MODELING IN MECHANICAL, ELECTRICAL, FLUID, AND THERMAL SYSTEMS – PART 3/3

(IN THE TIME DOMAIN)

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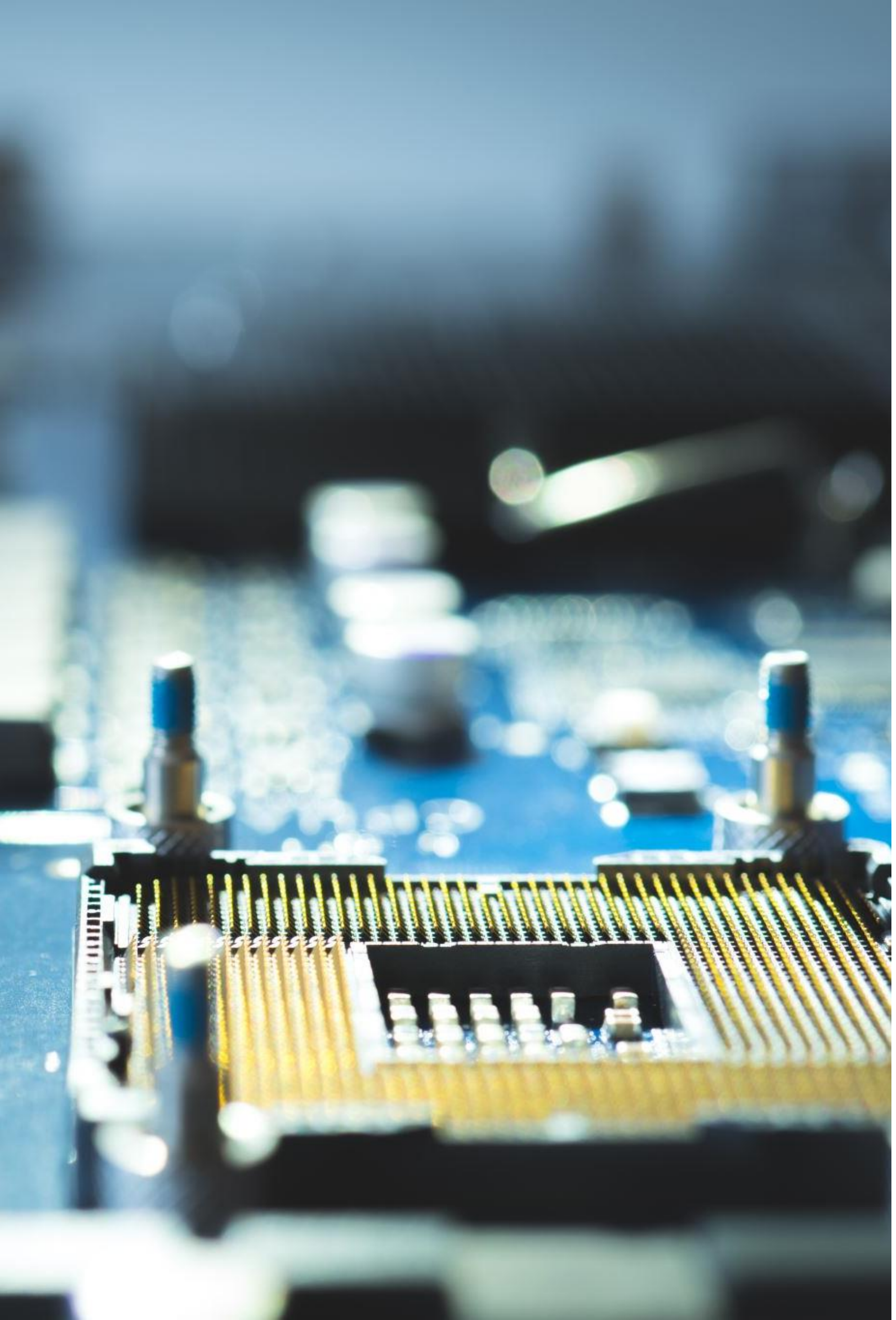


Learning Outcomes

- Describe the State-Space Model for Representing Physical Systems.
- Describe the State-Space Model for the Liquid-Level System - Example.
- Describe the State-Space Model for the Thermal System – Example.

Part #3:

State-Space Model for Thermal Systems

State-Space Model for the Thermal System

- Thermal systems are those that involve the transfer of heat from one substance to another.
- There are three different ways heat can flow from one substance to another: conduction, convection, and radiation. (Here we consider only conduction and convection).
- For conduction or convection heat transfer, the heat flow is:

$$q = K\Delta\theta \qquad \text{Eq. (3-18)}$$

State-Space Model for the Thermal System

Where:

q:= is the heat flow rate, $^{kcal}/_{sec}$,

$\Delta\theta$:= is the temperature difference, °C.

K:= is the coefficient, $^{kcal}/_{\mathrm{sec.}\,°\mathrm{C}}$.

- *For conduction*, the coefficient $K = \frac{kA}{\Delta X}$ where:

  k:= is the thermal conductivity, $^{kcal}/_{m\ \mathrm{sec}\ °\mathrm{C}}$

  A:= is the area normal to heat flow, m^2.

  ΔX:= is the thickness of the conductor, m.

State-Space Model for the Thermal System

- *For convection*, the coefficient $K = HA$ where:

  H:= is the convection coefficient, ${}^{kcal}/_{m^2 \text{ sec} \,°\text{C}}$
  A:= is the area normal to heat flow, m^2.

- The thermal resistance R for heat transfer between two substances may be defined as follows:

$$R = \frac{change\ in\ the\ temperature\ difference, °\text{C}}{change\ in\ \ the\ heat\ flow\ rate, {}^{kcal}/_{°\text{C}}} \qquad \text{Eq. (3-19)}$$

State-Space Model for the Thermal System

- The thermal resistance for conduction or convection heat transfer is given by:

$$R = \frac{d(\Delta\theta)}{dq} = \frac{1}{K} \qquad \text{Eq. (3-20)}$$

- Since the thermal conductivity and convection coefficients are almost constant, the thermal resistance for either conduction or convection is constant.

State-Space Model for the Thermal System

- The thermal capacitance C is defined by:

$$C = \frac{change\ in\ heat\ stored, kcal}{change\ in\ temperature, ℃} \qquad \text{Eq. (3-21)}$$

Or:

$$C = mc \qquad \text{Eq. (3-22)}$$

Where:

m:= mass of substance considered, kg

c:= specific heat of substance, $^{kcal}/_{kg\ ℃}$

State-Space Model for the Thermal System

- Consider the system shown in Figure 3-3:

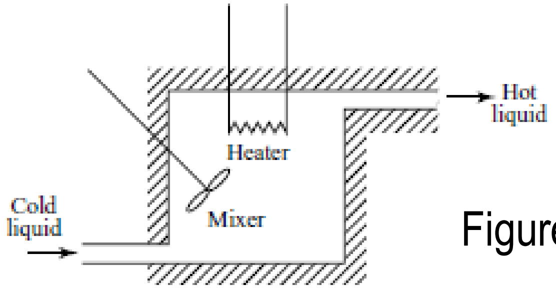


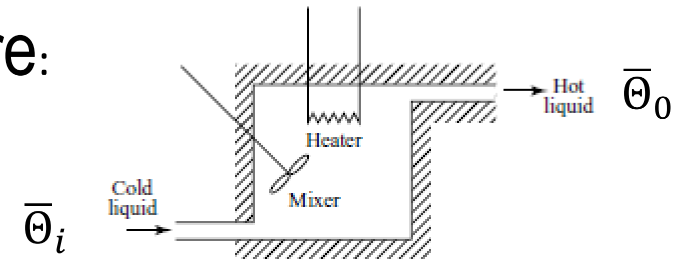
Figure 3-3

- Assumptions made:
  - tank is insulated to eliminate heat loss to the surrounding air.
  - there is no heat storage in the insulation
  - the liquid in the tank is perfectly mixed so that it is at a uniform temperature.

- *Therefore, a single temperature is used to describe the temperature of the liquid in the tank and of the outflowing liquid.*

State-Space Model for the Thermal System

- In Figure 3-3, the variables are:



$\bar{\Theta}_i$:= steady-state temperature of inflowing liquid, °C
$\bar{\Theta}_0$:= steady-state temperature of outflowing liquid, °C
G:= steady-state liquid flow rate, $^{kg}/_{sec}$
M:= mass of liquid in the tank, kg
c:= specific heat of liquid, $^{kcal}/_{kg\ °C}$
R:= thermal resistance, $^{°C\ sec}/_{kcal}$
C:= thermal capacitance, $^{kcal}/_{°C}$
$\bar{H}$:= steady-state heat input rate, $^{kcal}/_{sec}$

State-Space Model for the Thermal System

- We can obtain the dynamic equation given the following conditions:
  1. The temperature of the inflowing liquid is suddenly changed from $\bar{\Theta}_i$ to $\bar{\Theta}_i + \theta_i$ while the heat input rate H.
  2. The liquid flow rate G are kept constant.
  3. The heat outflow rate is changed from $\bar{H}$ to $\bar{H} + h_0$.
  4. The temperature of the outflowing liquid will be changed from $\bar{\Theta}_0$ to $\bar{\Theta}_0 + \theta_0$.

- For these conditions, the *heat balance equation* for this case is: $C\frac{d\theta}{dt} = Gc\theta_i - h_0$ Eq. (3-23)

- Substituting Eq. (3-20) into Eq. (3-23), yields: $RC\frac{d\theta}{dt} + \theta = \theta_i$ Eq. (3-24)

State-Space Model for the Thermal System: An Example

- Now, we derive the *state-space model for thermal control* system considering small deviations from steady-state operation. Assume that the heat loss to the surroundings is negligible.
- ***Solution:***
- From Eq. (3-24): $RC\frac{d\theta}{dt} + \theta = \theta_i$
- From Eq. (3-20): $Rdq = d\theta$
- The heat transfer results in a change in the system's temperature. So, taking the heat flow and temperature changes over a time interval, dt, Eq. (3-20) becomes:

$$R\frac{dq}{dt} = \frac{d\theta}{dt}$$

State-Space Model for the Thermal System: An Example

- Solution (cont.): Changing notation:

$$RC\frac{d\theta}{dt} + \theta(t) = \theta_i(t) \quad \rightarrow \quad RC\dot{\theta}(t) + \theta(t) = \theta_i(t) \qquad \text{(a)}$$

$$R\frac{dq}{dt} = \frac{d\theta}{dt} \quad \rightarrow \quad R\dot{q}(t) = \dot{\theta}(t) \qquad \text{(b)}$$

- From Eq. (a):

$$\dot{\theta}(t) = -\frac{1}{RC}\theta(t) + \frac{1}{RC}\theta_i(t) \qquad \text{(c)}$$

- ***Note***: Recall that $C = \frac{dq}{d\theta}$ and deriving $q = mc\theta$, we get $\mathrm{d}q = mcd\theta$ (or $\frac{dq}{d\theta} = mc$), where c is the specific heat of the liquid and m is the mass of liquid in the tank. Then:

$$C = mc \qquad \text{(d)}$$

$$q = C\theta \qquad \text{(e)}$$

State-Space Model for the Thermal System: An Example

- Solution (cont.): Substituting Eq (b) and Eq. (e) into Eq. (c), we get:

$$R\dot{q}(t) = \dot{\theta}(t) \qquad q(t) = C\theta(t) \qquad \dot{\theta}(t) = -\frac{1}{RC}\theta(t) + \frac{1}{RC}\theta_i(t)$$

$$\rightarrow R\dot{q}(t) = -\left(\frac{1}{RC}\right)\left(\frac{1}{C}\right)q(t) + \frac{1}{RC}\theta_i(t) \qquad \therefore\ \dot{q}(t) = -\frac{1}{R^2C^2}q(t) + \frac{1}{R^2C}\theta_i(t) \qquad (f)$$

- Let us define state variables $x_1(t)$ and $x_2(t)$ as:

$$x_1(t) = \theta(t) \qquad (g)$$

$$x_2(t) = q(t) \qquad (h)$$

State-Space Model for the Thermal System: An Example

- Solution (cont.):
- From Eq. (g): $\dot{x}_1(t) = \dot{\theta}(t) \Rightarrow$ $\dot{x}_1(t) = \dot{\theta}(t) = -\frac{1}{RC}\theta(t) + \frac{1}{RC}\theta_i(t)$ Eq. (i)
- From Eq. (h): $\dot{x}_2(t) = \dot{q}(t)$ $\dot{x}_2(t) = \dot{q}(t) = -\frac{1}{R^2C^2}q(t) + \frac{1}{R^2C}\theta_i(t)$ Eq. (j)
- Dropping t from Eq. (i) & Eq. (h), the state-space model in matrix form is:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

State Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & -\frac{1}{(RC)^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{R^2C} \end{bmatrix} u$$

Output Equation:

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State-Space Model for Representing Physical Systems

- In the state-space model representation, the numbers of state variables is equal with the highest-order of the ordinary differential equation (ODE) describing the dynamic system.
- The set of state variables is not unique, and they may be defined in terms of physical variables which can be measured.
- *State-space models* are the cornerstone of modem control theory.

THANK YOU

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