat is transferred ($Q = 0$).
- The temperature of the working substance increases from T_C back to T_H.
- Work is done on the system ($W_{41} < 0$).
- For any reversible adiabatic process, the change in entropy is zero:

$$\Delta S_{41} = 0$$

Total Entropy Change for a Reversible Cycle

- One of the key implications of the Second Law of Thermodynamics for any reversible cycle is that the net change in entropy over the complete cycle is zero.

$$\Delta S_{\text{cycle}} = 0$$

- The total entropy change for the Carnot cycle is the sum of the entropy changes for each of the four reversible processes:

$$\Delta S_{\text{cycle}} = \Delta S_{12} + \Delta S_{23} + \Delta S_{34} + \Delta S_{41}$$

Putting It Together: The Derivation

- Substitute the expressions for the entropy changes into the equation for the total entropy change:

$$0 = \frac{Q_H}{T_H} + 0 + \left(-\frac{Q_C}{T_C}\right) + 0$$

- This simplifies to:

$$0 = \frac{Q_H}{T_H} - \frac{Q_C}{T_C}$$

- Rearranging the terms:

$$\frac{Q_H}{T_H} = \frac{Q_C}{T_C}$$

The Result

- We can rearrange the previous equation to obtain the desired relation:

$$\frac{Q_C}{Q_H} = \frac{T_C}{T_H}$$

- Since this relation was derived for a reversible cycle (the Carnot cycle), it specifically applies to reversible power cycles:

$$\left(\frac{Q_C}{Q_H}\right)_{\text{rev cycle}} = \frac{T_C}{T_H}$$

Conclusion: Significance of the Relation

- We have demonstrated that for any reversible power cycle operating between two thermal reservoirs, the ratio of the heat rejected to the cold reservoir (Q_C) and the heat absorbed from the hot reservoir (Q_H) is equal to the ratio of their absolute temperatures (T_C/T_H).
- This relation is a cornerstone of thermodynamics and leads directly to the definition of the Kelvin temperature scale and the Carnot efficiency, which represents the maximum possible efficiency for any heat engine operating between T_H and T_C.