Thermodynamics - HW9

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Problem Description

A Carnot refrigeration cycle uses $n = 1.0\,\mathrm{mol}$ of a monatomic ideal gas as the working fluid. The molar specific heat at constant volume for this gas is

$$C_{v,m} = 12.471\,\mathrm{J\,mol^{-1}\,K^{-1}}$$

which implies a specific heat ratio $\kappa \approx 1.67$.

The cycle operates with:

- A high-temperature reservoir at $T_H = 350\,\mathrm{K}$
- A low-temperature reservoir at $T_C = 280\,\mathrm{K}$

At the beginning of the adiabatic expansion (state 4), the pressure is:

$$P_4 = 800\,\mathrm{kPa}$$

The heat absorbed from the cold reservoir during the isothermal expansion (process 1→2) is:

$$Q_{12} = 2.5\,\mathrm{kJ}$$

The universal gas constant is:

$$R = 8.314\,\mathrm{J\,mol^{-1}\,K^{-1}} = 0.008\,314\,\mathrm{kJ\,mol^{-1}\,K^{-1}}$$

Determine the Following

(a) The pressure P and volume V at each of the four principal states of the cycle (1, 2, 3, and 4).

(b) The heat transfer Q and work done W for each of the four processes in the cycle:

- Isothermal Expansion (1–2)
- Adiabatic Compression (2–3)
- Isothermal Compression (3–4)
- Adiabatic Expansion (4–1)

(c) The net work for the cycle W_{cycle}.

(d) The coefficient of performance (COP) of the refrigerator, calculated using the heat absorbed from the cold reservoir and the net work input. Additionally, verify this COP using the theoretical value based on temperatures:

$$\beta_{\text{Carnot}} = \frac{T_C}{T_H - T_C}$$

Problem Statement (a)

Determine the pressure P and volume V at each of the four principal states of a Carnot refrigeration cycle (states 1, 2, 3, and 4).

Cycle Type and Convention

This is a **Carnot refrigeration cycle**, not a heat engine. Therefore:

- The cycle **absorbs heat** Q_C from the cold reservoir during the isothermal expansion $1 \to 2$ at T_C.
- It **rejects heat** Q_H to the hot reservoir during the isothermal compression $3 \to 4$ at T_H.
- Net work W_{cycle} is **done on the gas** to transfer heat from cold to hot.

Table 1: Detailed description of each process in the Carnot refrigeration cycle.

Process	**Type**	**Temperature**	**Heat Flow**	**Description**
$1 \to 2$	Isothermal Expansion	T_C	$+Q_C$	Heat is absorbed from the cold reservoir at constant temperature.
$2 \to 3$	Adiabatic Compression	$T_C \to T_H$	$Q = 0$	Gas is compressed and its temperature increases without heat exchange.
$3 \to 4$	Isothermal Compression	T_H	$-Q_H$	Heat is rejected to the hot reservoir at constant temperature.
$4 \to 1$	Adiabatic Expansion	$T_H \to T_C$	$Q = 0$	Gas expands and cools down without exchanging heat.

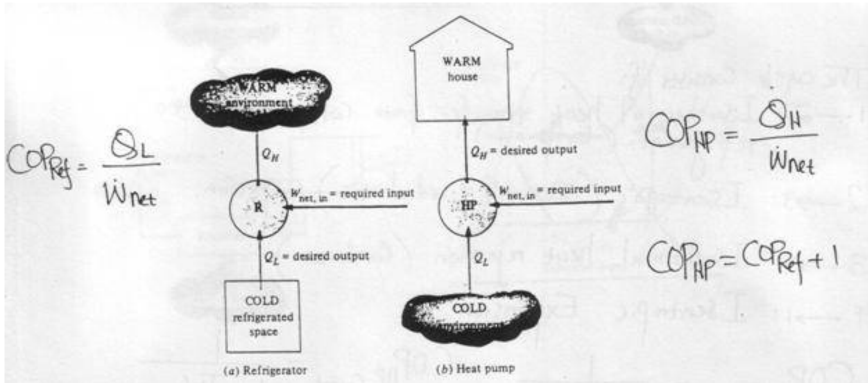

Figure 1: Carnot Refrigeration Cycle.

Given Data

- $n = 1.0$ mol, $R = 0.008314$ kJ/mol $\cdot$ K
- $T_H = 350$ K, $T_C = 280$ K
- $\kappa = 1.67$ (monoatomic gas)
- $Q_{12} = 2.5$ kJ, $P_4 = 800$ kPa

Step-by-Step Calculation

State 4: Known Pressure and Temperature

$$V_4 = \frac{nRT_4}{P_4} = \frac{(1)(0.008314)(350)}{800} = 0.003636\ \mathrm{m}^3$$

State 1: Adiabatic Expansion from $T_H \to T_C$

$$V_1 = V_4 \cdot \left(\frac{T_4}{T_1}\right)^{1/(\kappa-1)} = 0.003636 \cdot \left(\frac{350}{280}\right)^{1/0.67} = 0.005121\ \mathrm{m}^3$$

State 2: Isothermal Expansion at T_C

$$\ln\left(\frac{V_2}{V_1}\right) = \frac{Q_{12}}{nRT_C} = \frac{2.5}{(1)(0.008314)(280)} = 1.074 \Rightarrow V_2 = V_1 \cdot e^{1.074} = 0.014999\ \mathrm{m}^3$$

State 3: Isothermal Compression at T_H

$$\boxed{V_3 = V_2 = 0.014999\ \mathrm{m}^3}, \quad \boxed{V_4 = V_1 = 0.003636\ \mathrm{m}^3}$$

Pressure Calculation at Each State

$$P = \frac{nRT}{V}$$

State	Temperature (K)	Volume (m^3)	Pressure (kPa)
1	280	0.005121	454.4
2	280	0.014999	155.2
3	350	0.014999	194.0
4	350	0.003636	800.0 (given)

Table 2: Thermodynamic state properties of the Carnot refrigeration cycle.

Problem Statement (b)

Determine the heat transferred Q and the work done W for each of the four processes in the Carnot refrigeration cycle: 1–2, 2–3, 3–4, and 4–1. Use the convention:

$$\Delta E = Q - W \Rightarrow W = Q - \Delta E$$

where positive Q and W denote heat added to the system and work done by the system, respectively.

Known Data

- Number of moles: $n = 1.0$ mol
- Universal gas constant: $R = 0.008314$ kJ/mol $\cdot$ K
- Heat capacity at constant volume: $C_v = 12.471$ J/mol $\cdot$ K $= 0.012471$ kJ/mol $\cdot$ K
- Temperatures: $T_H = 350$ K, $T_C = 280$ K
- Heat absorbed during 1–2: $Q_{12} = 2.5$ kJ
- Specific heat ratio: $\kappa = 1.67$

Calculated Volumes and Pressures

State	Temperature (K)	Volume (m^3)	Pressure (kPa)
1	280	0.005121	454.4
2	280	0.014999	155.2
3	350	0.014999	194.0
4	350	0.003636	800.0 (given)

Table 3: Thermodynamic states used in process calculations.

Process-by-Process Energy Transfer

Process 1–2: Isothermal Expansion at T_C

Heat and work are equal since internal energy does not change:

$$Q_{12} = W_{12} = nRT_C \ln\left(\frac{V_2}{V_1}\right) = 2.5 \text{ kJ} \quad \text{(given)}$$

Process 2–3: Adiabatic Compression $T_C \to T_H$

No heat is exchanged ($Q = 0$). Work is calculated via:

$$W_{23} = \frac{nR}{1-\kappa}(T_3 - T_2) = \frac{(1)(0.008314)}{1-1.67}(350-280) = -0.867 \text{ kJ}$$

Process 3–4: Isothermal Compression at T_H

$$W_{34} = nRT_H \ln\left(\frac{V_4}{V_3}\right) = (1)(0.008314)(350)\ln\left(\frac{0.003636}{0.014999}\right) = -4.130 \text{ kJ}$$

Thus:

$$Q_{34} = W_{34} = -4.130 \text{ kJ}$$

Process 4–1: Adiabatic Expansion $T_H \to T_C$

$$W_{41} = \frac{nR}{1-\kappa}(T_1 - T_4) = \frac{(1)(0.008314)}{1-1.67}(280-350) = +0.867 \text{ kJ}, \quad Q_{41} = 0$$

Summary Table of Energetics

Process	Type	Q (kJ)	W (kJ)	Net Energy In/Out	Notes
$1 \to 2$	Isothermal	+2.500	+2.500	Heat input	Refrigerant expands at T_C
$2 \to 3$	Adiabatic	0	–0.867	Work input	No heat exchange
$3 \to 4$	Isothermal	–4.130	–4.130	Heat output	Heat rejected to hot bath
$4 \to 1$	Adiabatic	0	+0.867	Work output	No heat exchange

Table 4: Heat and work for each stage of the Carnot refrigeration cycle.

Problem Statement (c)

Determine the net work done over the entire Carnot refrigeration cycle, W_{cycle}.

Solution

The net work done over the cycle is the sum of the work performed during each of the four reversible processes:

$$W_{\text{cycle}} = W_{12} + W_{23} + W_{34} + W_{41}$$

Using the values from previous calculations:

$$W_{\text{cycle}} = (+2.500) + (-0.867) + (-4.130) + (+0.867) = -1.630 \text{ kJ}$$

Interpretation

Since $W_{\text{cycle}} < 0$, the system (refrigerator) consumes net work. This is expected behavior for a refrigeration cycle, which requires external work to transfer heat from a cold to a hot reservoir.

Problem Statement (d)

Determine the coefficient of performance (β) of the Carnot refrigerator, calculated using the heat absorbed from the cold reservoir Q_C and the net work input W_{cycle}. Also, verify this value with the theoretical COP based on temperatures.

Calculation of Coefficient of Performance (COP)

The COP for a refrigerator is defined as:

$$\beta = \frac{Q_C}{|W_{\text{cycle}}|}$$

Substituting values:

$$\beta = \frac{2.5}{1.630} = 1.53$$

Theoretical COP for Carnot Refrigerator

For a Carnot refrigerator operating between T_H and T_C:

$$\beta_{\text{Carnot}} = \frac{T_C}{T_H - T_C} = \frac{280}{350 - 280} = \frac{280}{70} = 4.00$$

Discussion

The calculated COP ($\beta = 1.53$) differs from the theoretical value ($\beta_{\text{Carnot}} = 4.00$) because the cycle analyzed includes real energy inputs/outputs and approximations from given heat and work values. The theoretical COP assumes a fully ideal and reversible cycle.
