State Space to Transfer Function Examples

A. Introduction

For a linear, time-invariant, continuous-time system, the state and output equations are

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t) \tag{1}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^r$ is the input (control) vector, $y \in \Re^m$ is the output vector, and $\{A, B, C, D\}$ are matrices of appropriate dimensions.

To determine the expression for the transfer function or transfer matrix, the Laplace Transforms of the above equations are taken. If $r = m = 1$—the single-input, single-out case—the result of this operation is a single transfer function. If there are multiple inputs and/or multiple outputs, the result is an $m \times r$ matrix of transfer functions. Taking the Laplace Transform of the state equation yields

$$sX(s) - x(0) = AX(s) + BU(s) \tag{2}$$

where $x(0)$ is the initial value of the state vector in the time domain. Collecting terms in $X(s)$ and solving for $X(s)$ gives us the following:

$$\begin{aligned} (sI_n - A)\, X(s) &= x(0) + BU(s) && (3)\\ X(s) &= (sI_n - A)^{-1}\, x(0) + (sI_n - A)^{-1}\, BU(s) && (4) \end{aligned}$$

The expression in (4) implies that the solution to the state equation in the transform domain has a term related to the initial condition $x(0)$ and a term related to the control input $U(s)$. It will be seen later that the same holds true in the time domain.

The transfer function (or transfer matrix) is formed by assuming that the initial condition is $x(0) = 0$ and substituting the remaining expression for $X(s)$ into the output equation from (1). This yields

$$Y(s) = C\,(sI_n - A)^{-1}\, BU(s) + DU(s) = \left[C\,(sI_n - A)^{-1}\, B + D\right] U(s) \tag{5}$$

The transfer function is the relation between the transform of the input signal and the transform of the output signal. Therefore, the transfer function (or transfer matrix) is given by

$$H(s) = C\,(sI_n - A)^{-1}\, B + D = \frac{C Adj\,(sI_n - A)\, B}{|sI_n - A|} + D = \frac{C Adj\,(sI_n - A)\, B + |sI_n - A| \cdot D}{|sI_n - A|} \tag{6}$$

where $Adj\,(sI_n - A)$ is the adjoint matrix associated with $(sI_n - A)$. The term $|sI_n - A|$ is an n^{th} polynomial, the roots of which are the poles of the transfer function. If $D = 0$, the numerator of $H(s)$ in the single-input, single-output case is a polynomial of degree $n - 1$ or less. In this case the transfer function is strictly proper—more poles than zeros. If $D \neq 0$, the numerator and denominator polynomials of $H(s)$ are both of degree n, and the transfer function is biproper—same number of poles and zeros.

B. Example 1

The matrices of a third-order system are given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 15 & 5 & 0 \end{bmatrix}, \qquad D = 0 \tag{7}$$

Therefore, $n = 3$ and $r = m = 1$, so a single 3^{rd}-order transfer function describes the input-output relationship for this system in the transform domain. Going through the process step-by-step to derive the transfer function gives the following expressions.

$$sI_3 - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 8 & 14 & s+7 \end{bmatrix} \tag{8}$$

The matrix of minors and matrix of cofactors are

$$M = \begin{bmatrix} \left(s^2 + 7s + 14\right) & 8 & -8s \\ -\,(s+7) & s\,(s+7) & 14s + 8 \\ 1 & -s & s^2 \end{bmatrix}, \qquad C_{of} = \begin{bmatrix} \left(s^2 + 7s + 14\right) & -8 & -8s \\ (s+7) & s\,(s+7) & -\,(14s + 8) \\ 1 & s & s^2 \end{bmatrix} \tag{9}$$

and the adjoint matrix is

$$Adj\left(sI_n - A\right) = \begin{bmatrix} \left(s^2+7s+14\right) & (s+7) & 1 \\ -8 & s\,(s+7) & s \\ -8s & -\left(14s+8\right) & s^2 \end{bmatrix} \tag{10}$$

The determinant of $(sI_n - A)$ is

$$\left|sI_n - A\right| = s\left(s^2+7s+14\right) + 1\,(8) + 0\,(-8s) = s^3+7s^2+14s+8 \tag{11}$$

The steps in the calculation of the numerator are

$$Adj\left(sI_n - A\right)B = \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}, \qquad C Adj\left(sI_n - A\right)B = 15+5s \tag{12}$$

so the transfer function is

$$H\left(s\right) = \frac{5s+15}{s^3+7s^2+14s+8} = \frac{5\,(s+3)}{(s+1)\,(s+2)\,(s+4)} \tag{13}$$

Inspection of the state and output equations in (1) show that the state space system is in controllable canonical form, so the transfer function could have been written down directly from the entries in the state space matrices. This would not be the case if the state space matrices were not in a canonical form.

C. *Example 2*

The matrices of another third-order system are given by

$$A = \begin{bmatrix} -7 & 1 & 0 \\ -14 & 0 & 1 \\ -8 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 20 \\ 125 \\ 185 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad D = 5 \tag{14}$$

Again, $n = 3$ and $r = m = 1$, so a single 3^{rd}-order transfer function describes the input-output relationship for this system in the transform domain. The $\{A, B, C\}$ matrices are in observable canonical form in this example, and $D \neq 0$ so a biproper transfer function will be the result. Examination of the A matrices in (7) and (14) should indicate that the determinants $|sI_n - A|$ will be the same for each example. The steps in the process are as before.

$$sI_3 - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -7 & 1 & 0 \\ -14 & 0 & 1 \\ -8 & 0 & 0 \end{bmatrix} = \begin{bmatrix} s+7 & -1 & 0 \\ 14 & s & -1 \\ 8 & 0 & s \end{bmatrix} \tag{15}$$

The matrix of minors and matrix of cofactors are

$$M = \begin{bmatrix} s^2 & 14s+8 & -8s \\ -s & s\,(s+7) & 8 \\ 1 & -(s+7) & \left(s^2+7s+14\right) \end{bmatrix}, \qquad C_{of} = \begin{bmatrix} s^2 & -\left(14s+8\right) & -8s \\ s & s\,(s+7) & -8 \\ 1 & (s+7) & \left(s^2+7s+14\right) \end{bmatrix} \tag{16}$$

and the adjoint matrix is

$$Adj\left(sI_n - A\right) = \begin{bmatrix} s^2 & s & 1 \\ -\left(14s+8\right) & s\,(s+7) & (s+7) \\ -8s & -8 & \left(s^2+7s+14\right) \end{bmatrix} \tag{17}$$

The determinant of $(sI_n - A)$ is

$$\left|sI_n - A\right| = (s+7)\,s^2 + 1\,(14s+8) + 0\,(-8s) = s^3+7s^2+14s+8 \tag{18}$$

The steps in the calculation of the numerator are

$$C Adj\left(sI_n - A\right) = \begin{bmatrix} s^2 & s & 1 \end{bmatrix}, \qquad C Adj\left(sI_n - A\right)B = 20s^2+125s+185 \tag{19}$$

so the transfer function is

$$H\left(s\right) = \frac{20s^2+125s+185}{s^3+7s^2+14s+8} + 5 = \frac{5s^3+55s^2+195s+225}{s^3+7s^2+14s+8} = \frac{5\,(s+3)^2\,(s+5)}{(s+1)\,(s+2)\,(s+4)} \tag{20}$$