Thermodynamics - HW6

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Problem Statement

A gas in a piston-cylinder assembly undergoes an expansion process for which the relationship between pressure and volume is given by

$$pV^n = \text{constant}.$$

The initial pressure is 5 bar, the initial volume is $0.15\,\mathrm{m}^3$, and the final volume is $0.25\,\mathrm{m}^3$. Determine the work for the process, in kJ, if:

(a) $n = 1.5$,

(b) $n = 1.0$,

(c) $n = 0$.

1 What is a Polytropic Exponent?

The **polytropic exponent** (n) is a parameter that describes how the pressure and volume of a gas change during a process known as a **polytropic process**. This type of process is characterized by the relationship:

$$PV^n = \text{constant}$$

Where:

- P = Pressure of the gas.
- V = Volume of the gas.
- n = Polytropic exponent.

1.1 Physical Interpretation of the Polytropic Exponent (n)

The value of n can vary depending on the specific conditions of the process, such as heat transfer, efficiency, and the presence or absence of heat exchange with the surroundings. Some common values are:

- $n = 0$: **Isobaric Process (Constant Pressure)**. The pressure remains unchanged while the volume varies. Example: Free expansion of a gas in an open container.
- $n = 1$: **Isothermal Process (Constant Temperature)**. The temperature remains constant, and heat is transferred so that the gas expands or compresses without changing temperature. Example: Slow compression in a piston with conductive walls.
- $n = \gamma$ (where $\gamma = \frac{C_p}{C_v}$): **Adiabatic Reversible Process**. There is no heat exchange with the surroundings, and work is done at the expense of internal energy. Example: Rapid compression of a gas with no time for heat exchange.
- $n \to \infty$: **Isochoric Process (Constant Volume)**. The volume remains constant, and any change in pressure is due solely to temperature changes. Example: Heating of a gas in a rigid container.
- n with any other value: **General Polytropic Process**. It may involve heat loss or gain, friction, or inefficiencies. It is a flexible model that covers a wide range of thermodynamic processes.

1.2 Summary

The polytropic exponent n defines the type of process that occurs when a gas expands or compresses. Changing the value of n affects how pressure and volume are related during the process.

2 General Formula for Work in a Polytropic Process

In a polytropic process, the relationship between pressure, volume, and the polytropic exponent n is:

$$PV^n = \text{constant}$$

This implies:

$$P = CV^{-n}$$

Where C is a constant. During an expansion or compression process, the differential work dW is:

$$dW = P\,dV$$

Replacing the expression for P:

$$dW = CV^{-n}dV$$

Since $C = P_1V_1^n = P_2V_2^n$, we can integrate both sides:

$$W = \int_{V_1}^{V_2} P\,dV = \int_{V_1}^{V_2} CV^{-n}\,dV$$

$$W = C\int_{V_1}^{V_2} V^{-n}\,dV$$

The integral of V^{-n} is:

$$\int V^{-n}\,dV = \frac{V^{1-n}}{1-n}$$

Thus:

$$W = C\left(\frac{V_2^{1-n} - V_1^{1-n}}{1-n}\right)$$

Now, substituting $C = P_1V_1^n = P_2V_2^n$:

$$W = \frac{P_2V_2 - P_1V_1}{1-n}$$

2.1 Important Observations

- This formula is valid for any polytropic process as long as $n \neq 1$. If $n = 1$, it corresponds to an isothermal process, and the formula changes to $W = PV\ln\left(\frac{V_2}{V_1}\right)$.
- If $n = 0$, it is an isobaric process, and the formula simplifies to $W = P(V_2 - V_1)$.

3 Given Data

- Initial Pressure: $p_1 = 5$ bar
- Initial Volume: $V_1 = 0.15\ \mathrm{m^3}$
- Final Volume: $V_2 = 0.25\ \mathrm{m^3}$

4 Calculations

4.1 Case (a): $n = 1.5$

In a polytropic process, the relationship $PV^n = \text{constant}$ applies. Therefore:

$$P_1 V_1^n = P_2 V_2^n$$

Solving for P_2:

$$P_2 = P_1 \left(\frac{V_1}{V_2} \right)^n$$

$$P_2 = 500 \times \left(\frac{0.15}{0.25} \right)^{1.5}$$

$$P_2 \approx 232.38 \text{ kPa}$$

Calculation of Work W:

$$W = \frac{P_2 V_2 - P_1 V_1}{1 - n}$$

$$W = \frac{(232.38 \times 0.25) - (500 \times 0.15)}{1 - 1.5}$$

$$W \approx \frac{58.095 - 75}{-0.5}$$

$$W \approx \frac{-16.905}{-0.5}$$

$$W \approx 33.81 \text{ kJ}$$

4.2 Case (b): $n = 1.0$ (Isothermal Process)

When $n = 1$, the process is isothermal, and the formula for work is:

$$W = PV \ln \left(\frac{V_2}{V_1} \right)$$

Since P is not constant, we use $P_1 V_1 = P_2 V_2$. Therefore:

$$W = P_1 V_1 \ln \left(\frac{V_2}{V_1} \right)$$

$$W = 500 \times 0.15 \times \ln \left(\frac{0.25}{0.15} \right)$$

$$W \approx 500 \times 0.15 \times 0.5108$$

$$W \approx 38.31 \text{ kJ}$$

4.3 Case (c): $n = 0$ (Isobaric Process)

When $n = 0$, the process is isobaric, meaning the pressure is constant:

$$W = P(V_2 - V_1)$$

$$W = 500 \times (0.25 - 0.15)$$

$$W = 500 \times 0.10$$

$$W = 50 \text{ kJ}$$

5 Summary of the Results

The results for each case are:

- Case (a): $W = 33.81$ kJ
- Case (b): $W = 38.31$ kJ
- Case (c): $W = 50$ kJ