Mathematical Modeling of Control Systems

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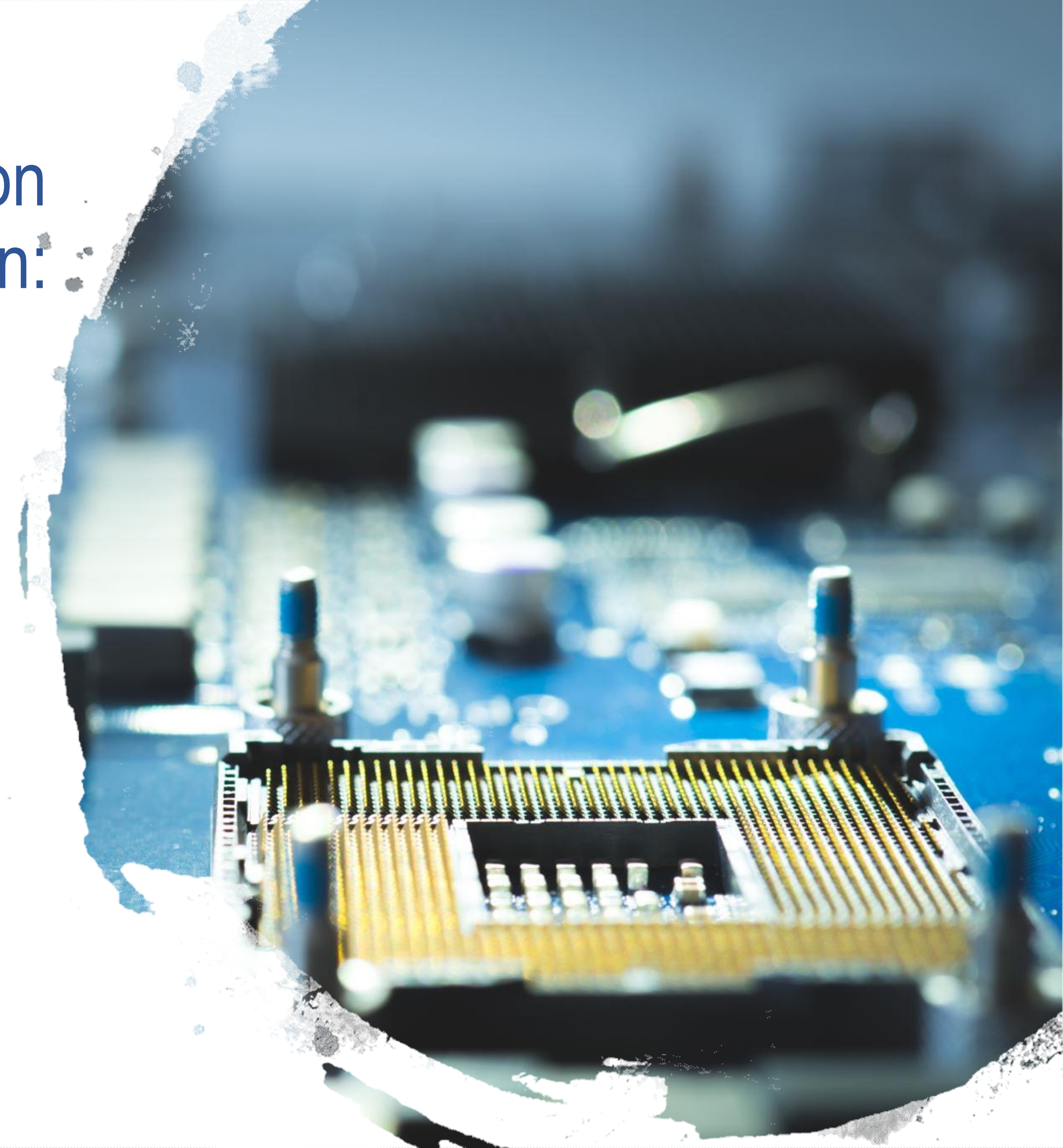
Learning Outcomes

- Convert a Transfer Function to State-Space Representation.
- Convert a State-Space Representation to a Transfer Function Model.
- Linear Approximation of Nonlinear Mathematical Models

Lesson #1: Transfer Function to State-Space Representation

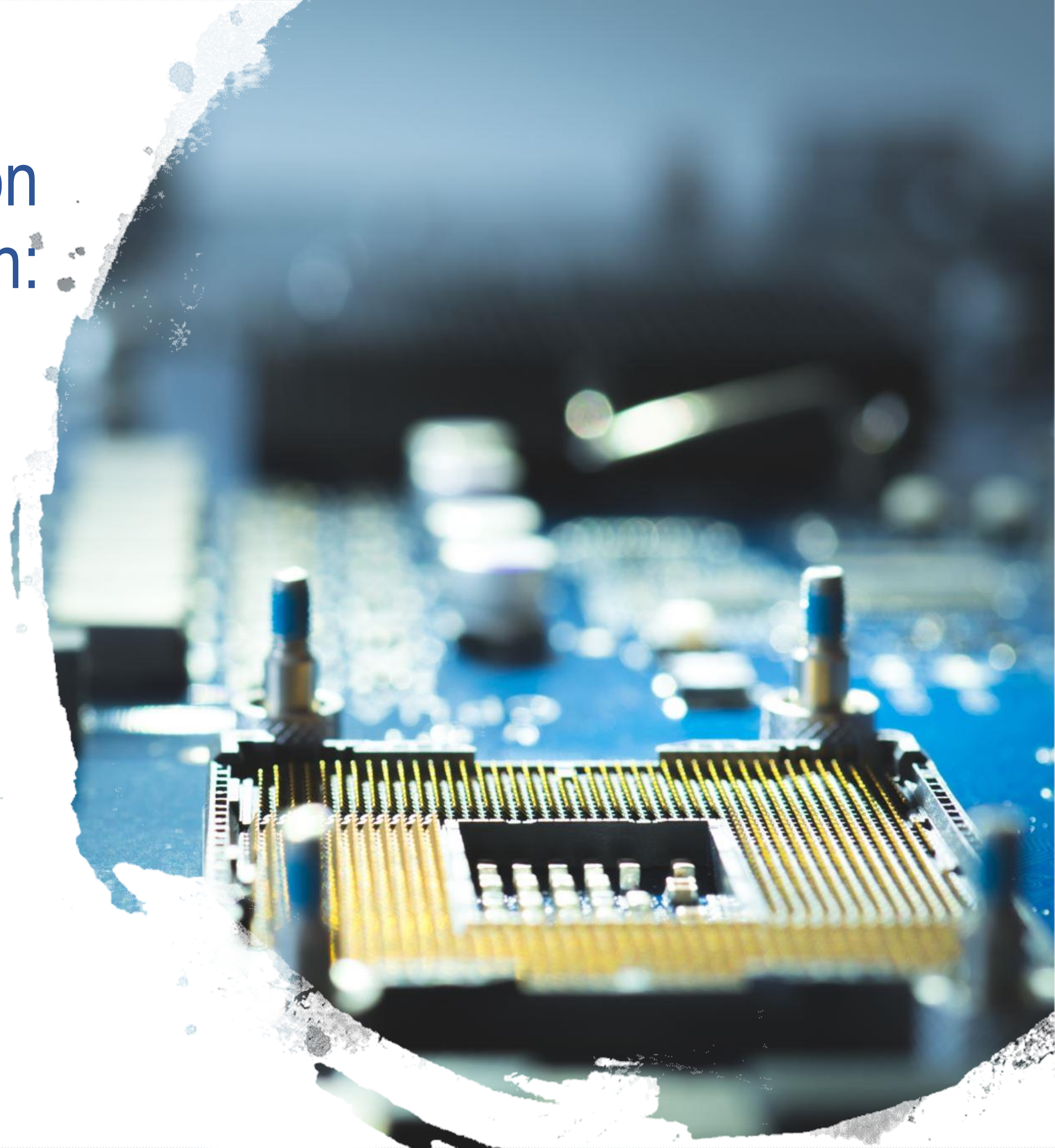
Conversion a Transfer Function to State-Space Representation:

- As previously remarked, a mathematical model of the system's differential equation to a transfer function algebraically relates a representation of the output to a representation of the input, $H(s)$.
- This approach is known as classical or frequency-domain technique.
- In this module, we will explore the conversion of a transfer function model of the system to state-space representation with an example of 2^{nd}-order differential equation to get started. Later, we'll generalize this approach to include n^{th}-order differential equations.
- Also, we will use the tf2ss(num, den) function in MATLAB® where num is the numerator polynomial and the den is the denominator polynomial of the transfer function $H(s)$, respectively.



Conversion a Transfer Function to State-Space Representation:

- A major advantage of the classical approach is that they rapidly provide stability and transient response information.
- We can immediately see the effects of varying system parameters until an acceptable design is met (See Module #6 for detailed information).
- The primary disadvantage of the classical approach is that it can be applied only to linear, time-invariant (LTI) systems (or systems that can be approximated as LTI).



Transfer Function Model of the Translational Mechanical System – An Example.

- In Module #2, we have the differential equation (DE) of a Translational Mechanical System as:

$$m\ddot{x}(t) + kx(t) + d\dot{x}(t) = f(t)$$

- Suppose that we have a constant input function $f(t) = F_0$, and the initial displacement is x_0 = 0 and the initial velocity is $\dot{x}_0$ = 0; if we apply the Laplace transform on both sides as (check the Laplace Transform Table):

$$m[s^2X(s) - \cancel{x_0}s - \cancel{\dot{x}_0}] + kX(s) + d[sX(s) - \cancel{x_0}] = F_0$$

$$s^2X(s) + ({}^k/_m)\,X(s) + ({}^d/_m)\,sX(s) = \frac{F_0}{m}$$

$$H(s) = \frac{X(s)}{F(s)} = \frac{\frac{F_0}{m}}{\left(s^2 + \left({}^d/_m\right)s + \left({}^k/_m\right)\right)}$$

Transfer Function Model of the Translational Mechanical System – An Example.

Where: $\frac{F_0}{m}$ is the numerator polynomial and, the $s\left(s^2 + \left({}^{d}/_{m}\right) s + \left({}^{k}/_{m}\right)\right)$ is the denominator polynomial of the transfer function of the Translational Mechanical System.

- So, to convert a transfer function to state-space representation, we start with the 2nd-orden linear differential equation; and then determine the transfer function, $H(s)$.

- For the implementation in the MATLAB®, we will define the variables num and den as follows:

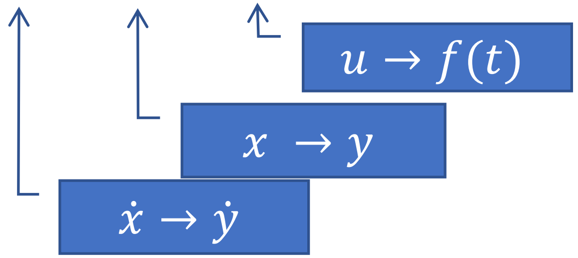
$$num = \frac{F_0}{m} \quad \rightarrow \quad \left[\frac{F_0}{m}\right]$$

$$\text{den} = \left(s^2 + \left({}^{d}/_{m}\right) s + \left({}^{k}/_{m}\right)\right) \rightarrow \left[1 \quad \left({}^{d}/_{m}\right) \quad \left({}^{k}/_{m}\right)\right]$$

Transfer Function Model of the Translational Mechanical System – An Example.

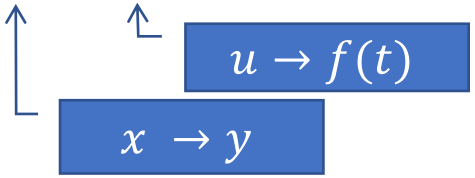
- So, the state-space model is formed by state equations:

$$\dot{x} = \boldsymbol{A}x + \boldsymbol{B}u$$



- And the output equation, y, is:

$$y = \boldsymbol{C}x + \boldsymbol{D}u$$



- There are two energy storage elements, so we expect two state equations. The energy storage elements are the spring, k, the mass, m. Therefore, we choose as our state variables x (the energy in spring is $\frac{1}{2}ky^2$), and the velocity of mass, $\dot{y}$ (the energy in the mass m is $\frac{1}{2}mv^2$, where v is the first derivative of y):

Transfer Function Model of the Translational Mechanical System – An Example.

$$\ddot{y}(t) = -({}^{d}/_{m})\dot{y}(t) - ({}^{k}/_{m})y(t) + \frac{1}{m}f(t) \qquad \text{Eq. (5-1)}$$

$$x_2 = \dot{y} \qquad x_1 = y \qquad u = f(t)$$

- In the Eq. (5-1), we picked y(t) and its derivatives to express the differential equation, since x is already a state variable; then, choosing the state variables x_1 and x_2 as defined above, and differentiating both sides yields:

$$\dot{x}_1 = \dot{y} \qquad \text{Eq. (5-2)}$$

$$\dot{x}_2 = \ddot{y} \qquad \text{Eq. (5-3)}$$

- Now, substituting the definitions of Eq. (5-2) and Eq. (5-3) into Eq. (5-1), the state equations are evaluated as:

$$\dot{x}_2 = -\left({}^{d}/_{m}\right)x_2 - \left({}^{k}/_{m}\right)x_1 + \frac{1}{m}u$$

$$\therefore \dot{x}_2 = -\left({}^{k}/_{m}\right)x_1 - \left({}^{d}/_{m}\right)x_2 + ({}^{1}/_{m})u$$

Eq. (5-4)

Transfer Function Model of the Translational Mechanical System – An Example.

- In vector-matrix form, Eq. (5-4) become:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\left(k/m\right) & -\left(d/m\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(1/m\right) \end{bmatrix} u$$

Eq. (5-5)

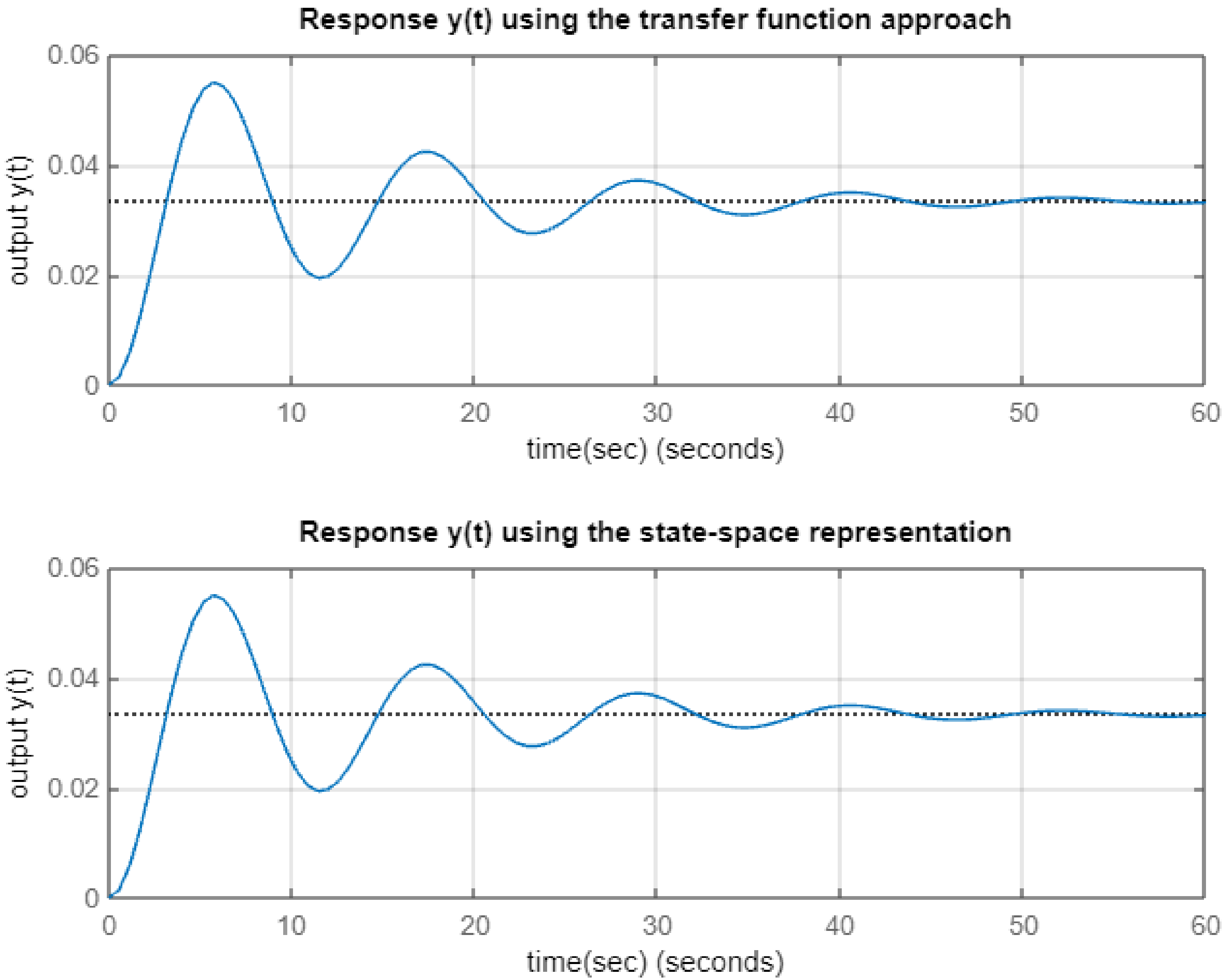
- And the output equation, y, is:

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Eq. (5-6)

- In the MATLAB®: A numerical example is available in the files of this module titled as “mass_spring_damper.m” in Blackboard using the built-in tf2ss(num, den) function. A plot of the response using this technique is shown in the next slide.

Transfer Function Model of the Translational Mechanical System – Plot.



Transfer Function to State-Space Representation – Generalizing to n^th^-Order Linear Differential Equations.

- To represent a general, n^th^-order, linear differential equation with constant coefficients, a_i and b_0, in state- space representation, we consider the differential equation in the form:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_0 u(\mathrm{t})$$

- An appropriate way to choose state variables is to choose the output, $y(t)$, and its $(n-1)$ derivatives as the state variables. Dropping the time, t, this become:

$$\frac{d^n y}{dt^n} = -a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} - \cdots - a_1\frac{dy}{dt} - a_0 y + b_0 u$$

$$x_n = \frac{d^n y}{dt^n} \qquad x_{n-1} = \frac{d^{n-1}y}{dt^{n-1}} \qquad x_2 = \frac{dy}{dt} \qquad x_1 = y \qquad u$$

Eq. (5-7)

Transfer Function to State-Space Representation – Generalizing to nth-Order Linear Differential Equations.

- Then, differentiating both sides of Eq. (5-7), yields:

$$\dot{x}_1 = \frac{dy}{dt}$$
$$\dot{x}_2 = \frac{d^2y}{dt^2}$$
$$\dot{x}_3 = \frac{d^3y}{dt^3}$$
$$\vdots$$
$$\dot{x}_n = \frac{d^ny}{dt^n}$$

Eq. (5-8)

- Substituting the definitions of Eq. (5-7) and Eq. (5-8) into Eq. (5-1), the state equations are evaluated as:

Transfer Function to State-Space Representation – Generalizing to nth-Order Linear Differential Equations.

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
x_n &= -a_0 x_1 - a_1 x_2 - a_2 x_3 \cdots a_{n-1} x_n + b_0 u
\end{aligned}
$$

Eq. (5-9)

- In vector-matrix form, Eq. (5-9) become:

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u
$$

Eq. (5-10)

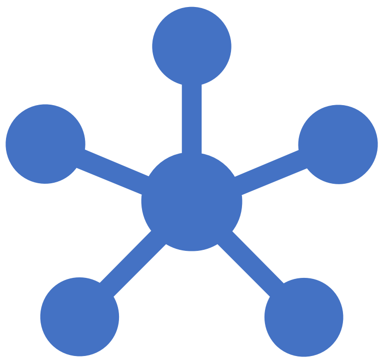
Transfer Function to State-Space Representation – Generalizing to nth-Order Linear Differential Equations.

- And the output equation, y, become:

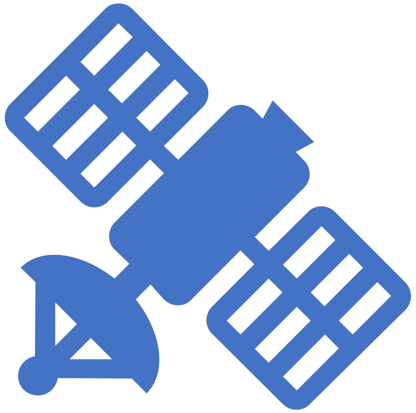
$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \qquad \text{Eq. (5-11)}$$

Remember: The dynamics of many systems, whether they are mechanical, electrical, thermal, economic, biological, and so on, may be described in terms of:

Differential Equation Model

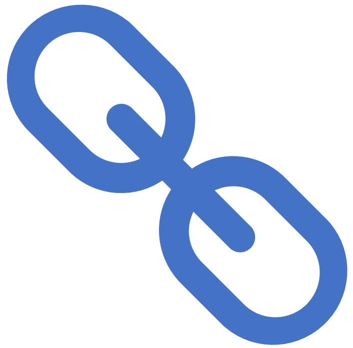


Transfer Function Model



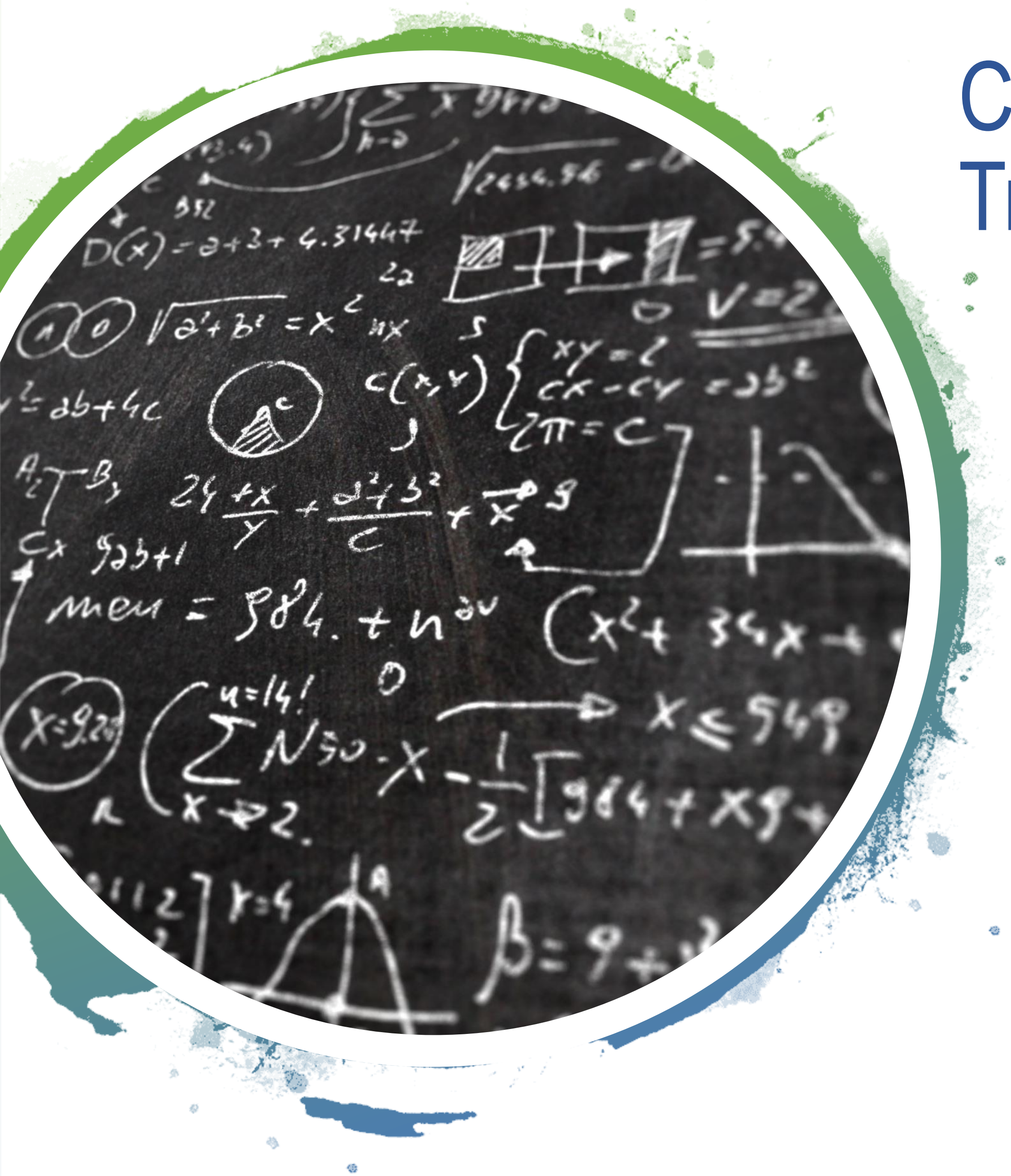
Or

State Space Model



Lesson #2:

Conversion State-Space to Transfer Function Representation

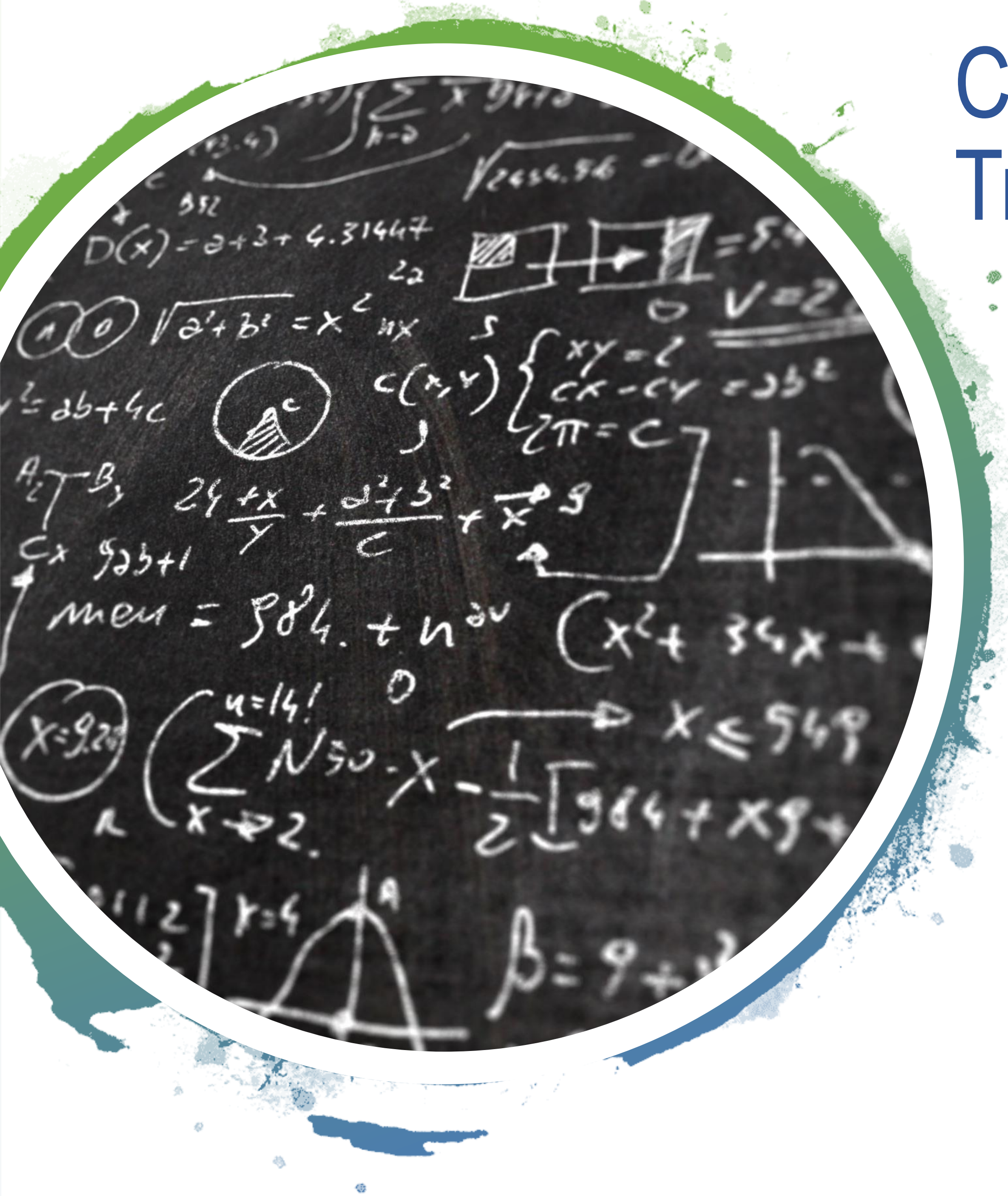


Conversion of State-Space to Transfer Function Representation

- Next, we'll focus on how to derive the transfer function of a single-input, single-output (SISO) system from the state-space equations.

- Let us consider the system whose transfer function is given by:

$$G(s) = \frac{Y(s)}{U(s)} \qquad \text{Eq. (5-12)}$$



Conversion of State-Space to Transfer Function Representation

- This system may be represented in state space by the following equations:

$$\dot{x} = \boldsymbol{A}x + \boldsymbol{B}u$$
$$y = \boldsymbol{C}x + \boldsymbol{D}u \qquad \text{Eq. (5-13)}$$

- The Laplace Transforms of Eq. (5-12) and (5-13) are given by:

$$sX(s) - x(0) = \boldsymbol{A}X(s) + \boldsymbol{B}U(s)$$
$$Y(s) = \boldsymbol{C}X(s) + \boldsymbol{D}U(s) \qquad \text{Eq. (5-14)}$$

Conversion of State-Space to Transfer Function Representation

- In Eq. (5-14), we set $x(0)$ to be zero; then, we have:

$$sX(s) - \cancelto{0}{x(0)} = \boldsymbol{A}X(s) + \boldsymbol{B}U(s)$$

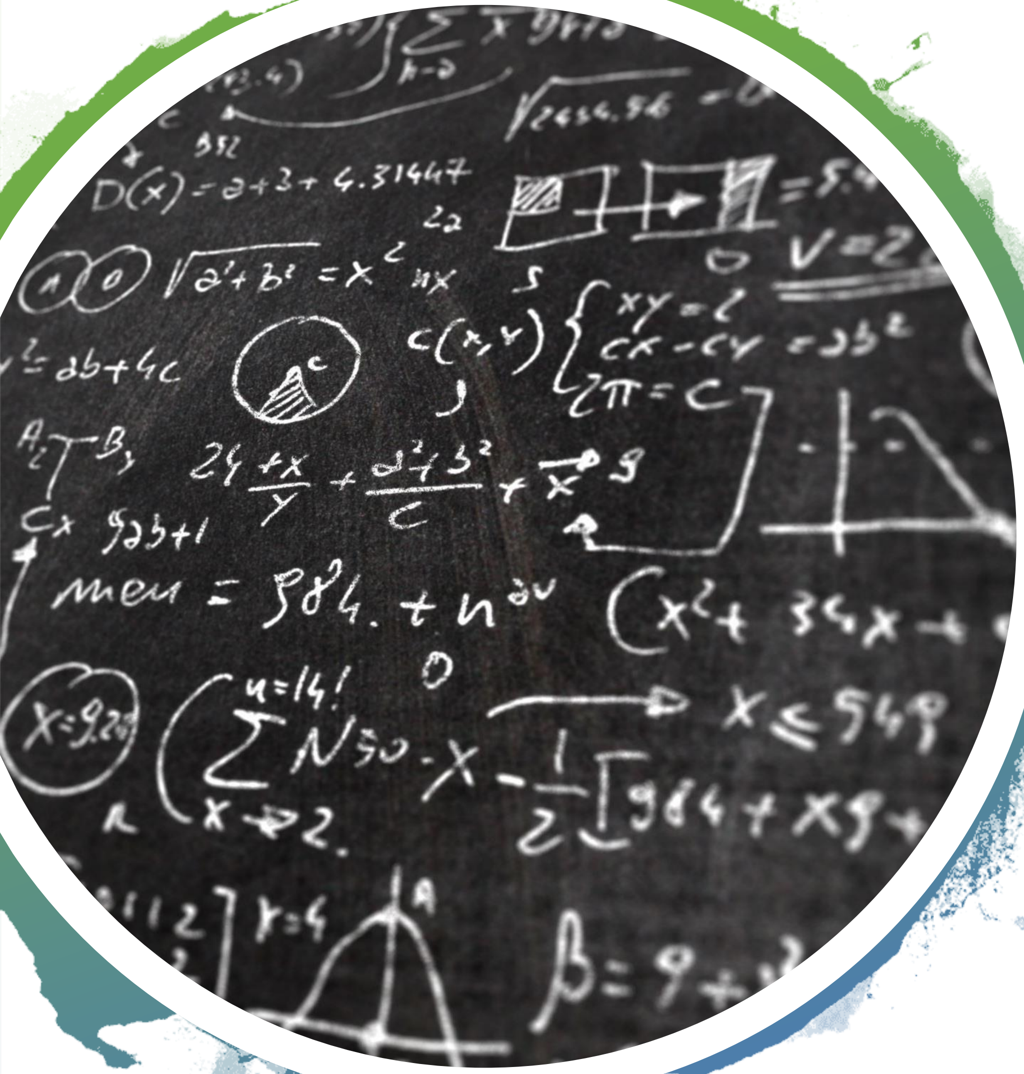
$$\therefore sX(s) - \boldsymbol{A}X(s) = \boldsymbol{B}U(s)$$

$$\rightarrow (sI - \boldsymbol{A})X(s) = \boldsymbol{B}U(s) \qquad \text{Eq. (5-15)}$$

- Pre-multiplying $(sI - \boldsymbol{A})^{-1}$ to both sides of Eq. (5-15), we obtain:

$$(sI - \boldsymbol{A})^{-1}(sI - \boldsymbol{A})X(s) = (sI - \boldsymbol{A})^{-1}\boldsymbol{B}U(s)$$

$$\rightarrow X(s) = (sI - \boldsymbol{A})^{-1}\boldsymbol{B}U(s) \qquad \text{Eq. (5-16)}$$



Conversion of State-Space to Transfer Function Representation

- The output $Y(s)$ can be found by substituting Eq. (5–16) into Eq. (5–14):

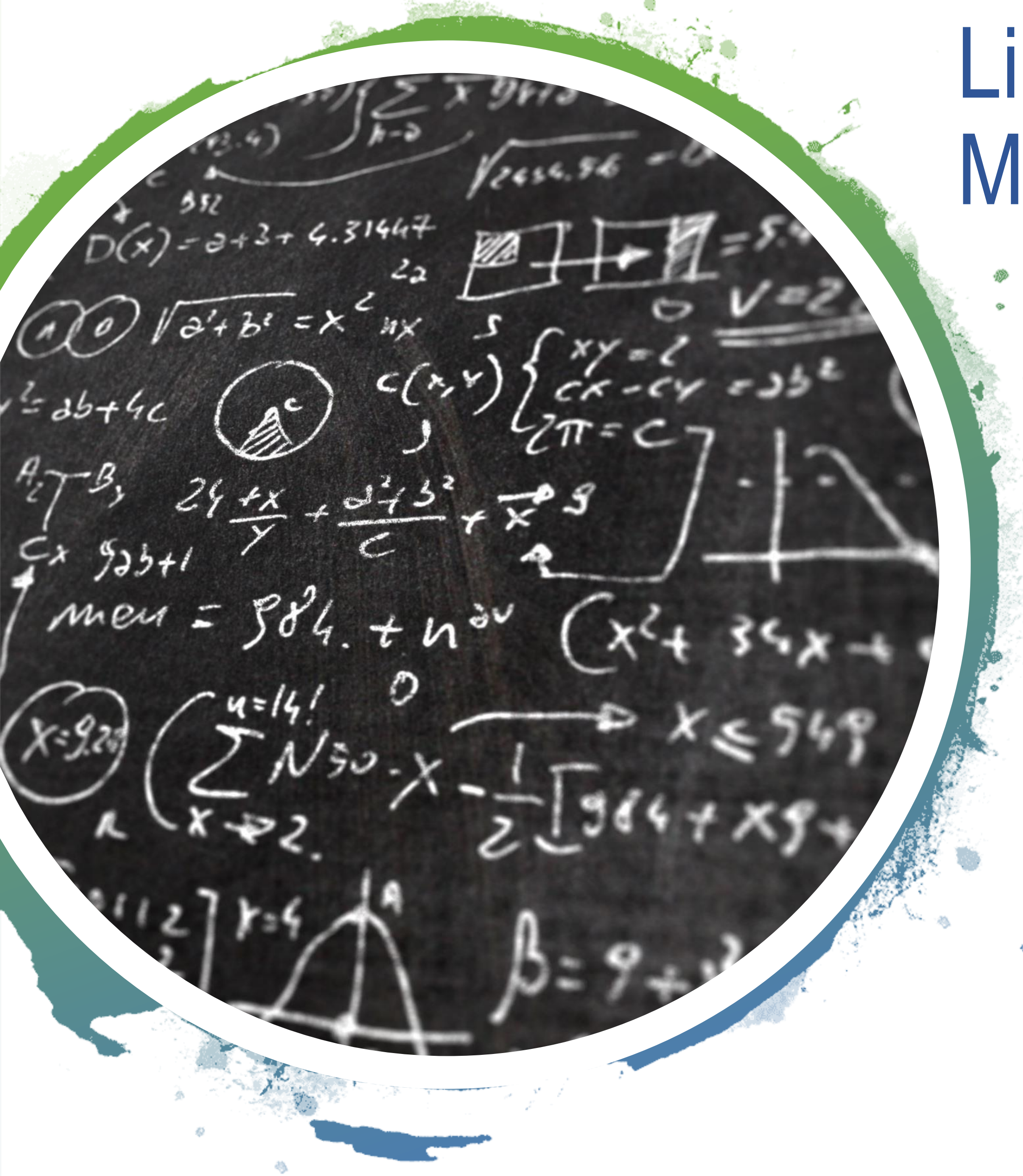
$$\rightarrow Y(s) = [C(sI - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}]\,U(s)$$

Eq. (5-17)

- Comparing Eq. (5–17) with Eq. (5-12), we see that the transfer function of the system, $G(s)$, is:

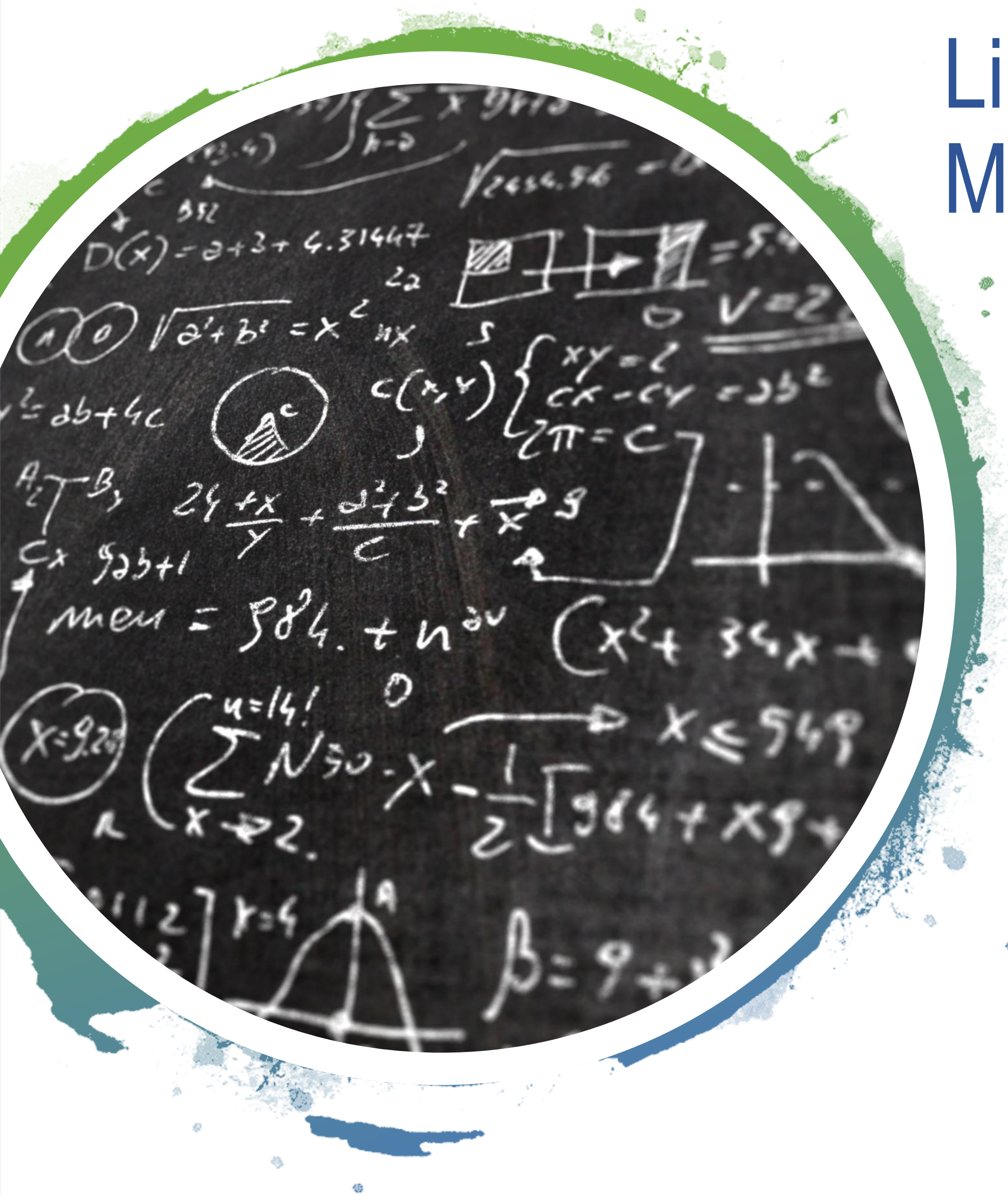
$$\rightarrow G(s) = [C(sI - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}] \qquad \text{Eq. (5-18)}$$

- In Blackboard, download a file titled “State Space to Transfer Function Examples” to have access to a numerical examples on the application of the Eq. (5-18). Also, there is a file available in the files of this module titled as “ss_2_tf.m” in Blackboard using the built-in ss2tf(A,B,C,D) function.



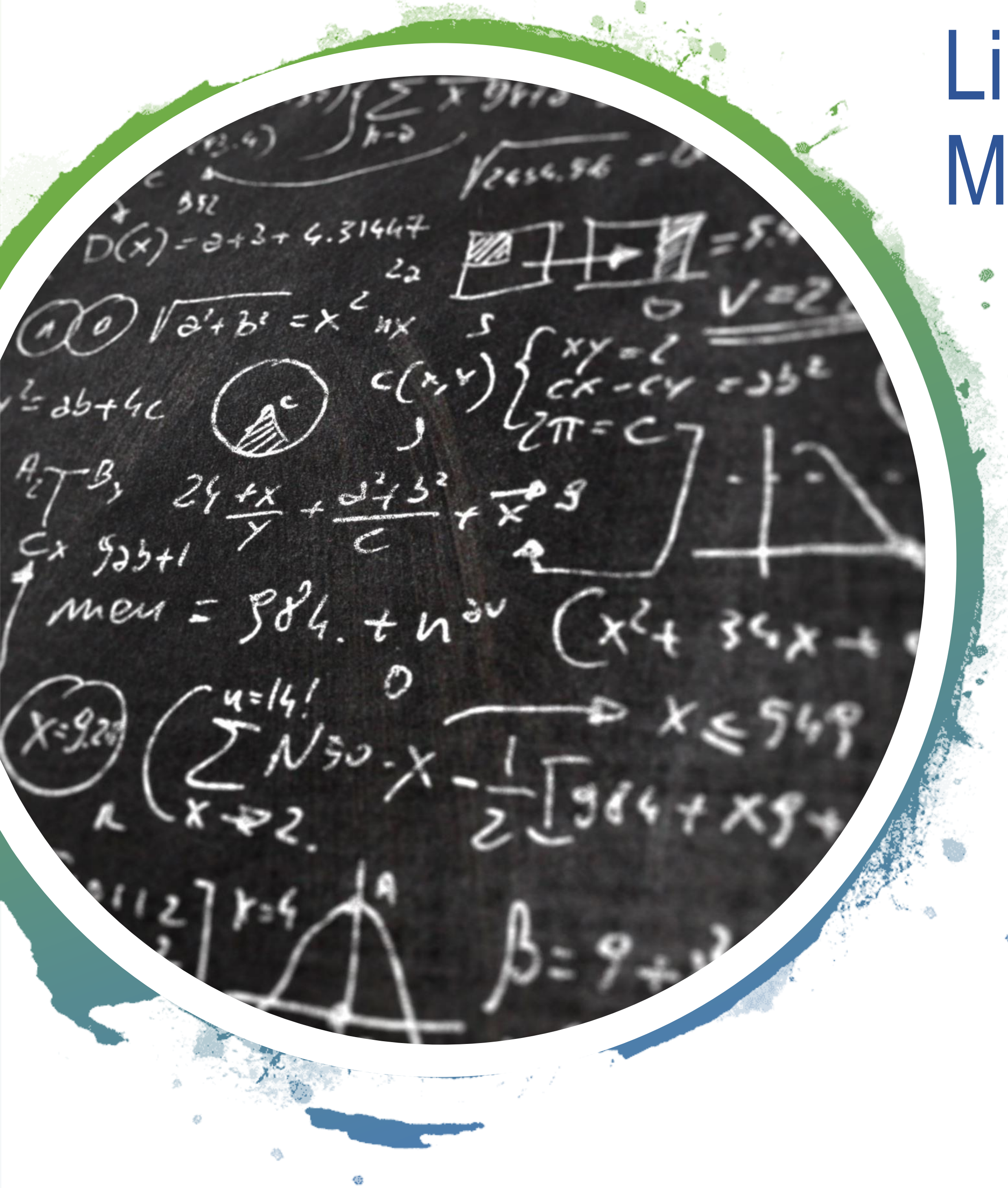
Linear Approximation of Nonlinear Mathematical Models

- In practice, many electromechanical systems, hydraulic systems, pneumatic systems, and so on, involve non-linear relationships among the variables.
- If such system operates around an equilibrium point and if the signals involved are small signals, then it is possible to approximate the nonlinear system by a linear system that works within a limited operating range.
- The linearization procedure is based on the expansion of nonlinear function into a Taylor series about the operating point and the retention of only the linear term.



Linear Approximation of Nonlinear Mathematical Models

- The higher-order terms neglected terms in Taylor series must be small enough; that is, the variables deviate only slightly from the operating condition.
- In summary, to obtain a linear approximation to the nonlinear system, we will expand the equations that represent the system into a Taylor series about the normal operating point.



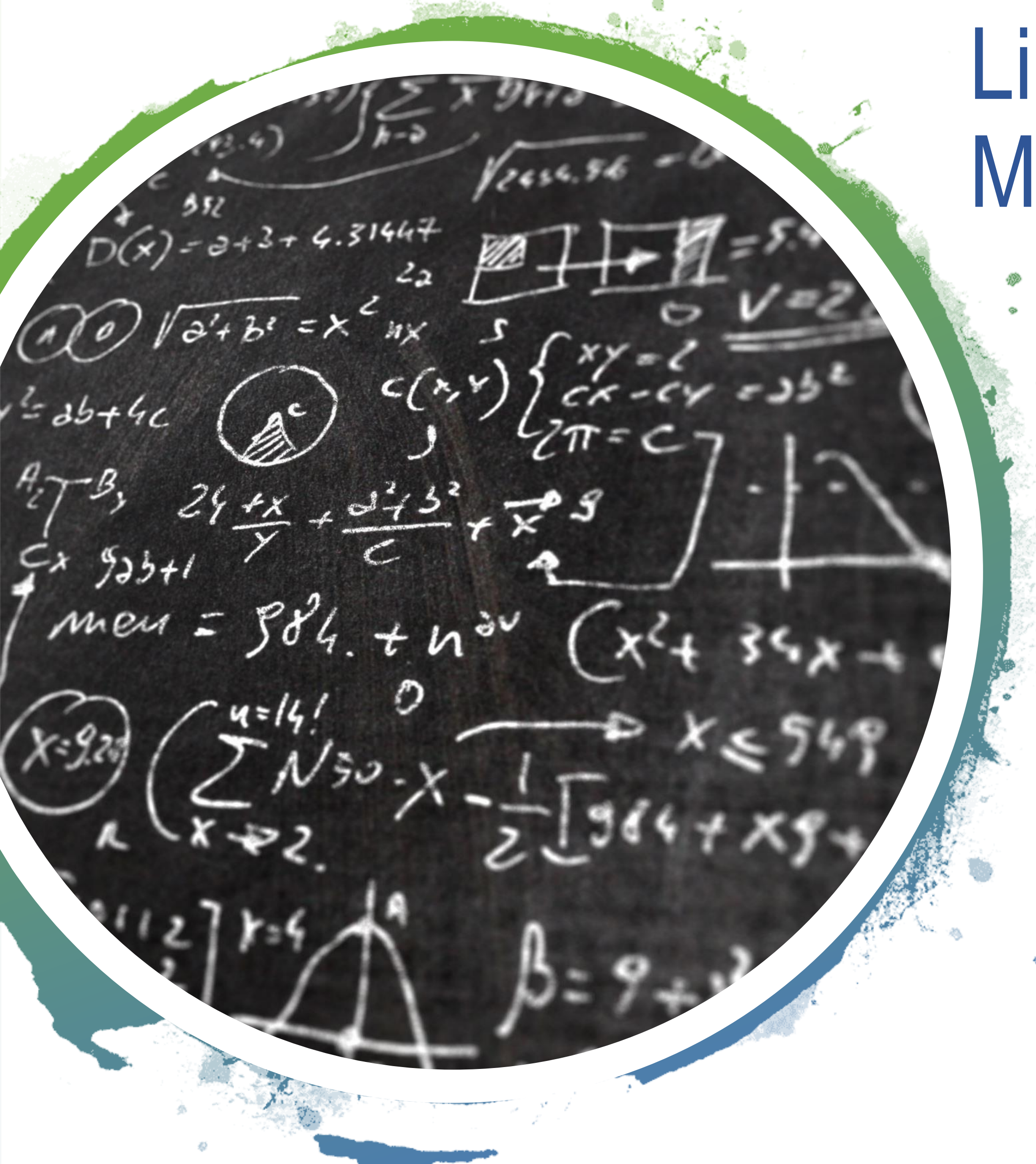
Linear Approximation of Nonlinear Mathematical Models

- The Taylor series expansion of nonlinear function is as follows:

$$y = f(\bar{x}_1, \bar{x}_2) + \left[\frac{\partial f}{\partial x_1}(x_1 - \bar{x}_1) + \frac{\partial f}{\partial x_2}(x_2 - \bar{x}_2)\right]$$

$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x_1^2}(x_1 - \bar{x}_1)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - \bar{x}_1)(x_2 - \bar{x}_2) + \frac{\partial^2 f}{\partial x_2^2}(x_2 - \bar{x}_2)^2\right] + \cdots$$

where the partial derivatives are evaluated at $x_1 = \bar{x}_1$ and $x_2 = \bar{x}_2$. Near the normal operating point, the higher-order terms may be neglected. The linear mathematical model of this nonlinear system about the normal operating condition is then given by:



Linear Approximation of Nonlinear Mathematical Models

$$y-\bar{y}=a(x_1-\bar{x}_1)+\mathrm{b}(x_2-\bar{x}_2)$$

- Where:

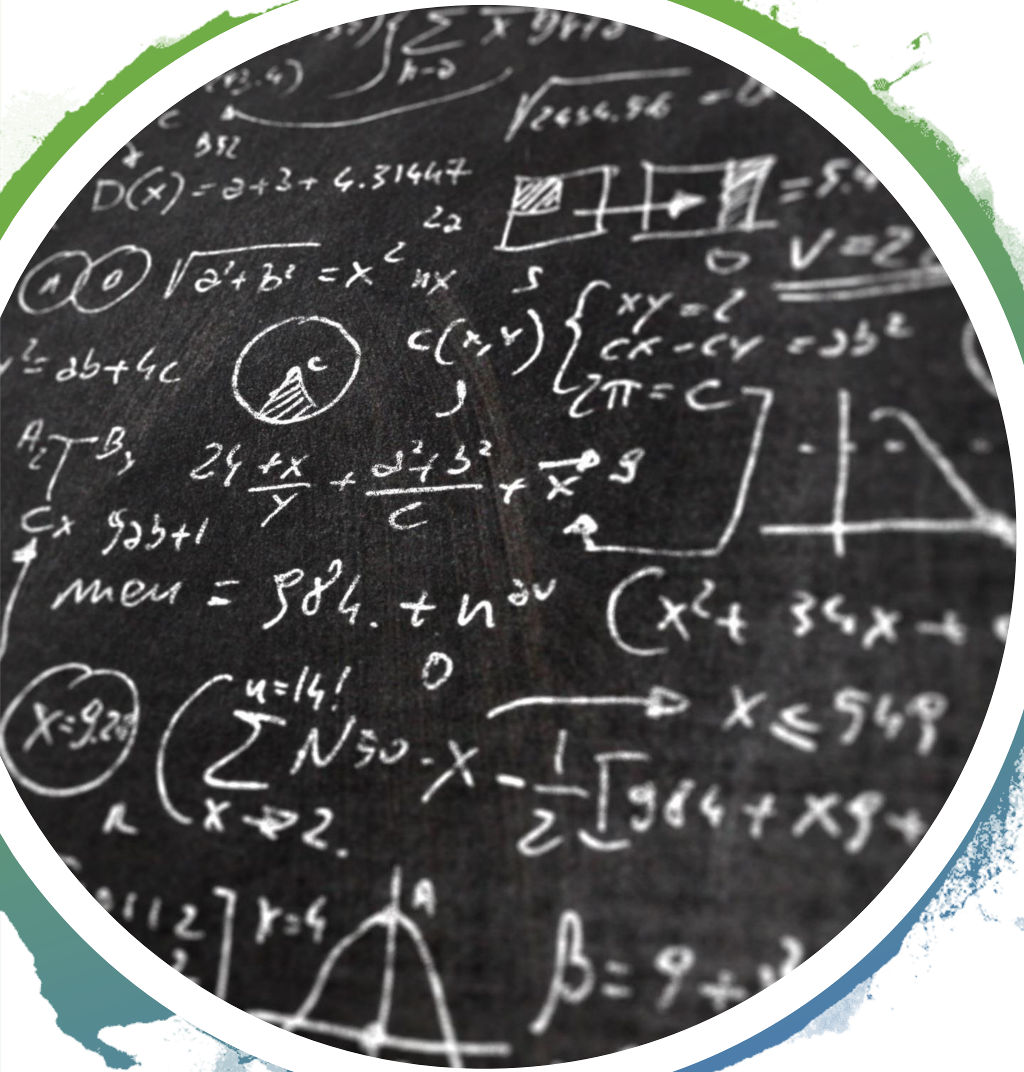
$$\bar{y}=f(\bar{x}_1,\bar{x}_2)$$

$$a=\frac{\partial f}{\partial x_1}|_{x_1=\bar{x}_1,x_2-\bar{x}_2}$$

And

$$b=\frac{\partial f}{\partial x_2}|_{x_1=\bar{x}_1,x_2-\bar{x}_2}$$

- As an example, we linearize the nonlinear equation $z=xy$ in the region $5\leq x\geq 7$, $10\leq \mathrm{y}\geq 12$; and find the error if the linearized equation is used to calculate the value of z when $x=5$, $y=10$.



Linear Approximation of Nonlinear Mathematical Models

- Since the region considered is given by $5 \leq x \geq 7, 10 \leq \mathrm{y} \geq 12$, we can choose $\bar{x} = 6$, and $\bar{y} = 11$ as the operating point. Then, we obtain a linearized equation for the nonlinear equation near a point $\bar{x} = 6$, and $\bar{y} = 11$.
- Expanding the nonlinear equation into a Taylor series about point $\mathrm{x} = \bar{x}$, and $\mathrm{y} = \bar{y}$ and neglecting the higher-order terms, we have:

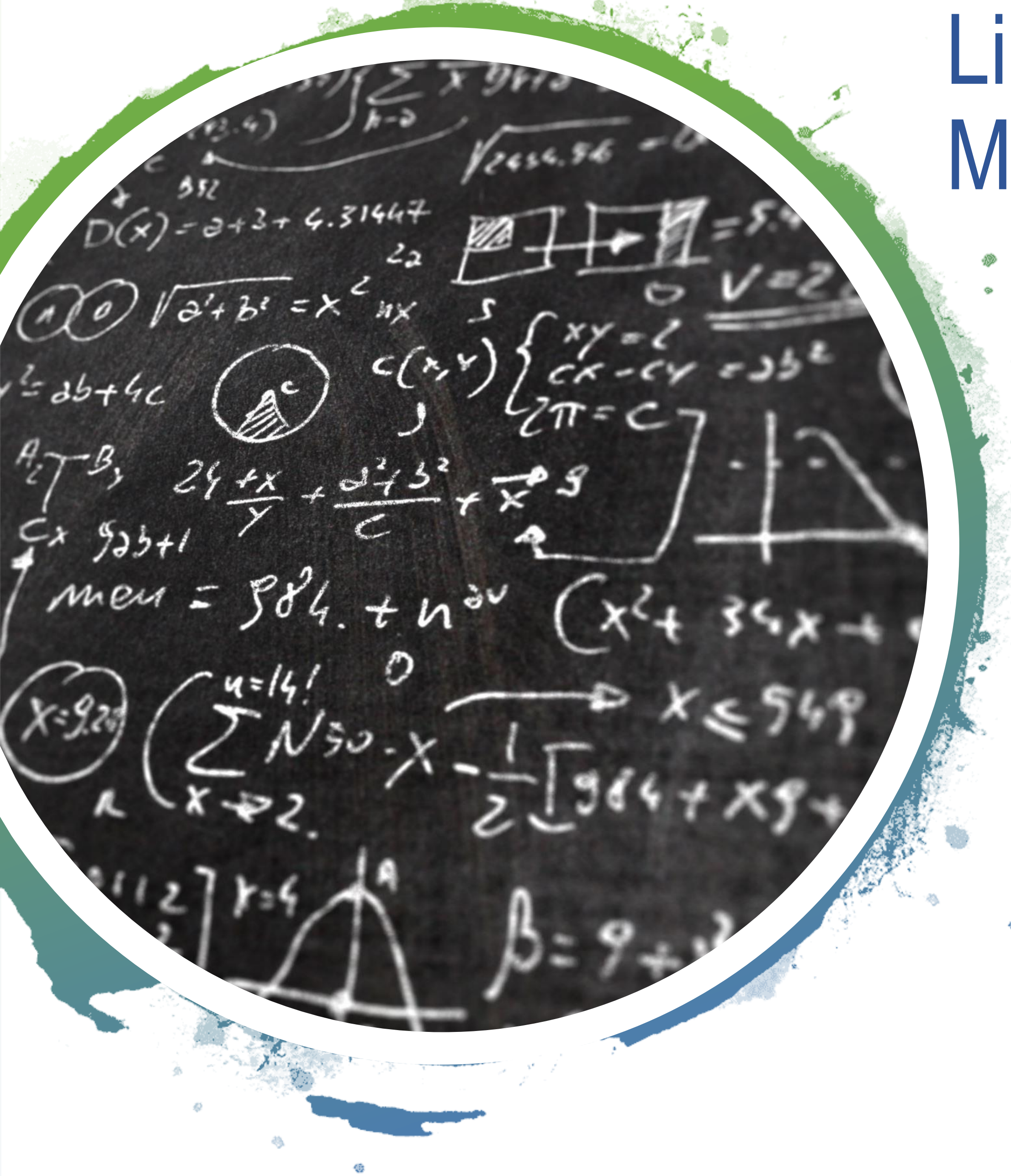
$$z - \bar{z} = a(x - \bar{x}) + \mathrm{b}(\mathrm{y} - \bar{y})$$

$$\bar{z} = \bar{x}\bar{y} = (6)(11) = 66$$

$$a = \frac{\partial(xy)}{\partial x}\Big|_{\mathrm{x}=\bar{x},\ \mathrm{y}=\bar{y}} = \bar{y} = 11$$

$$b = \frac{\partial(xy)}{\partial y}\Big|_{\mathrm{x}=\bar{x},\ \mathrm{y}=\bar{y}} = \bar{x} = 6$$

Linear Approximation of Nonlinear Mathematical Models



- Then, the linearized equation is:

$$z - 66 = 11(x - 6) + 6(y - 11) \text{ or}$$

$$z = 11x + 6y - 66$$

- When $x = 5$ and $y = 10$, the value of z given by the linearized equation is:

$$z = 11x + 6y - 66$$
$$= 11(5) + 6(10) - 66 = 49$$

- The exact value of $z = xy = (5)(10) = 50$. So, the error is $e = 50 - 49 = 1$. In terms of percentage, the error is 2%.

Thank You

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