SECOND-ORDER SYSTEMS:

> GOVERNING EQUATION:

The differential equation is the form:

$$M\ddot{y} + b\dot{y} + ky = kx(t) \qquad (1) \qquad y(t) \equiv \text{output or response}$$

$$\text{or } M\frac{d^2y(t)}{dt^2} + b\frac{dy}{dt} + Ky(t) = Kx(t) \qquad (2) \qquad x(t) \equiv \text{input function}$$

> The CHARACTERISTIC EQUATION:

The characteristic equation corresponds to the denominator of the transfer function $Y(s)/X(s)$ equal to zero; If we assume zero initial conditions, the transfer function $Y(s)/X(s)$ is:

$$(Ms^2 + bs + K)\,Y(s) = KX(s)$$

$$\therefore \frac{Y(s)}{X(s)} = \frac{K}{Ms^2 + bs + K}$$

So, the characteristic equation of a second-order system will be:

$$Ms^2 + bs + K = 0 \qquad (3)$$

The system's response can be quantified in terms of the roots of the characteristic equation. In the case of a second-order system, the roots can be obtained from the evaluation of the quadratic formula as:

$$s_{1,2} = \frac{-b \pm b^2 - 4MK}{2M}$$

$$\therefore s_{1,2} = \frac{-b}{2M} \pm \frac{\sqrt{b^2 - 4MK}}{2M} \qquad (4)$$

Now, from Eq. (4) and from the type of roots evaluated, the response of the second-order system can be clasiffied as:

(1) Under-damped response
(2) Over-damped response, or
(3) Critically-damped response

1. UNDER-damped response :
In this case, the value $b^2 - 4MK$ is negative ; that $b^2 < 4MK$.
The roots $s_{1,2}$ are then complex conjugates: From Eq (4):

$$s_{1,2} = \frac{-b}{2M} \pm j\frac{\sqrt{4MK - b^2}}{2M} \quad \text{(complex-conjugate roots)}$$

Let $\sigma = \frac{b}{2M}$ and $w_d = \frac{\sqrt{4MK - b^2}}{2M}$, then

$$s_{1,2} = -\sigma \pm j w_d$$

The time-domain response $y(t)$ will be of the form:

$$y(t) = e^{-\sigma t} \{ c_1 \sin(w_d t) + c_2 \cos(w_d t) + X$$

where c_1 and c_2 are constants.

2. Over-damped response :
The response will be over-damped (non-oscillatory) if :

$$b^2 > 4MK$$

Once again, we use the quadratic formula to obtain the roots:

$$s_{1,2} = \frac{-b}{2M} \pm \frac{\sqrt{b^2 - 4MK}}{2M} \quad \text{or}$$

$$s_{12} = -\sigma \pm w_d \quad \text{where} \quad \sigma = \frac{-b}{2M} \ \& \ w_d = \frac{\sqrt{b^2 - 4MK}}{2M}$$

So, the roots are real and distinct. The time-domain response $y(t)$ is of the form: $y(t) = c_1 e^{(-\sigma + w_d)t} + c_2 e^{-(\sigma - w_d)t} + X$

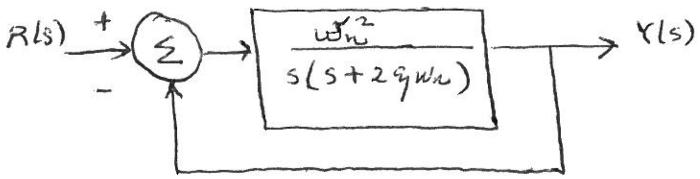
Note : There are not sin() or cos() terms in the response $y(t)$, so the system does not oscillate.

3. Critically-damped response:
Occurs when $b^2 = 4MK$, so the roots will be real and equal to : $s_{1,2} = \frac{-b}{2M}$. The response is of the form :

$$y(t) = c_1 e^{-\sigma t} + c_2 t e^{-\sigma t} + X.$$

TRANSIENT RESPONSE SPECIFICATIONS:

The performance characteristics of a control system can be specified in terms of the transient response to a unit-step input. Typically, it is assumed that the initial conditions $y(0)$ and $y'(0)$ are both zero. this way, the response characteristics can be compared to the canonical form of the transfer function of a second-order single loop feedback systems:



where:

$\xi \equiv$ damping ratio and

$\omega_n \equiv$ natural frequency, rad/s

So, the canonical form of a second-order system in terms of ξ and ω_n is:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \qquad (5)$$

the characteristic equation from (5) is then: $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

From Eq (3) obtained before, we have the characteristic equation of the 2nd order system in terms of parameters of the physical system M, b and K. This means that we can express the damping ratio and natural frequency in terms of the physical values of M, b and K. For example, if M = mass, b = friction coefficient and K = spring's coefficient, then we can easily compute the damping ratio and natural frequency of this system:

From Eq (3) $\frac{Y(s)}{X(s)} = \frac{K}{Ms^2 + bs + K}$ or $\frac{K}{M\left(s^2 + \frac{b}{M}s + \frac{K}{M}\right)}$

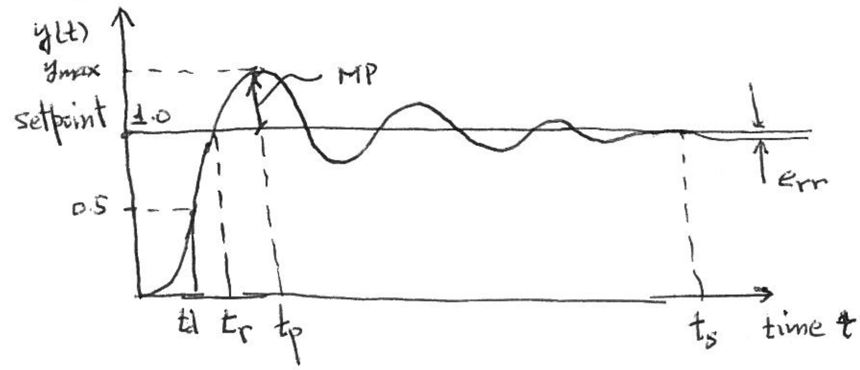
$$\therefore \frac{Y(s)}{X(s)} = \frac{K/M}{s^2 + \frac{b}{M}s + \frac{K}{M}} \qquad (6)$$

Comparing the terms in (5) and (6), we see that:

$$2\xi\omega_n = \frac{b}{M} \quad \& \quad \omega_n^2 = \frac{K}{M} \Rightarrow \omega_n = \sqrt{\frac{K}{M}} \ (\text{rad/s})$$

$$\xi = \frac{b}{2M\omega_n} = \frac{b}{2\sqrt{KM}} \quad (\text{dimensionless})$$

In practice, the transient response of a second-order system exhibits damped oscillations before reaching steady-state. as shown in the graph:



The following characteristics are commonly specified:

1. Delay time, t_d : time required for the response to reach about 50% of setpoint for the first time
2. Rise time, t_r : time required for the response to rise from 10% to 90% or 5% to 95% or 0% to 100%
3. Peak time, t_p : time required to reach to the peak of the overshoot
4. Maximum Overshoot, in percentage, %MP : max. value of the response measured from 1.
5. Settling time, t_s : time required to reach steady-state
6. Steady-state error, %err : measured value of the response at steady-state from 1.0.

> Calculating the peak time, t_p:

For an under-damped (oscillatory response), the response equation is

$$y(t) = 1 - e^{-\zeta \omega_n t}\left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t\right) \qquad (7)$$

The maximum value y_{max} is expressed as $\frac{dy}{dt} = 0$. So the peak time, t_p, corresponds to the time when $\frac{dy}{dt} = 0$. That is:

From (7):

$$\frac{dy}{dt} = \zeta \omega_n e^{-\zeta \omega_n t}\left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t\right) + e^{-\zeta \omega_n t}\left(\omega_d \sin \omega_d t - \frac{\zeta \omega_d}{\sqrt{1-\zeta^2}} \cos \omega_d t\right) = \underline{0}$$

since $\omega_d = \omega_n \sqrt{1-\zeta^2}$, the cosine terms cancel each other. So:

$$\frac{dy}{dt} = (\sin \omega_d t) \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t} = 0 \qquad (8)$$

Evaluating Eq 8 at $t = t_p$ yields:

$$\left.\frac{dy}{dt}\right|_{t=t_p} = (\sin w_d t_p)\frac{w_n}{\sqrt{1-\xi^2}}\, e^{-\xi w_n t_p} = 0.$$

We know ξ and w_n are not zero and $e^{-\xi w_n t_p} \neq 0$, so:

$\sin w_d t_p = 0$ in order to satisfy Eq 8:

$\sin()$ is zero at $0, \pi, 2\pi, 3\pi \ldots n\pi$ $n \in$ integer

Since the peak time corresponds to the first overshoot (max. overshoot)

$\therefore\ w_d t_p = \pi \quad \Rightarrow \quad \boxed{t_p = \frac{\pi}{w_d}}$

> Calculating the maximum overshoot:

As previously defined $M_p = y_{max} - 1$ (see graph also)

$$= y(t_p) - 1$$

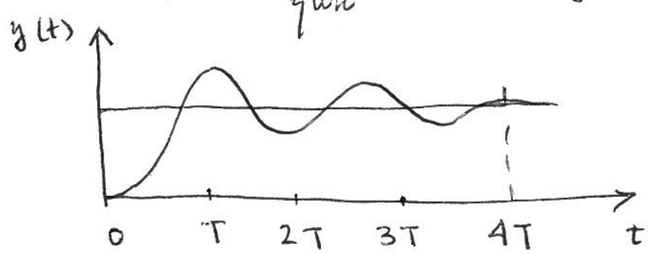
$$\therefore\ M_p = -e^{-\xi w_n\left(\frac{\pi}{w_d}\right)}\left(\cancel{\cos(\pi)}^{-1} + \frac{\xi}{\sqrt{1-\xi^2}}\cancel{\sin(\pi)}^{0}\right)$$

$\therefore\ M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$ In percentage $\%M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \times 100\%$

NOTE: If the final value $y(t\to\infty)$ is not unity, then the following equation must be used: $\%M_p = \frac{y(t_p) - y(\infty)}{y(\infty)}$

> Calculating the settling time, t_s:

The settling is obtained from the evaluation of Eq. (7) at multiple values of $T = \frac{1}{\xi w_n}$ in the range $[0\ n]$;



$t_s \approx 4T = \frac{4}{\xi w_n}$ (2% tolerance)

>> Calculating the Rise time, t_r :
The rise time is defined as the time required to reach the final value (setpoint) for the first time. So, given this definition and the plot shown on page 4, we know that $y(t_r) = 1.0$. That is,

$$y(t_r) = 1 \Rightarrow 1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right) = \underline{\underline{1}} \quad (9)$$

Since $e^{-\zeta\omega_n t} \neq 0$, Eq (9) becomes

$$\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t) = 0 \qquad (10)$$

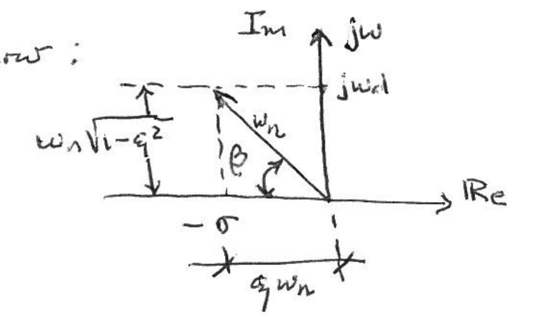
Using the following definitions of variables as before:

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\sigma = \zeta\omega_n$, from (10):

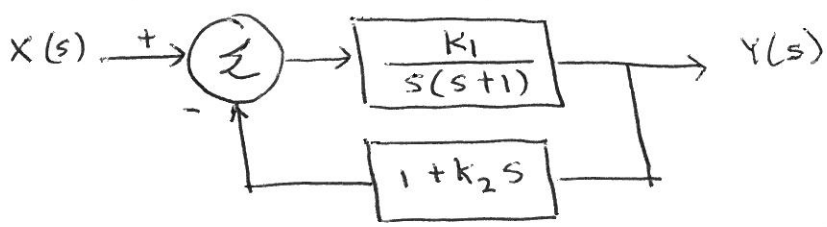
$$\tan(\omega_d t_r) = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$

So, $t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_d}{\sigma}\right) = \frac{\pi - \beta}{\omega_d}$ where β is defined

in the figure below:



Ex: For the system shown in the block diagram, determine the values of K_1 and K_2 so that the maximum overshoot is 40% and the peak time is 1.0 sec. Assume that the input is a unit-step function, $u(t)$.



Solution:

MAX. OVERSHOOT : 40%

$$\%M_P = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 \Rightarrow 0.4 = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\ln(0.4) = \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} \Rightarrow \text{squaring both sizes} \quad \frac{\zeta^2\pi^2}{1-\zeta^2} = 0.8395$$

$\Rightarrow$ solving for ζ $\Rightarrow$ $\zeta = 0.28$ //

PEAK TIME, t_p : 1 sec.

$$t_p = \frac{\pi}{\omega_d} = 1.0 \Rightarrow \omega_d = 3.1416 \text{ rad/s}$$

We know that $\omega_d = \omega_n\sqrt{1-\zeta^2} \Rightarrow \omega_n = \omega_d / \sqrt{1-\zeta^2}$

Sub. $\omega_d = 3.1416$ and $\zeta = 0.28$ from the previous calculation, we get :

$$\omega_n = \frac{3.1416}{\sqrt{1-0.28^2}} = 3.27 \text{ rad/s} \quad *$$

For small damping ratios, it is expected that $\omega_n \sim \omega_d$.

>> In order to obtain K_1 and K_2 to satisfy the imposed constraints $\%M_P = 40\%$ and $t_p = 1.0$ sec, we need to obtain the closed-Loop transfer function of the system:

Since the system is a single-loop feedback, we can use the feedback formula:

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)H(s)} \quad \text{where} \quad G(s) = \frac{K_1}{s(s+1)} \ \& \ H(s) = 1 + K_2 s$$

as given in the block diagram.

$$\therefore \frac{Y(s)}{X(s)} = \frac{\frac{K_1}{s(s+1)}}{\frac{1}{1} + \frac{K_1(1+K_2 s)}{s(s+1)}} = \frac{K_1}{s^2 + (K_1K_2+1)s + K_1} \quad (i)$$

From (i), the characteristic equation of this control system is:

$$s^2 + (K_1K_2+1)s + K_1 = 0 \quad (ii)$$

Comparing Eq. (ii) to the canonical form: $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

we find:

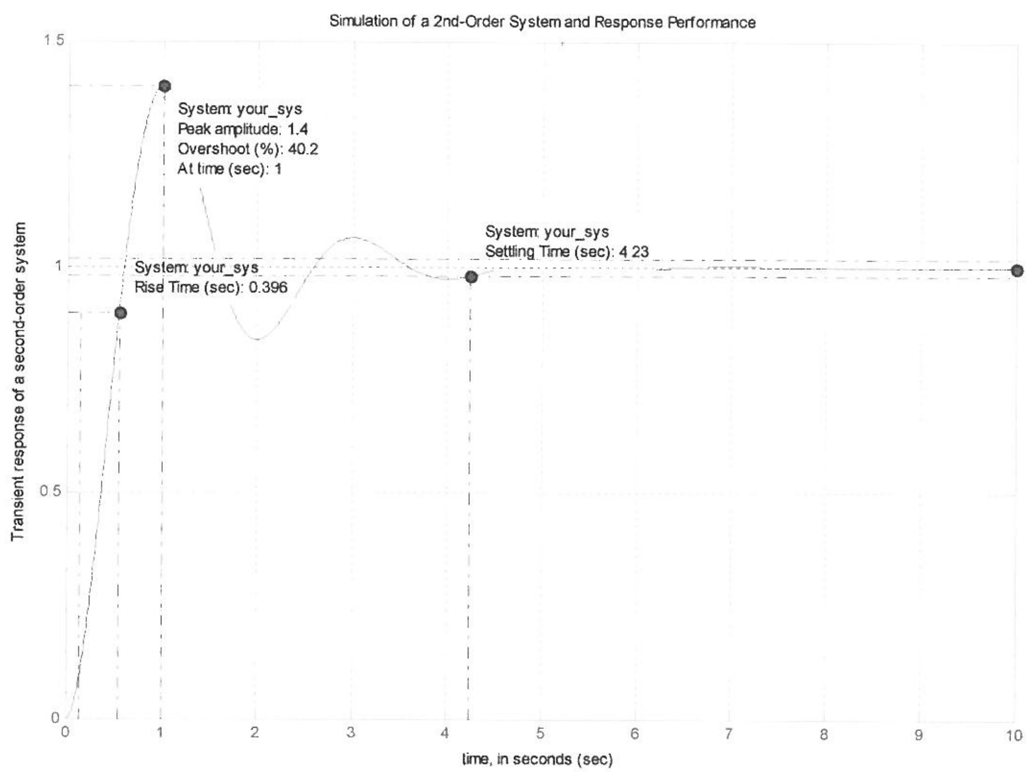
1. $\omega_n^2 = K_1 \Rightarrow K_1 = (3.27)^2 = 10.69$ //

2. $K_1 K_2 + 1 = 2\zeta\omega_n \Rightarrow K_2 = (2\zeta\omega_n - 1)/K_1$

$\therefore K_2 = 0.077$

The gains $K_1 = 10.69$ and $K_2 = 0.077$ will make the response to comply with the constraints: 40% max. overshoot and peak time, $t_p = 1.0$ sec. This performance can be observed if we plot the simulation results using Matlab on page #9:

% Simulation Script ⇒ Results
% See page #9 for implementation of the script in Matlab. next



```
function sec_order_sys_ex1()

%define the gains K1 and K2:
K1 = 10.69;
K2 = 0.077;

%define the numerator of the system's transfer function:
num = K1;
den = [1 (K1*K2+1) K1];

%use tf built-in function in Matlab to obtain the transfer function
%of the system using the numerator and denominator polynomials defined
%above:

your_sys = tf(num, den);

%define the simulation time as a list of time values:
tsim = 0:0.01:10; %the tsim variable is a list starting at zero, ending
                  %10 seconds later with incremental steps of 0.01↙
seconds

%obtain the response of the system to a unit-step function input using
%the "step" built-in Matlab function as shown below. The "step" output is↙
a
%plot of the response for the specified simulation time tsim above.

step(your_sys, tsim);
xlabel('time, in seconds'); ylabel('Transient response of a second-order↙
system');
title('Simulation of a 2nd-Order System and Response Performance');
grid on;
```