TABLE OF LAPLACE TRANSFORMS
Revision J

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Operation Transforms		
N	F(s)	f (t) , t > 0
1.1	$Y(s) = \int_0^{\infty} \exp(-st) y(t) dt$	y(t) , definition of Laplace transform
1.2	$Y(s)$	$y(t) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \exp(st) Y(s) ds$ inversion formula
1.3	$sY(s) - y(0)$	$y'(t)$, first derivative
1.4	$s^2 Y(s) - sy(0) - y'(0)$	$y''(t)$, second derivative
1.5	$s^n Y(s) - s^{n-1}[y(0)]$ $- s^{n-2}[y'(0)] - \ldots - s[y^{(n-2)}(0)]$ $-[y^{(n-1)}(0)]$	$y^{(n)}(t)$, nth derivative
1.6	$\frac{1}{s} F(s)$	$\int_0^t Y(\tau) d\tau$, integration
1.7	F(s)G(s)	$\int_0^t f(t-\tau) g(\tau) d\tau$, convolution integral
1.8	$\frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$	$f(\alpha t)$, scaling
1.9	$F(s - \alpha)$	$\exp(\alpha t) f(t)$, shifting in the s plane
1.10	$\frac{1}{\alpha} F\left(\frac{s}{\alpha} - \beta\right)$	$\exp(\alpha \beta t) f(\alpha t)$, combined scaling and shifting

Function Transforms		
N	F(s)	f (t) , t > 0
2.1	1	$\delta(t)$, unit impulse at t = 0
2.2	s	$\frac{d}{dt}\delta(t)$, doublet impulse at t = 0
2.3	$\exp(-\alpha s),\ \alpha \geq 0$	$\delta(t-\alpha)$
2.4	$\frac{1}{s}$	u(t) , unit step
2.5	$\frac{1}{s}\exp(-\alpha s)$	$u(t-\alpha)$
2.6	$\frac{1}{s^2}$	t
2.7	$\frac{1}{s^n}$, $n = 1, 2, 3, ...$	$\frac{t^{n-1}}{(n-1)!}$
2.8	$\frac{n!}{s^{n+1}}$, $n = 1, 2, 3, ...$	t^n
2.9	$\frac{1}{s^k}$, k is any real number > 0	$\frac{t^{k-1}}{\Gamma(k)}$, the Gamma function is given in Appendix A
2.10	$\frac{1}{s+\alpha}$	$\exp(-\alpha t)$
2.11	$\frac{1}{(s+\alpha)^2}$	$t\exp(-\alpha t)$

Function Transforms		
N	F(s)	f (t) , t > 0
2.12	$\frac{1}{(s+\alpha)^n}$, $n = 1, 2, 3, \ldots$	$\left[\frac{t^{n-1}}{(n-1)!}\right]\exp(-\alpha t)$
2.13	$\frac{\alpha}{s(s+\alpha)}$	$1-\exp(-\alpha t)$
2.14	$\frac{1}{(s+\alpha)(s+\beta)}$, $\alpha \neq \beta$	$\frac{1}{(\beta-\alpha)}[\exp(-\alpha t)-\exp(-\beta t)]$
2.15	$\frac{1}{s(s+\alpha)(s+\beta)}$, $\alpha \neq \beta$	$\frac{1}{\alpha\beta}+\frac{\exp(-\alpha t)}{\alpha(\alpha-\beta)}+\frac{\exp(-\beta t)}{\beta(\beta-\alpha)}$
2.16	$\frac{s}{(s+\alpha)(s+\beta)}$, $\alpha \neq \beta$	$\frac{1}{(\alpha-\beta)}[\alpha\exp(-\alpha t)-\beta\exp(-\beta t)]$
2.17	$\frac{s}{(s+\alpha)^2}$	$[1-\alpha t]\exp(-\alpha t)$
2.18	$\frac{1}{s(s+\alpha)^2}$	$\frac{1}{\alpha^2}\{1-[1+\alpha t]\exp(-\alpha t)\}$
2.19	$\frac{\alpha}{s^2+\alpha^2}$	$\sin(\alpha t)$
2.20	$\frac{[\sin(\phi)]s+[\cos(\phi)]\alpha}{s^2+\alpha^2}$	$\sin(\alpha t+\phi)$
2.21	$\frac{s}{s^2+\alpha^2}$	$\cos(\alpha t)$
2.22	$\frac{s^2-\alpha^2}{\left(s^2+\alpha^2\right)^2}$	$t\cos(\alpha t)$
2.23	$\frac{1}{s\left(s^2+\alpha^2\right)}$	$\frac{1}{\alpha^2}[1-\cos(\alpha t)]$

Function Transforms		
N	F(s)	f (t) , t > 0
2.24	$\frac{1}{\left(s^2+\alpha^2\right)^2}$	$\frac{1}{2\alpha^3}[\sin(\alpha t)-\alpha t\cos(\alpha t)]$
2.25	$\frac{s}{\left(s^2+\alpha^2\right)^2}$	$\frac{1}{2\alpha}[t\sin(\alpha t)]$
2.26	$\frac{s^2}{\left(s^2+\alpha^2\right)^2}$	$\frac{1}{2\alpha}[\sin(\alpha t)+\alpha t\cos(\alpha t)]$
2.27	$\frac{1}{\left(s^2+\omega^2\right)\left(s^2+\alpha^2\right)}$, $\alpha \neq \omega$	$\left\{\frac{1}{\omega^2-\alpha^2}\right\}\left\{\frac{1}{\alpha}\sin(\alpha t)-\frac{1}{\omega}\sin(\omega t)\right\}$
2.28	$\frac{\alpha}{s^2(s+\alpha)}$	$t-\frac{1}{\alpha}[1-\exp(-\alpha t)]$
2.29	$\frac{\beta}{(s+\alpha)^2+\beta^2}$	$\exp(-\alpha t)\sin(\beta t)$
2.30	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$	$\exp(-\alpha t)\cos(\beta t)$
2.31	$\frac{s+\lambda}{(s+\alpha)^2+\beta^2}$	$\exp(-\alpha t)\left\{\cos(\beta t)+\left[\frac{\lambda-\alpha}{\beta}\right]\sin(\beta t)\right\}$
2.32	$\frac{s+\alpha}{s^2+\beta^2}$	$\frac{\sqrt{\alpha^2+\beta^2}}{\beta}\sin(\beta t+\phi), \quad \phi=\arctan\left(\frac{\beta}{\alpha}\right)$

Function Transforms		
N	F(s)	f (t) , t > 0
2.33	$\frac{1}{s^2 - \alpha^2}$	$\frac{1}{\alpha}\sinh(\alpha t)$
2.34	$\frac{s}{s^2 - \alpha^2}$	$\cosh(\alpha t)$
2.35	$\arctan\left(\frac{\alpha}{s}\right)$	$\frac{1}{t}\sin(\alpha t)$
2.36	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
2.37	$\frac{1}{\sqrt{s+\alpha}}$	$\frac{1}{\sqrt{\pi t}}\exp(-\alpha t)$
2.38	$\frac{1}{\sqrt{s^3}}$	$2\sqrt{\frac{t}{\pi}}$

Function Transforms		
N	F(s)	f (t) , t > 0
3.1	$\frac{1}{\sqrt{s^2+\alpha^2}}$	$J_o(\alpha t)$, Bessel function given in Appendix A
3.2	$\frac{1}{\left(s^2+\alpha^2\right)^{3/2}}$	$\left(\frac{t}{\alpha}\right) J_1(\alpha t)$
3.3	$\frac{1}{\sqrt{s^2-\alpha^2}}$	$I_o(\alpha t)$, Modified Bessel function given in Appendix A
3.4	$\frac{1}{\left(s^2-\alpha^2\right)^{3/2}}$	$\left(\frac{t}{\alpha}\right) I_1(\alpha t)$
3.5	$\sqrt{s-\alpha}-\sqrt{s-\beta}$	$\frac{1}{2t\sqrt{\pi t}}\left[\exp(\beta t)-\exp(\alpha t)\right]$

Exponential Function Transforms		
N	F(s)	f (t) , t > 0
4.1	$\exp(-\alpha s),\ \alpha \geq 0$	$\delta(t-\alpha)$
4.2	$\frac{1}{s}\exp(-\alpha s)$	$u(t-\alpha)$
4.3	$\left\{\frac{\lambda}{(s+\alpha)^2+\beta^2}\right\}\exp(-\rho s)$	$\left[\frac{\lambda}{\beta}\right]\exp[-\alpha(t-\rho)]\sin[\beta(t-\rho)]$, $t > \rho$
4.4	$\left\{\frac{s+\lambda}{(s+\alpha)^2+\beta^2}\right\}\exp(-\rho s)$	$\exp[-\alpha(t-\rho)]\left\{\cos[\beta(t-\rho)]+\left[\frac{-\alpha+\lambda}{\beta}\right]\sin[\beta(t-\rho)]\right\}$, $t > \rho$

<u>References</u>

1. Jan Tuma, Engineering Mathematics Handbook, McGraw-Hill, New York, 1979.
2. F. Oberhettinger and L. Badii, Table of Laplace Transforms, Springer-Verlag, N.Y.,1972.
3. M. Abramowitz and I. Stegun, editors, Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Washington, D.C., 1964.

APPENDIX A

Gamma Function

Integral

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt, \qquad x > 0 \tag{A-1}$$

Series

$$\Gamma(x) = \lim_{n \to \infty} \left\{ \frac{n^x\, n!}{x(x+1)(x+2)\ldots(x+n)} \right\} \tag{A-2}$$

Bessel Function of the First Kind

Zero Order

$$J_o(x) = 1 - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} - \frac{(x/2)^6}{(3!)^2} + \ldots \tag{A-3}$$

First Order

$$J_1(x) = \frac{x}{2}\left[1 - \frac{(x/2)^2}{2(1!)^2} + \frac{(x/2)^4}{3(2!)^2} - \frac{(x/2)^6}{4(3!)^2} + \ldots\right] = -\frac{d}{dx}\left[J_o(x)\right] \tag{A-4}$$

Modified Bessel Function of the First Kind

Zero Order

$$I_o(x) = 1 + \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} + \frac{(x/2)^6}{(3!)^2} + \ldots \tag{A-5}$$

First Order

$$I_1(x) = \frac{x}{2}\left[1 + \frac{(x/2)^2}{2(1!)^2} + \frac{(x/2)^4}{3(2!)^2} + \frac{(x/2)^6}{4(3!)^2} + \ \ldots\right] = \frac{d}{dx}\left[I_o(x)\right] \tag{A-6}$$