# Carnot Refrigeration Cycle Analysis

Step-by-Step Solution (Revised)

May 15, 2025

# Problem Statement

**KNOWN:**

- Mass of air, $m = 0.1$ kg.
- Air executes a Carnot refrigeration cycle.
- Specific heat ratio for air, $k = 1.4$.

**FIND:**

- (a) The pressure and volume at each of the four principal states.
- (b) The work for each of the four processes.
- (c) The coefficient of performance ($\beta$).

# Schematic & Given Data

**Given Data:**

- $m = 0.1$ kg
- $V_4 = 0.01$ m$^3$
- $Q_{12} = 3.4$ kJ (Heat absorbed from cold reservoir)
- $k = 1.4$
- $T_H = T_3 = T_4 = 300$ K $(27^\circ\text{C})$
- $T_C = T_1 = T_2 = 250$ K $(-23^\circ\text{C})$
- Gas constant for air, $R = \frac{8.314}{28.97} \approx 0.287$ kJ/kg K

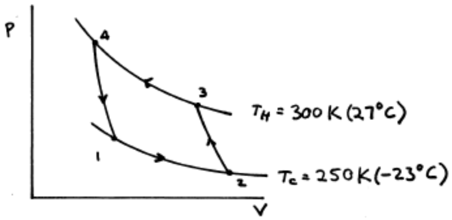


Figure: Enter Caption

*P-V Diagram (Carnot Refrigerator)

**Assumptions:**

1. Air is modeled as an ideal gas.
2. Volume change is the only work mode.
3. Specific heat ratio $k = 1.4$ is constant.

**Ideal Gas Law:** $PV = mRT$ **Adiabatic Process ($Q = 0$):**

- $PV^k = \text{constant}$
- $TV^{k-1} = \text{constant}$
- $\frac{T_b}{T_a} = \left(\frac{P_b}{P_a}\right)^{(k-1)/k}$

# Carnot Cycle Specific Relations

For any Carnot cycle operating between two temperatures:

- The two isothermal processes have equal volume ratios:

$$\frac{V_2}{V_1} = \frac{V_3}{V_4}$$

(where 1-2 is isothermal at $T_C$, and 3-4 is isothermal at $T_H$)

- This can also be written as (used in problem image):

$$V_4 V_2 = V_3 V_1$$

For a refrigeration cycle, work is input to move heat from $T_C$ to $T_H$.

# Part (a): Strategy for Pressures & Volumes

We will determine the pressure (P) and volume (V) at each state (1, 2, 3, 4) step-by-step:

1. Calculate $P_4$ using known $V_4$, $T_4$.
2. Calculate $P_1$ using adiabatic relation for process 4-1.
3. Calculate $V_1$ using ideal gas law at state 1.
4. Use $Q_{12}$ (isothermal process 1-2) to find volume ratio $V_2/V_1$.
5. Calculate $V_2$.
6. Calculate $P_2$ using ideal gas law or isothermal relation.
7. Calculate $V_3$ using Carnot cycle volume relation.
8. Calculate $P_3$ using ideal gas law at state 3.

# State 4: Pressure ($P_4$)

Given: $V_4 = 0.01\ \text{m}^3$, $T_4 = T_H = 300\ \text{K}$, $m = 0.1\ \text{kg}$. Using the ideal gas equation $P_4 V_4 = mRT_4$:

$$P_4 = \frac{mRT_4}{V_4}$$

$$P_4 = \frac{(0.1\ \text{kg})\left(\frac{8.314}{28.97}\frac{\text{kJ}}{\text{kg K}}\right)(300\ \text{K})}{0.01\ \text{m}^3}$$

$$P_4 = \frac{(0.1)(8.314/28.97)(300)}{0.01}\ \text{kPa}$$

$$P_4 \approx 861.0\ \text{kPa}$$

# State 1: Pressure ($P_1$)

Process 4-1 is an adiabatic expansion ($T_4 = T_H \to T_1 = T_C$).

$$\frac{T_4}{T_1} = \left(\frac{P_4}{P_1}\right)^{(k-1)/k}$$

$$P_1 = P_4 \left(\frac{T_1}{T_4}\right)^{k/(k-1)}$$

$$P_1 = (861.0 \text{ kPa}) \left(\frac{250 \text{ K}}{300 \text{ K}}\right)^{1.4/(1.4-1)}$$

$$P_1 = (861.0) \left(\frac{250}{300}\right)^{1.4/0.4} = (861.0) \left(\frac{5}{6}\right)^{3.5}$$

$$P_1 \approx 454.9 \text{ kPa}$$

# State 1: Volume ($V_1$)

Given: $P_1 \approx 454.9$ kPa, $T_1 = T_C = 250$ K. Using the ideal gas equation $P_1 V_1 = mRT_1$:

$$V_1 = \frac{mRT_1}{P_1}$$

$$V_1 = \frac{(0.1 \text{ kg}) \left( \frac{8.314}{28.97} \frac{\text{kJ}}{\text{kg K}} \right) (250 \text{ K})}{454.9 \text{ kPa}}$$

$$V_1 \approx 0.01577 \text{ m}^3$$

# Process 1-2: Volume Ratio ($V_2/V_1$)

Process 1-2 is isothermal expansion at $T_C = 250$ K. For an ideal gas, $\Delta U = 0$, so $Q_{12} = W_{12}$.

$$W_{12} = mRT_C \ln\left(\frac{V_2}{V_1}\right)$$

Given $Q_{12} = 3.4$ kJ, so $W_{12} = 3.4$ kJ.

$$\begin{aligned}
\ln\left(\frac{V_2}{V_1}\right) &= \frac{W_{12}}{mRT_C} \\
\ln\left(\frac{V_2}{V_1}\right) &= \frac{3.4 \text{ kJ}}{(0.1 \text{ kg})\left(\frac{8.314}{28.97}\frac{\text{kJ}}{\text{kg K}}\right)(250 \text{ K})} \\
\ln\left(\frac{V_2}{V_1}\right) &\approx \frac{3.4}{(0.1)(0.286952)(250)} \approx 0.47391 \\
\frac{V_2}{V_1} &= \exp(0.47391) \approx 1.6062
\end{aligned}$$

# State 2: Volume ($V_2$)

From the previous calculation, $\frac{V_2}{V_1} \approx 1.6062$. With $V_1 \approx 0.01577 \text{ m}^3$:

$$V_2 = 1.6062 \times V_1$$
$$V_2 = 1.6062 \times (0.01577 \text{ m}^3)$$
$$V_2 \approx 0.02533 \text{ m}^3$$

# State 2: Pressure ($P_2$)

Given: $V_2 \approx 0.02533 \text{ m}^3$, $T_2 = T_C = 250$ K. Using the ideal gas equation $P_2 V_2 = mRT_2$:

$$P_2 = \frac{mRT_2}{V_2}$$

$$P_2 = \frac{(0.1 \text{ kg}) \left( \frac{8.314}{28.97} \frac{\text{kJ}}{\text{kg K}} \right) (250 \text{ K})}{0.02533 \text{ m}^3}$$

$$P_2 \approx 283.2 \text{ kPa}$$

Alternatively, for isothermal process 1-2: $P_1 V_1 = P_2 V_2$

$$P_2 = P_1 \frac{V_1}{V_2} = (454.9 \text{ kPa}) \frac{1}{1.6062} \approx 283.2 \text{ kPa}$$

# State 3: Volume ($V_3$)

Using the Carnot cycle relation $V_4 V_2 = V_3 V_1$, which can be rearranged as used in the problem image: $V_3 = V_2 \left(\frac{V_4}{V_1}\right)$.

$$V_3 = (0.02533 \text{ m}^3) \left(\frac{0.01 \text{ m}^3}{0.01577 \text{ m}^3}\right)$$
$$V_3 \approx (0.02533) \times (0.634115)$$
$$V_3 \approx 0.01606 \text{ m}^3$$

(Note: $\frac{V_2}{V_1} \approx 1.6062$ and $\frac{V_3}{V_4} = \frac{0.01606}{0.01} = 1.606$. Consistent.)

# State 3: Pressure ($P_3$)

Given: $V_3 \approx 0.01606 \text{ m}^3$, $T_3 = T_H = 300$ K. Using the ideal gas equation $P_3 V_3 = mRT_3$:

$$P_3 = \frac{mRT_3}{V_3}$$

$$P_3 = \frac{(0.1 \text{ kg})\left(\frac{8.314}{28.97} \frac{\text{kJ}}{\text{kg K}}\right)(300 \text{ K})}{0.01606 \text{ m}^3}$$

$$P_3 \approx 536.1 \text{ kPa}$$

# Part (a): Summary of Pressures and Volumes

**Pressures:**

- $P_1 \approx 454.9$ kPa
- $P_2 \approx 283.2$ kPa
- $P_3 \approx 536.1$ kPa
- $P_4 = 861.0$ kPa

**Volumes:**

- $V_1 \approx 0.01577$ m$^3$
- $V_2 \approx 0.02533$ m$^3$
- $V_3 \approx 0.01606$ m$^3$
- $V_4 = 0.01$ m$^3$

# Part (b): Strategy for Work Calculation

**Isothermal Process (e.g., 1-2 at $T_C$):**

$$W = mRT \ln\left(\frac{V_{final}}{V_{initial}}\right)$$

For an ideal gas, $\Delta U = 0$, so $Q = W$. **Adiabatic Process (e.g., 2-3, $Q = 0$):**

$$W = -\Delta U = -mc_v(T_{final} - T_{initial})$$

where $c_v = \frac{R}{k-1}$. We will use $c_v = 0.717$ kJ/kg K (consistent with image solution).

# Work $W_{12}$ (Process 1-2, Isothermal Expansion)

Process 1-2 is isothermal at $T_C = 250$ K. Heat $Q_{12} = 3.4$ kJ is absorbed.

$$W_{12} = Q_{12} = 3.4 \text{ kJ}$$

This is work done *by* the system (expansion).

# Specific Heat $c_v$ Calculation

The specific heat at constant volume, $c_v$, is needed for adiabatic work.

$$c_v = \frac{R}{k-1}$$

Using $R = \frac{8.314}{28.97} \frac{\text{kJ}}{\text{kg K}}$ and $k = 1.4$:

$$c_v = \frac{(8.314/28.97)\,\frac{\text{kJ}}{\text{kg K}}}{1.4 - 1}$$
$$c_v \approx \frac{0.286952}{0.4} \approx 0.71738 \frac{\text{kJ}}{\text{kg K}}$$

The provided image solution uses $c_v = 0.717$ kJ/kg K. We will use this value for consistency in work calculations for adiabatic processes.

# Work $W_{23}$ (Process 2-3, Adiabatic Compression)

Process 2-3: $T_2 = T_C = 250 \text{ K} \rightarrow T_3 = T_H = 300 \text{ K}$. $Q_{23} = 0$.

$$
\begin{aligned}
W_{23} &= -mc_v(T_3 - T_2) \\
W_{23} &= -(0.1 \text{ kg})(0.717 \text{ kJ/kg K})(300 \text{ K} - 250 \text{ K}) \\
W_{23} &= -(0.1)(0.717)(50) \text{ kJ} \\
W_{23} &= -3.585 \text{ kJ}
\end{aligned}
$$

Work is done *on* the system (compression).

# Work $W_{34}$ (Process 3-4, Isothermal Compression)

Process 3-4 is isothermal at $T_H = 300$ K. $V_3 \approx 0.01606$ m$^3$, $V_4 = 0.01$ m$^3$.

$$W_{34} = mRT_H \ln\left(\frac{V_4}{V_3}\right)$$
$$W_{34} = (0.1\text{ kg})\left(\frac{8.314}{28.97}\frac{\text{kJ}}{\text{kg K}}\right)(300\text{ K})\ln\left(\frac{0.01}{0.01606}\right)$$
$$W_{34} \approx (0.1)(0.286952)(300)\ln(0.622665)$$
$$W_{34} \approx (8.60856)(-0.47375)$$
$$W_{34} \approx -4.079\text{ kJ}$$

Work is done *on* the system (compression).

# Work $W_{41}$ (Process 4-1, Adiabatic Expansion)

Process 4-1: $T_4 = T_H = 300$ K $\rightarrow T_1 = T_C = 250$ K. $Q_{41} = 0$.

$$W_{41} = -mc_v(T_1 - T_4)$$
$$W_{41} = -(0.1 \text{ kg})(0.717 \text{ kJ/kg K})(250 \text{ K} - 300 \text{ K})$$
$$W_{41} = -(0.1)(0.717)(-50) \text{ kJ}$$
$$W_{41} = 3.585 \text{ kJ}$$

Work is done *by* the system (expansion). Note: $|W_{23}| = |W_{41}|$ for a Carnot cycle with ideal gas.

**Work for each process:**

- $W_{12} = 3.4$ kJ (Work by system)
- $W_{23} \approx -3.585$ kJ (Work on system)
- $W_{34} \approx -4.079$ kJ (Work on system)
- $W_{41} \approx 3.585$ kJ (Work by system)

**Net work for the cycle ($W_{cycle}$):**

$$W_{cycle} = W_{12} + W_{23} + W_{34} + W_{41}$$
$$W_{cycle} = 3.4 - 3.585 - 4.079 + 3.585 \text{ kJ}$$
$$W_{cycle} = 3.4 - 4.079 = -0.679 \text{ kJ}$$

Negative $W_{cycle}$ means net work is done *on* the system (refrigerator).

# Part (c): Coefficient of Performance ($\beta$)

For a refrigeration cycle, the Coefficient of Performance ($\beta$) is defined as:

$$\beta = \frac{\text{Desired Output}}{\text{Required Input}} = \frac{\text{Heat removed from cold space}}{\text{Net Work Input}} = \frac{Q_{in}}{|W_{cycle}|}$$

For a Carnot refrigerator, it is also given by:

$$\beta_{max} = \frac{T_C}{T_H - T_C}$$

(Temperatures in Kelvin)

# Calculating $\beta$ using Temperatures

Using the reservoir temperatures: $T_C = 250$ K, $T_H = 300$ K.

$$\beta = \beta_{Carnot} = \frac{T_C}{T_H - T_C}$$
$$\beta = \frac{250\text{ K}}{300\text{ K} - 250\text{ K}}$$
$$\beta = \frac{250}{50}$$
$$\beta = 5.00$$

# Calculating $\beta$ using $Q_{in}$ and $W_{cycle}$

Using the calculated heat input and net work: $Q_{in} = Q_{12} = 3.4$ kJ
$|W_{cycle}| = |-0.679 \text{ kJ}| = 0.679$ kJ

$$\beta = \frac{Q_{in}}{|W_{cycle}|}$$
$$\beta = \frac{3.4 \text{ kJ}}{0.679 \text{ kJ}}$$
$$\beta \approx 5.00736$$

The image solution shows 5.01. The slight difference is due to rounding in intermediate values (primarily $W_{cycle}$).

# Conclusion

The Carnot refrigeration cycle analysis yielded:

- **Pressures and Volumes:** Determined for all four states.
- **Work per process:**
  - $W_{12} = 3.4$ kJ
  - $W_{23} \approx -3.585$ kJ
  - $W_{34} \approx -4.079$ kJ
  - $W_{41} \approx 3.585$ kJ
- **Net Work Input:** $|W_{cycle}| \approx 0.679$ kJ.
- **Coefficient of Performance:** $\beta \approx 5.0$.

The results align closely with theoretical Carnot cycle performance, with minor variations due to rounding.