Thermodynamics - HW8

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Total Value: 5% (1% Each Question)

1. Instructional Video

Review the following video for conceptual understanding of the Carnot cycle volume relationships:

`https://www.youtube.com/watch?v=ixRtSV3CXPA&t=3s`

2. Thermodynamic Identity Proof

Using the information from the video, demonstrate the following identity based on the volume-temperature relationship in the reversible Carnot cycle:

$$\left(\frac{Q_C}{Q_H}\right)_{\text{rev}} = \frac{T_C}{T_H}$$

Where:

- Q_C: Heat rejected to the cold reservoir
- Q_H: Heat absorbed from the hot reservoir
- T_C: Absolute temperature of the cold reservoir (in Kelvin)
- T_H: Absolute temperature of the hot reservoir (in Kelvin)

1. Isothermal Heat Transfers

In a reversible isothermal process for an ideal gas:

$$Q = nRT \ln\left(\frac{V_{\text{final}}}{V_{\text{initial}}}\right)$$

For the Carnot cycle:

- At T_H (expansion):

$$Q_H = nRT_H \ln\left(\frac{V_2}{V_1}\right)$$

- At T_C (compression):

$$Q_C = nRT_C \ln\left(\frac{V_3}{V_4}\right)$$

2. Adiabatic Process Relation

In adiabatic (reversible) processes:

$$TV^{\gamma-1} = \text{constant}$$

Derived from the fundamental thermodynamic principles:

A. First Law of Thermodynamics

The First Law of Thermodynamics for a closed system is:

$$dU = \delta Q - \delta W \tag{1}$$

In an **adiabatic process**, there is no heat transfer:

$$\delta Q = 0 \quad \Rightarrow \quad dU = -\delta W \tag{2}$$

B. Internal Energy of an Ideal Gas

For an ideal gas, the internal energy depends only on temperature:

$$dU = mc_v dT \tag{3}$$

The work done during a quasi-static process is:

$$\delta W = PdV \tag{4}$$

Substituting into the energy balance:

$$mc_v dT = -PdV \tag{5}$$

C. Substituting the Ideal Gas Law

Using the ideal gas law $PV = mRT$, we express pressure as:

$$P = \frac{mRT}{V} \tag{6}$$

Substitute into the energy equation:

$$mc_v dT = -\frac{mRT}{V} dV \tag{7}$$

Divide both sides by m:

$$c_v dT = -\frac{RT}{V} dV \tag{8}$$

D. Separation of Variables and Integration

$$\frac{dT}{T} = -\frac{R}{c_v} \frac{dV}{V} \tag{9}$$

Since $\gamma = \frac{c_p}{c_v}$ and $R = c_p - c_v$, it follows:

$$\frac{R}{c_v} = \gamma - 1 \tag{10}$$

Substituting:

$$\frac{dT}{T} = -(\gamma - 1)\frac{dV}{V} \tag{11}$$

Integrating both sides:

$$\int \frac{dT}{T} = -(\gamma - 1) \int \frac{dV}{V} \tag{12}$$

$$\ln T = -(\gamma - 1) \ln V + \text{constant} \tag{13}$$

E. Final Adiabatic Temperature-Volume Relation

Using logarithmic identities:

$$\ln(TV^{\gamma-1}) = \text{constant} \tag{14}$$

Exponentiating both sides:

$$\boxed{TV^{\gamma-1} = \text{constant}} \tag{15}$$

This is the fundamental temperature–volume relation for a reversible adiabatic process in an ideal gas.

Thus:

$$T_H V_2^{\gamma-1} = T_C V_3^{\gamma-1} \quad \text{and} \quad T_H V_1^{\gamma-1} = T_C V_4^{\gamma-1}$$

Then:

$$\frac{V_2}{V_1} = \frac{V_3}{V_4}$$

3. Ratio of Heat Transfers

$$\frac{Q_C}{Q_H} = \frac{nRT_C \ln\left(\frac{V_3}{V_4}\right)}{nRT_H \ln\left(\frac{V_2}{V_1}\right)} = \frac{T_C}{T_H}$$

Conclusion

$$\boxed{\left(\frac{Q_C}{Q_H}\right)_{\text{rev}} = \frac{T_C}{T_H}}$$

3. Efficiency Evaluation

An inventor claims to have developed a power cycle that delivers a total work output of 410 kJ for a heat input of 1000 kJ. The heat is transferred from a source at 500 K and rejected to the atmosphere at 300 K.

Evaluate the claim.

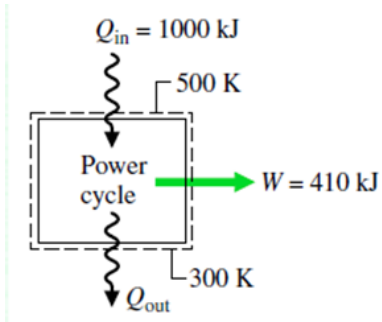

Figure 1: Carnot cycle

The thermal efficiency is defined as:

$$\eta_{\text{real}} = \frac{W}{Q_H} = \frac{410 \text{ kJ}}{1000 \text{ kJ}} = 0.41 = 41\%$$

Maximum Theoretical Efficiency (Carnot)

The Carnot efficiency sets the theoretical maximum limit for any heat engine operating between two thermal reservoirs:

$$\eta_{\text{Carnot}} = 1 - \frac{T_C}{T_H} = 1 - \frac{300}{500} = 1 - 0.6 = 0.4 = 40\%$$

Conclusion

$$\eta_{\text{real}} = 41\% > \eta_{\text{Carnot}} = 40\%$$

This result is physically impossible.

Therefore, the inventor's claim violates the second law of thermodynamics. No real or ideal power cycle can surpass the efficiency of the Carnot cycle operating between the same temperature limits.

4. Coefficient of Performance Comparison

A refrigerator maintains a freezer compartment at $-5\,°C$ while the surrounding air is at $22\,°C$. The rate of heat transfer from the freezer compartment to the refrigerant is:

$$\dot{Q}_{\text{in}} = 8000\,\text{kJ}\,\text{h}^{-1}$$

and the power input is:

$$W_{\text{in}} = 3200\,\text{kJ}\,\text{h}^{-1}$$

Tasks:

a) Determine the actual coefficient of performance (COP) of the refrigerator.

b) Compare it to the COP of a reversible refrigeration cycle operating between the same temperature limits.

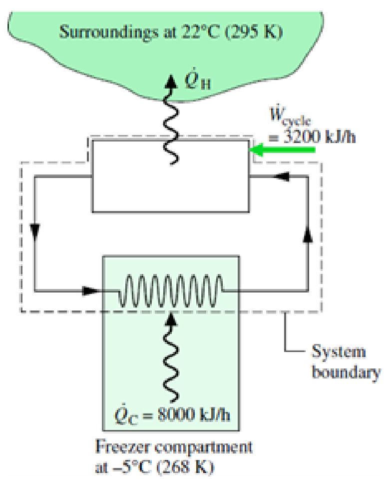

Figure 2: Carnot cycle

Part (a) - Actual COP

The actual coefficient of performance (COP) for a refrigerator is:

$$\text{COP}_{\text{real}} = \frac{\dot{Q}_C}{\dot{W}_{\text{in}}} = \frac{8000}{3200} = 2.5$$

Part (b) - Ideal (Carnot) COP

The maximum COP for a reversible (Carnot) refrigeration cycle is:

$$\text{COP}_{\text{Carnot}} = \frac{T_C}{T_H - T_C} = \frac{268}{295 - 268} = \frac{268}{27} \approx 9.926$$

The real COP of the refrigerator is 2.5, while the maximum theoretical COP (Carnot) is approximately 9.93. As expected, the actual system performs below the theoretical limit imposed by the second law of thermodynamics.

5. Heat Pump Work Requirement

A dwelling requires 5×10^5 kJ per day to maintain its interior temperature at 22 °C, while the outside temperature is 10 °C.

Task: If an electric heat pump is used, determine the **minimum theoretical work input** (in kJ) required for one full day of operation.

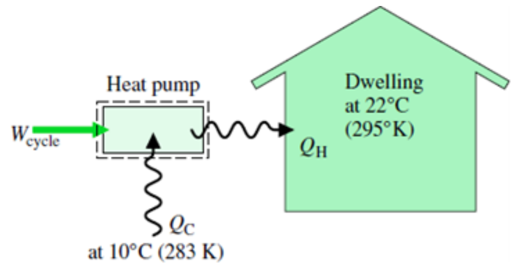

Figure 3: Carnot cycle

1. Carnot COP for Heat Pump

$$\mathrm{COP}_{\mathrm{Carnot}} = \frac{T_H}{T_H - T_C} = \frac{295}{295 - 283} = \frac{295}{12} \approx 24.58$$

2. Minimum Work Input

$$\mathrm{COP} = \frac{Q_H}{W_{\min}} \Rightarrow W_{\min} = \frac{Q_H}{\mathrm{COP}_{\mathrm{Carnot}}} = \frac{5 \times 10^5}{24.58} \approx 20\,335\,\mathrm{kJ}$$

The minimum theoretical work required is approximately $20\,335\,\mathrm{kJ}$ per day. This value represents the ideal limit of performance for a Carnot heat pump operating between 283 K and 295 K.