HW-4. Ogata B3-14 and B3-15

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1 Modeling of a mass and two springs system with double damping

Obtain a mathematical model for the system shown in figure 3-44

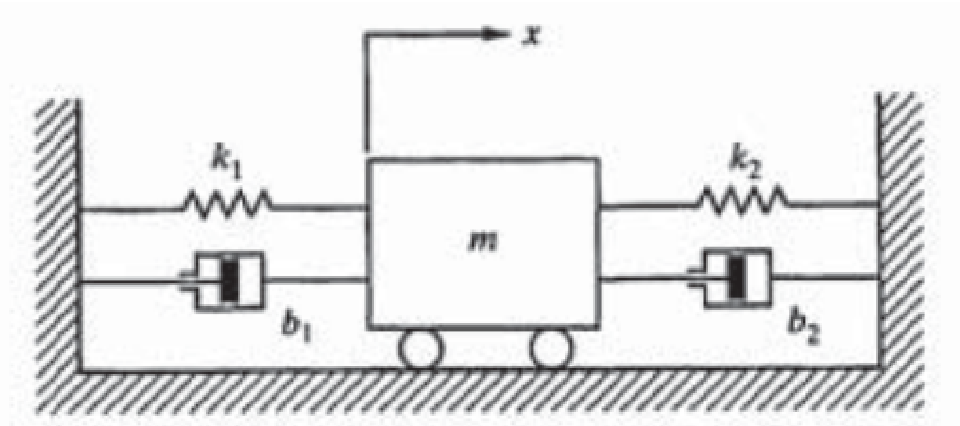

Figure 1: Figure 3-44

We are tasked with modeling the motion of a mass m positioned at the center between two rigid walls, connected to two springs with constants k_1 and k_2, and subjected to friction forces proportional to velocity, with coefficients b_1 and b_2. The equilibrium position is defined as $x = 0$. The mass is displaced to x_0 and released from rest. Using Newton's second law, we derive the governing equation, solve it using the Laplace transform method, and compute the inverse transform to express the displacement $x(t)$ explicitly.

Step 1: Free-Body Diagram and Newton's Second Law

The forces acting on the mass are:

- Restoring force due to the spring on the left: $-k_1 x$

- Restoring force due to the spring on the right: $-k_2 x$
- Frictional force due to velocity, left side: $-b_1 \dot{x}$
- Frictional force due to velocity, right side: $-b_2 \dot{x}$

Applying Newton's second law, the net force equals the mass times acceleration:

$$m\ddot{x} = -k_1 x - k_2 x - b_1 \dot{x} - b_2 \dot{x}. \tag{1}$$

Simplifying:

$$m\ddot{x} + (b_1 + b_2)\dot{x} + (k_1 + k_2)x = 0. \tag{2}$$

This is a second-order linear differential equation.

Step 2: Laplace Transform of the Differential Equation

Taking the Laplace transform of both sides (with zero initial velocity, $\dot{x}(0) = 0$, and initial displacement $x(0) = x_0$):

$$\mathcal{L}\{m\ddot{x}\} + \mathcal{L}\{(b_1 + b_2)\dot{x}\} + \mathcal{L}\{(k_1 + k_2)x\} = 0. \tag{3}$$

Using Laplace transform properties:

$$\begin{aligned} \mathcal{L}\{\ddot{x}\} &= s^2 X(s) - s x(0), \\ \mathcal{L}\{\dot{x}\} &= s X(s), \\ \mathcal{L}\{x\} &= X(s), \end{aligned}$$

where $X(s)$ is the Laplace transform of $x(t)$. Substituting:

$$m\big(s^2 X(s) - s x_0\big) + (b_1 + b_2) s X(s) + (k_1 + k_2) X(s) = 0. \tag{4}$$

Reorganizing:

$$X(s)\big[m s^2 + (b_1 + b_2)s + (k_1 + k_2)\big] = m s x_0. \tag{5}$$

Solving for $X(s)$:

$$X(s) = \frac{m s x_0}{m s^2 + (b_1 + b_2)s + (k_1 + k_2)}. \tag{6}$$

Step 3: Simplify the Transfer Function

The denominator is a quadratic equation:

$$m s^2 + (b_1 + b_2)s + (k_1 + k_2). \tag{7}$$

Let $B = b_1 + b_2$ and $K = k_1 + k_2$. Thus:

$$X(s) = \frac{m s x_0}{m s^2 + B s + K}. \tag{8}$$

Step 4: Inverse Laplace Transform

The solution is obtained by partial fraction decomposition and using standard Laplace transform tables. Rewrite the denominator:

$$ms^2 + Bs + K = m\big(s^2 + \frac{B}{m}s + \frac{K}{m}\big). \tag{9}$$

Define $\omega_n^2 = \frac{K}{m}$ (natural frequency) and $2\zeta\omega_n = \frac{B}{m}$ (damping ratio), where $\zeta = \frac{B}{2\sqrt{mK}}$. The denominator becomes:

$$s^2 + 2\zeta\omega_n s + \omega_n^2. \tag{10}$$

The transfer function is:

$$X(s) = \frac{x_0 s}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{11}$$

Using Laplace transform tables, the time-domain solution is:

$$x(t) = x_0 e^{-\zeta\omega_n t}\cos(\omega_d t), \tag{12}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency.

Final Solution

The displacement as a function of time is:

$$x(t) = x_0 e^{-\zeta\omega_n t}\cos\big(\omega_n\sqrt{1-\zeta^2}t\big). \tag{13}$$

2 Modeling a Mass-Spring-Damping System

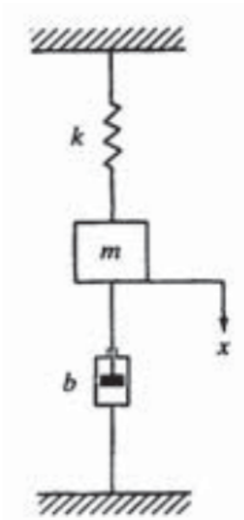

Figure 2: Figure 3-45

Consider the system shown in figure 3-45, where m = 2 kg, b = 4 Ns/m and k = 20 N/m. Assume that x(0) = 0.1 m, and v(0) = 0. The displacement is measured from the equilibrium position

We are tasked with modeling the motion of a mass m suspended vertically from a ceiling by a spring of stiffness k. The system is subjected to a damping force proportional to velocity with coefficient b. The equilibrium position is defined as $x = 0$, where the spring is stretched due to the weight of the mass. The mass is displaced to x_0 and released from rest. Using Newton's second law, we derive the governing equation, solve it using the Laplace transform method, and compute the inverse transform to express the displacement $x(t)$ explicitly.

Step 1: Free-Body Diagram and Newton's Second Law

The forces acting on the mass are:

- Restoring force due to the spring: $-kx$
- Damping force: $-b\dot{x}$
- Gravitational force: $-mg$

Applying Newton's second law:

$$m\ddot{x} = -kx - b\dot{x} - mg. \tag{14}$$

At equilibrium ($x = 0$), the spring force balances the weight:

$$kx_{\text{eq}} = mg \implies x_{\text{eq}} = \frac{mg}{k}. \tag{15}$$

Defining the displacement from equilibrium as $y = x - x_{\text{eq}}$, we substitute $x = y + x_{\text{eq}}$ into the equation:

$$m\ddot{y} = -k(y + x_{\text{eq}}) - b\dot{y} - mg,$$
$$m\ddot{y} = -ky - kx_{\text{eq}} - b\dot{y} - mg.$$

Using $kx_{\text{eq}} = mg$:

$$m\ddot{y} + b\dot{y} + ky = 0. \tag{16}$$

This is the governing equation for the displacement y from equilibrium.

Step 2: Laplace Transform of the Differential Equation

Taking the Laplace transform of both sides (with zero initial velocity, $\dot{y}(0) = 0$, and initial displacement $y(0) = x_0$):

$$\mathcal{L}\{m\ddot{y}\} + \mathcal{L}\{b\dot{y}\} + \mathcal{L}\{ky\} = 0. \tag{17}$$

Using Laplace transform properties:

$$\mathcal{L}\{\ddot{y}\} = s^2 Y(s) - sy(0),$$
$$\mathcal{L}\{\dot{y}\} = sY(s),$$
$$\mathcal{L}\{y\} = Y(s),$$

where $Y(s)$ is the Laplace transform of $y(t)$. Substituting:

$$m\big(s^2 Y(s) - sx_0\big) + bsY(s) + kY(s) = 0. \tag{18}$$

Reorganizing:

$$Y(s)\big[ms^2 + bs + k\big] = msx_0. \tag{19}$$

Solving for $Y(s)$:

$$Y(s) = \frac{msx_0}{ms^2 + bs + k}. \tag{20}$$

Step 3: Simplify the Transfer Function

The denominator is a quadratic equation:

$$ms^2 + bs + k. \tag{21}$$

Let $\omega_n^2 = \frac{k}{m}$ (natural frequency) and $2\zeta\omega_n = \frac{b}{m}$ (damping ratio), where $\zeta = \frac{b}{2\sqrt{mk}}$. The denominator becomes:

$$s^2 + 2\zeta\omega_n s + \omega_n^2. \tag{22}$$

The transfer function is:

$$Y(s) = \frac{x_0 s}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{23}$$

Step 4: Inverse Laplace Transform

Using Laplace transform tables, the time-domain solution is:

$$y(t) = x_0 e^{-\zeta\omega_n t}\cos(\omega_d t), \tag{24}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency.

Step 5: Final Solution with Parameters

For $m = 2$
, kg,
$k = 20$
, N/m,
$x_0 = 0.1$

, m,
$\dot{x}(0) = 0$,
$b = 4$
, Ns/m, we calculate:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{2}} = \sqrt{10}$$

, rad/s,

$$\zeta = \frac{b}{2\sqrt{mk}} = \frac{4}{2 \cdot \sqrt{2 \cdot 20}} = \frac{4}{4\sqrt{10}} = \frac{1}{\sqrt{10}},$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = \sqrt{10}\sqrt{1 - \frac{1}{10}} = \sqrt{10} \cdot \sqrt{\frac{9}{10}} = 3.$$

The displacement from equilibrium is:

$$x(t) = \frac{mg}{k} + x_0 e^{-\zeta \omega_n t} \cos(\omega_d t). \tag{25}$$

Substituting values:

$$x(t) = \frac{(2)(9.8)}{20} + 0.1e^{-\frac{1}{\sqrt{10}}\sqrt{10}t} \cos(3t) = 0.98 + 0.1e^{-t} \cos(3t). \tag{26}$$

The solution is represented in figure 3,

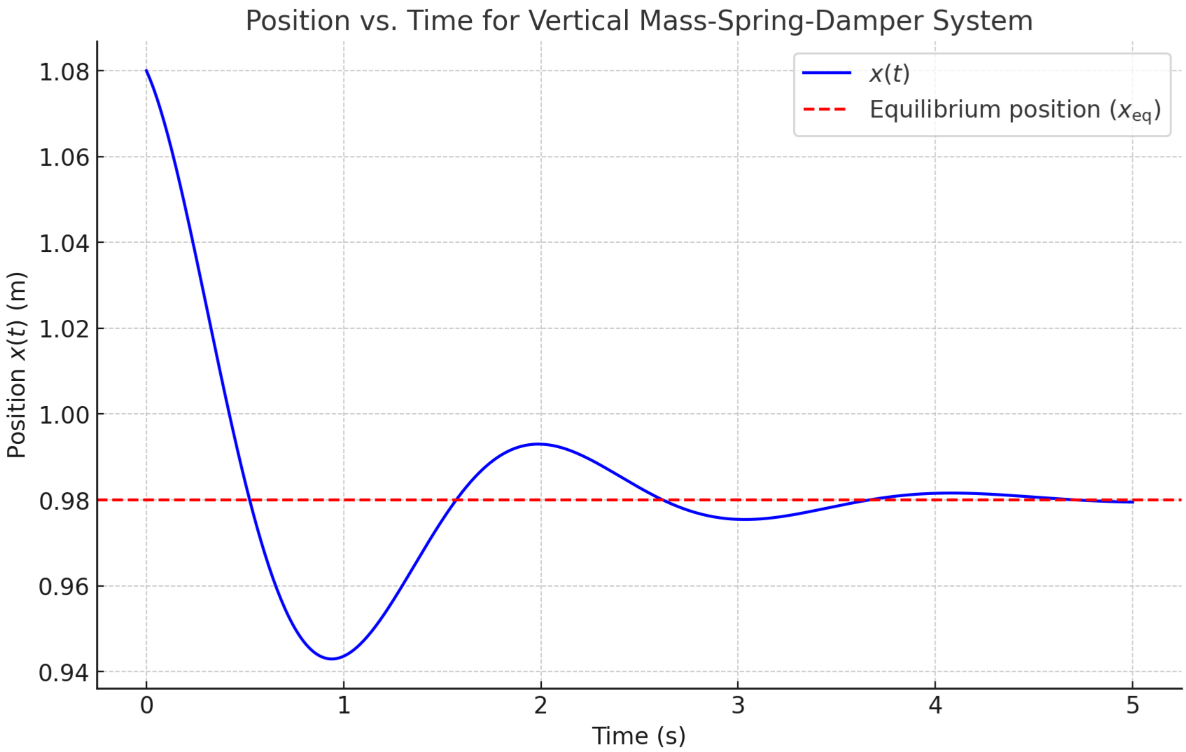

Figure 3: Position as a time function

redefining the zero position in the equilibrium position:

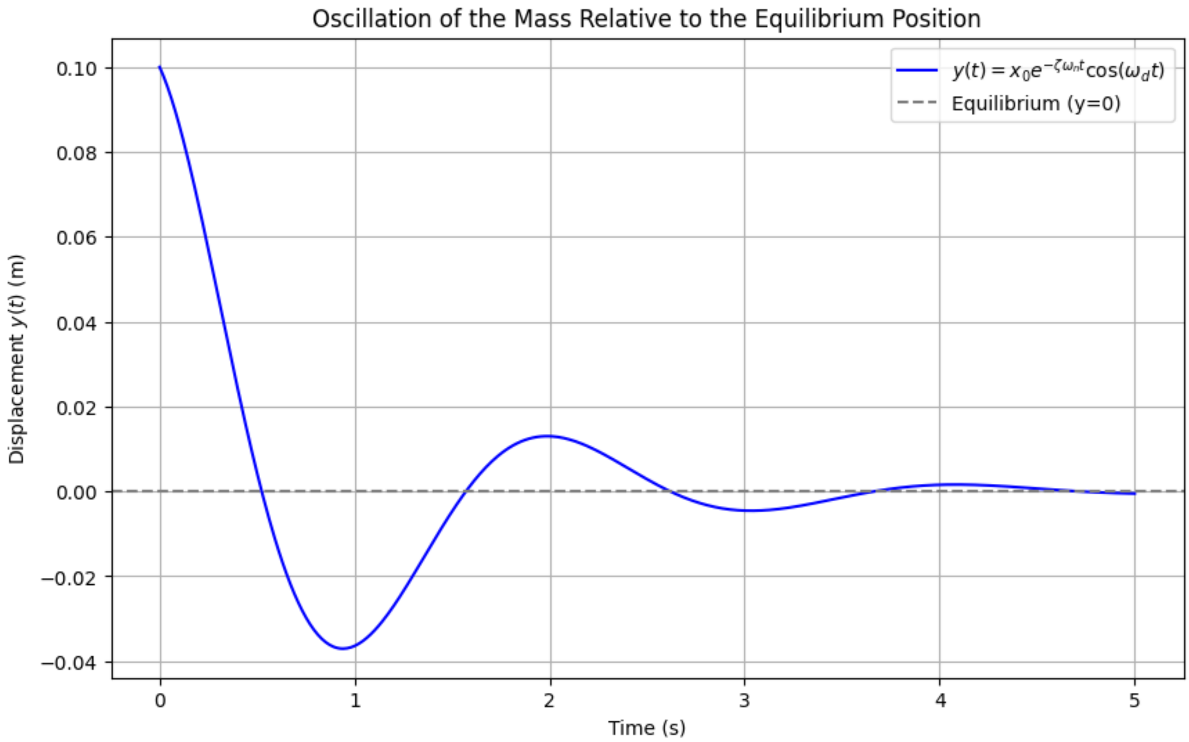

Figure 4: Position y as a time function

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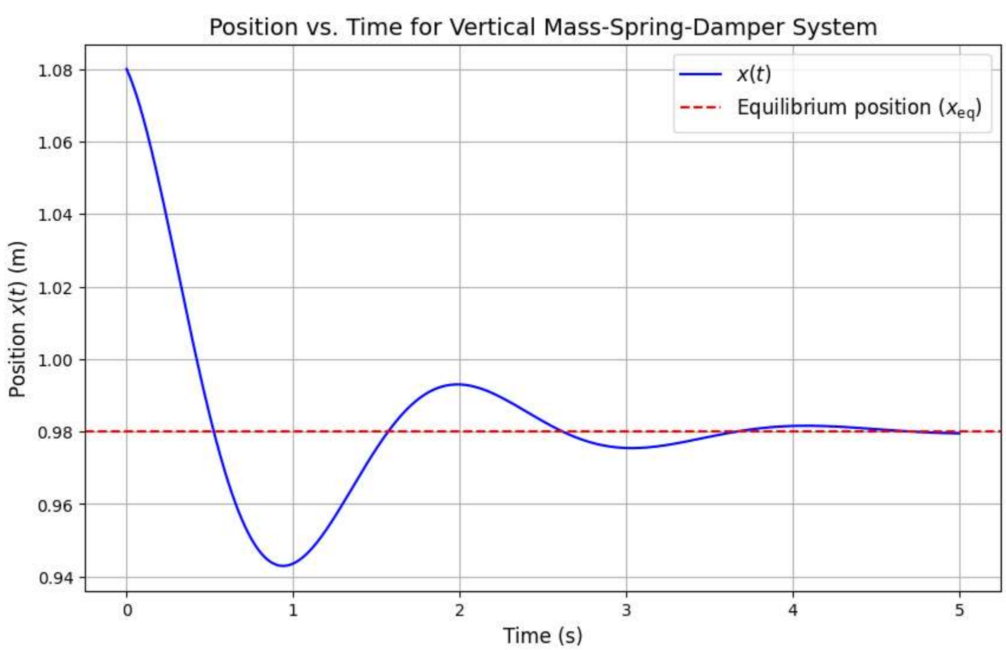
```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters
m = 2  # kg
k = 20  # N/m
b = 4  # Ns/m
x0 = 0.1  # m (initial displacement)
g = 9.8  # m/s^2 (gravitational acceleration)

# Derived parameters
omega_n = np.sqrt(k / m)  # natural frequency (rad/s)
zeta = b / (2 * np.sqrt(m * k))  # damping ratio
omega_d = omega_n * np.sqrt(1 - zeta**2)  # damped natural frequency
tau = 1 / (zeta * omega_n)  # characteristic decay time

# Time array: 0 to 5 times the decay time constant
t = np.linspace(0, 5 * tau, 500)

# Position as a function of time
x_eq = m * g / k  # equilibrium position due to gravity
x_t = x_eq + x0 * np.exp(-zeta * omega_n * t) * np.cos(omega_d * t)

# Plot the position over time
plt.figure(figsize=(10, 6))
plt.plot(t, x_t, label=r"$x(t)$", color="blue")
plt.axhline(x_eq, color="red", linestyle="--", label="Equilibrium position ($x_\mathrm{eq}$)")
plt.title("Position vs. Time for Vertical Mass-Spring-Damper System", fontsize=14)
plt.xlabel("Time (s)", fontsize=12)
plt.ylabel("Position $x(t)$ (m)", fontsize=12)
plt.legend(fontsize=12)
plt.grid(True)
plt.show()
```



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```
import numpy as np
import matplotlib.pyplot as plt

# Define parameters
m = 2.0  # kg
k = 20.0  # N/m
b = 4.0  # Ns/m
x_0 = 0.1  # m (initial displacement)
omega_n = np.sqrt(k / m)  # natural frequency
zeta = b / (2 * np.sqrt(m * k))  # damping ratio
omega_d = omega_n * np.sqrt(1 - zeta**2)  # damped natural frequency

# Define time range: up to 5 times the characteristic decay time (tau = 1 / (zeta * omega_n))
tau = 1 / (zeta * omega_n)
t = np.linspace(0, 5 * tau, 500)

# Compute y(t): displacement relative to equilibrium
y_t = x_0 * np.exp(-zeta * omega_n * t) * np.cos(omega_d * t)

# Plot y(t)
plt.figure(figsize=(10, 6))
plt.plot(t, y_t, label=r"$y(t) = x_0 e^{-\zeta \omega_n t} \cos(\omega_d t)$", color='blue')
plt.axhline(0, color='gray', linestyle='--', label="Equilibrium (y=0)")
plt.title("Oscillation of the Mass Relative to the Equilibrium Position")
plt.xlabel("Time (s)")
plt.ylabel("Displacement $y(t)$ (m)")
plt.legend()
plt.grid()
plt.show()
```