Cornerstones

Emmanuele DiBenedetto

Classical Mechanics

Theory and Mathematical Modeling

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Preface

Millennia of astronomical observations were fully understood only when the seminal ideas of Galileo, Kepler, and Newton were impacted by mathematics. The subsequent theoretical elaborations of the laws of motion by d'Alembert, Lagrange, Hamilton, Liouville, and others are at the basis of all mechanical devices that affect modern life in essentially all its practical aspects. Lagrange, in the preface to his *Traité de Mécanique Analytique* [101], sets his vision of mechanics as a branch of mathematics, building on physical principles.[1] Classical mechanics stands as perhaps the most successful example of what contemporary scientists call "interdisciplinary" research. The complexity of astronomy in Newton's day is countered today by the complexity of disciplines such as chemistry and biology. Classical mechanics is a chief example of the scientific method of organizing a "complex" collection of information into theoretically rigorous unifying principles. In this sense it represents one of the highest forms of modeling. The elegance and depth of the theoretical thinking coupled with its ubiquitous applications make it comparable, in applied sciences, to Euclid's geometry. It also has a foundational value comparable to calculus, both as a fundamental language of applied sciences and as a catalyst of new concepts and discoveries.

This book collects my lectures in rational mechanics delivered at the School of Engineering of the University of Rome, Tor Vergata, from 1986 to 1998. The main vision was mathematical modeling, and the layout is theoretical. The required background includes a working knowledge of linear algebra (vector calculus, algebra of matrices), multivariate calculus, basic theory of ordinary

[1] From the Preface of [101]: "*. . . On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expose ne demandent ni constructions, ni raisonnements géométriques ou mécaniques, mais seulement des opérations algébriques, assujetties á une marche régulière et uniforme. Ceux qui aiment l'Analyse, verront avec plaisir la Mécanique en devenir une nouvelle branche, . . .*

Tel est le plan que j'avais tâché de remplir . . .

On a conservé la notation ordinaire du Calcul différentiel, parce qu'elle répond au système des infiniment petits . . ."

differential equations, and elementary physics. While the Lebesgue integral is used, a working knowledge of it is not required. Its use is mainly intended as a unifying feature, bridging from discrete systems whose material properties are described by a series to weighted Dirac masses, to continuum systems. In practice, in problems and examples only the Riemann integral is used.

The geometry of rigid motions is presented along with some of its implications to mechanical devices and the theory of Poinsot precessions. The latter is remarkable, since it explains the phenomenon of equinoctial precessions only by the geometrical rolling of the Poinsot cones. While Lagrange's equations are often assumed as a principle, here they are derived and put on a mathematical footing. Conversely, the cardinal equations, which are often passed over in favor of the Lagrangian and Hamiltonian formalism, are shown to be the basis of such a formalism. Classical topics, such as gyroscopes, precessions, spinning tops, effects of rotation of Earth on gravitational motions, variational principles, n-body problem, and celestial mechanics, are revisited to underscore the role of mathematics, without which they can be only perceived but not fully explained. Attention is paid to the theory of small oscillations and Lyapunov's stability, including stability and instability of Poinsot precessions and celestial motions. The connection between mechanics and geometrical optics is traced to their common variational principles. The former leads to a maximum-rank Hamiltonian system, and the latter yields a degenerate Huygens canonical system. The degeneracy is overcome by Euler's theorem of homogeneous functions. The Hamilton variational formalism naturally leads to the symplectic formalism and canonical transformations. These identify, among the transformations that preserve the variational and canonical nature of the Hamilton equations, those that preserve the hidden geometry of the motion (Lie), such as Poisson and Lagrange brackets and volumes (Liouville's theorem). They also transform paths in phase space into "immobile points" in the transformed phase space, thereby providing an integration method of Hamiltonian systems (Jacobi). The basic techniques of integration of the Hamilton–Jacobi equations (complete integrals, envelopes, separability, etc), are presented in their natural symplectic formalism. Some classical facts have been given new proofs based on more modern mathematical language. These include the local minimality of the stationary points of the action, the notion of envelopes, and the Hamiltonian form of Huygens systems in geometrical optics. In the last chapter we give a very brief introduction to continuum and fluid mechanics, mainly to underscore the mathematical and modeling ideas needed in transitioning from finite degrees of freedom to nonrigid continuum systems.

Classical mechanics has evolved from these seminal principles into numerous fields, including statistical mechanics, relativity, ergodic theory, symplectic geometry, and continuum mechanics including elasticity and fluid dynamics. The methods of investigation are diverse, ranging from probability and measure theory and classical analysis to algebra, topology, and Riemannian geometry. We have refrained from going into any of them both because of the specialized and massive nature of each of them, and because of our goal of giving essentially a "calculus-type" introduction to the applied sciences. As such, all topics are mutually interlaced, each providing the background for the indicated several directions that mechanics takes on. However, some more mathematical parts might be omitted at a first reading. They include the parameteric equations of fixed and moving centrodes (§11c of Chapter 1); the theory of radial potential (§4.3c), which, while used for some atomic potentials, in this context has mostly a theoretical value; finding the principal axes of inertia of a planar rigid system (§5c of Chapter 4), and some mathematical remarks on the stationary points of a functional (§§1.1c–1.2c and §1.5c of Chapter 9).

I learned classical mechanics from Giorgio Sestini (1908–1991) at the University of Florence, Italy, back in the mid 1970s. Sestini's teachings impressed upon me the artful balance between physics and mathematics, neither being permitted to be self-absorbing. These notes reflect those lectures, including some exercises and problems taken from my old class notes.

I would like to extend special thanks and express my deep appreciation to my close collaborators Fabrizio Daví, Alessandro Tiero, Luciano Teresi, Luigi Chierchia and Vieri Mastropietro. These fine researchers helped me, over the years and at various stages, in teaching Classical Mechanics at the Univ. of Rome Tor Vergata. They conducted recitation sessions for the students and assisted them in practice and problem solving sessions. They suggested a number of ways of improving my draft notes into a usable tool and suggested topics and problems aimed at expanding the scope and clarity of the course. I am very much indebted to them.

I would like to thank the numerous students who, over the years, have pointed out misprints and imprecise statements, among them Giovanni Caruso, Roberto Cespa, Ernesto Mottola, Daniele Ludovisi, Giuseppe Vairo, Michael O'Leary, Luca Andreassi, Alessio Lupoi, Vito Introna, Luciano Cantone, Francesco Vivio, Sonia Marfia, and Riccardo Baudille. The help of Daniele Andreucci needs to be singled out. He read a large part of the manuscript, picking up imprecise statements and misprints and pointing out unclear passages; I am much indebted to Daniele.

I am also grateful to a number of people for having examined the manuscript at various stages and for providing suggestions and feedback on the material and its presentation, among them Bob Glassey, Steve Davis, Franco Maceri, Giovanni Galdi, Tinsley Oden, Morton Gurtin, David Kinderlehrer, Ralph Showalter, Avner Friedman, Don Saari, John Ockendon, David Hoff.

I would like to extend many thanks to Robert Sekerka for his critical reading of the chapter on canonical transformation and pointing unclear points. His remarks led to a rewriting of that chapter.

Special thanks go to James Serrin for having examined in detail the final version of the manuscript; he suggested changes in some terminology and presentation, made me aware of further references, pointed out a more general version of Stokes's theorem on the friction stress tensor in fluids [139], and made precise some thermodynamical facts in the chapter on fluid dynamics. I am very much grateful to James.

My special thanks go also to Franco Maceri, former dean of the School of Engineering of the University of Rome Tor Vergata, for his support and encouragement in writing these notes and for affording me the freedom of experimenting with less-traditional ways of teaching classical mechanics.

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Emmanuele DiBenedetto

1

GEOMETRY OF MOTION

1 Trajectories in $\mathbb{R}^3$ and Intrinsic Triads

A triple of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is positively oriented if $\mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_k$ for every cyclic permutation $\{i, j, k\}$ of the indices. A positive *triad* is a Cartesian reference system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in $\mathbb{R}^3$ with origin in O and positively oriented unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. A vector-valued function

$$(a, b) \ni t \to \mathbf{v}(t) = x_j(t)\mathbf{e}_j = \sum_{j=1}^{3} x_j(t)\mathbf{e}_j$$

from an interval $(a, b) \subset \mathbb{R}$ into $\mathbb{R}^3$ is continuous or differentiable at a point of (a, b) or in the whole (a, b) if so are the scalar functions $x_j(\cdot)$ for $j = 1, 2, 3$.

The summation notation will be used throughout, that is, a monomial expression with repeated indices is intended to be summed over those indices.

Calculus operations are effected in terms of components. If $\mathbf{v}$ and $\mathbf{w}$ are differentiable vector-valued functions, then

$$(\mathbf{v} \wedge \mathbf{w})' = \mathbf{v}' \wedge \mathbf{w} + \mathbf{v} \wedge \mathbf{w}', \qquad (\mathbf{v} \cdot \mathbf{w})' = \mathbf{v}' \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}'.$$

The latter equality implies that $(\|\mathbf{v}\|^2)' = 2\mathbf{v} \cdot \mathbf{v}'$. Therefore if $\mathbf{v}$ has constant length, then $\mathbf{v}$ and $\mathbf{v}'$ are orthogonal. Let $t \to P(t)$ be a continuously differentiable vector-valued function defined in (a, b) and such that

$$\|\dot{P}\| = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2} > 0 \quad \text{in } (a, b).$$

The set of points $\{P(t)\}_{t\in(a,b)}$ is a curve γ in $\mathbb{R}^3$ and the vector P' is tangent to γ in P. The intrinsic parameter of γ is its arc length

$$s = \int_a^t \|P'(\tau)\| d\tau, \qquad t \in (a, b).$$

With improper but suggestive symbolism, we denote by $s \to P(s)$ the function $s \to \widetilde{P}(s) = P(t(s))$, to stress that corresponding values of the parameters t

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and s identify the same geometric point P on γ. In terms of s the unit tangent $\mathbf{t}(s)$ and unit normal $\mathbf{n}(s)$ to γ in $P(s)$ are

$$\mathbf{t}(s) = \frac{d}{ds}P(s) = \frac{\dot{P}(t)}{\|\dot{P}(t)\|}, \qquad \mathbf{n}(s) = \frac{\mathbf{t}'(s)}{\|\mathbf{t}'(s)\|}, \quad \text{provided } \mathbf{t}'(s) \neq 0.$$

If $\mathbf{t}' = 0$ the normal $\mathbf{n}$ is not defined. Since $\mathbf{t}$ has constant length, $\mathbf{t} \cdot \mathbf{t}' = 0$. Therefore $\mathbf{n}$ is normal to $\mathbf{t}$ and in this sense is the unit normal to γ. Set

$$\mathbf{t}' = \kappa \mathbf{n}, \qquad \kappa = \|\mathbf{t}'\|, \qquad \mathbf{b} = \mathbf{t} \wedge \mathbf{n}.$$

The quantity $\kappa(s)$ is the curvature of γ at $P(s)$ and it measures by how much γ deviates from $\mathbf{t}(s)$ by an infinitesimal variation of s. This formula also determines the orientation of $\mathbf{n}$ and hence the orientation of $\mathbf{b}$. The unit vector $\mathbf{b}$ is the *binormal* to γ at P. The triple of unit vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is positive and the triad $\{P; \mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is called the *intrinsic triad* to γ at P. Taking derivatives of $\|\mathbf{b}\|^2 \equiv 1$ and $\mathbf{b} \cdot \mathbf{t} \equiv 0$ with respect to the parameter s gives

$$\mathbf{b} \cdot \mathbf{b}' = 0, \qquad \mathbf{t} \cdot \mathbf{b}' + \mathbf{b} \cdot \mathbf{t}' = 0.$$

These imply that $d\mathbf{b}/ds$ is orthogonal to both $\mathbf{b}$ and $\mathbf{t}$. Therefore $\mathbf{b}'$ is parallel to $\mathbf{n}$. Thus there exists a scalar function $s \to \lambda(s)$ such that

$$\mathbf{b}' = \lambda \mathbf{n}, \qquad |\lambda| = \|\mathbf{b}'\|.$$

The quantity $\lambda(s)$ is the *torsion* of γ at $P(s)$ and it measures by how much the curve γ deviates from the plane through $P(s)$ and normal $\mathbf{t}(s) \wedge \mathbf{n}(s)$ by an infinitesimal variation of the parameter s. These relations between tangent $\mathbf{t}$, normal $\mathbf{n}$, and binormal $\mathbf{b}$ of a curve γ are called Frenet formulas [58]. If the parameter t is time, the map $t \to P(t)$ is interpreted as a point moving along its trajectory γ. Its velocity $\mathbf{v} = \dot{P}$ and its acceleration $\mathbf{a} = \ddot{P}$ may be expressed in terms of a Cartesian triad $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ or in terms of its intrinsic triad as

$$\begin{aligned}
\dot{P} &= \mathbf{v} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)^t = \mathbf{t}\dot{s}, \\
\|\dot{P}\| &= \|\mathbf{v}\| = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2} = |\dot{s}|, \\
\ddot{P} &= \mathbf{a} = (\ddot{x}_1, \ddot{x}_2, \ddot{x}_3)^t = \ddot{s}\mathbf{t} + \kappa \dot{s}^2 \mathbf{n}, \\
\|\mathbf{a}\| &= \sqrt{\ddot{x}_1^2 + \ddot{x}_2^2 + \ddot{x}_3^2} = \sqrt{\ddot{s}^2 + \kappa^2 \dot{s}^4}.
\end{aligned}$$

Note that the acceleration $\mathbf{a}$ has zero component along the binormal $\mathbf{b}$. If $\ddot{s} \equiv 0$, the motion is *uniform*. If $\kappa \equiv 0$, it is a *straight-line* motion. The *uniform straight-line* motions are those for which $\kappa \equiv 0$ and $\ddot{s} \equiv 0$.

Throughout we will avoid specifying the range of variation of t, and *for all times* will mean *for all times within the range of variation of* t.

2 Areolar Velocity and Central Motions

A motion is planar if its trajectory lies in a plane π. Select a Cartesian system $\{O;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ with origin $O\in\pi$ and $\mathbf{e}_3\perp\pi$ and set

$$P-O=\rho\mathbf{u},\quad \rho=\|P-O\|,\quad \mathbf{u}=\cos\varphi\mathbf{e}_1+\sin\varphi\mathbf{e}_2,\quad \varphi=\widehat{\mathbf{e}_1\,\mathbf{u}},$$

where φ from $\mathbf{e}_1$ to $P-O$ is spanned counterclockwise. By time differentiation,

$$\dot{\mathbf{u}}=\dot\varphi\mathbf{u}^\perp,\qquad \ddot{\mathbf{u}}=-\dot\varphi^2\mathbf{u}+\ddot\varphi\mathbf{u}^\perp.$$

One verifies that $\mathbf{u}^\perp=\mathbf{e}_3\wedge\mathbf{u}$ and therefore the triple $\{\mathbf{u},\mathbf{u}^\perp,\mathbf{e}_3\}$ is positive. From this one computes the expression of velocity and acceleration in terms of its radial (i.e., along $\mathbf{u}$) and transversal (i.e., along $\mathbf{u}^\perp$) components as

$$\dot P=\dot\rho\mathbf{u}+\rho\dot\varphi\mathbf{u}^\perp,\qquad \ddot P=\left(\ddot\rho-\rho\dot\varphi^2\right)\mathbf{u}+(2\dot\rho\dot\varphi+\rho\ddot\varphi)\,\mathbf{u}^\perp. \tag{2.1}$$

For a planar motion $t\to P(t)$ and a fixed point O, denote by $2\mathbf{A}$ the moment of the velocity $\dot P$ with respect to the pole O, i.e.,

$$2\mathbf{A}=(P-O)\wedge\dot P=\rho^2\dot\varphi\mathbf{u}\wedge(\mathbf{e}_3\wedge\mathbf{u})=\rho^2\dot\varphi\mathbf{e}_3. \tag{2.2}$$

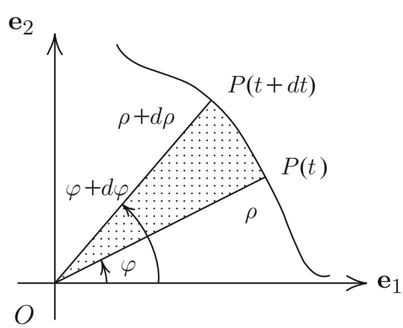

Fig. 2.1.

The vector $\mathbf{A}$ is normal to the plane of the motion, and it is called *areolar velocity*. Consider the area swept out by the vector radius $P-O$ in an elemental time dt during which the angle φ undergoes a variation $d\varphi$. Such an area is given by $\frac12\rho^2d\varphi$ up to terms of higher order in dt. Formally dividing by dt gives $\|\mathbf{A}\|=\frac12\rho^2|\dot\varphi|$. This justifies the name areolar velocity given to $\mathbf{A}$. A motion $t\to P(t)$ is *central* if there exists a point O, called *center of motion*, such that

$$(P-O)\wedge\ddot P=0\quad\text{ for all times.}$$

Equivalently, the motion is central if the vector radius $P - O$ and the acceleration $\ddot{P}$ are parallel at all times. If $t \to P(t)$ is central, then

$$2\dot{\mathbf{A}} = \frac{d}{dt}\big[(P-O)\wedge \dot{P}\big] = 0, \qquad \text{which implies} \qquad \mathbf{A} = \mathbf{const}.$$

Therefore, apart from the case $\mathbf{A} = 0$, the motion takes place in the plane π through O and normal to $\mathbf{A}$. This implies that central motions are planar. The vector radius $P-O$ sweeps out equal areas in equal times and it keeps its rotation orientation at all times. Also, there exists a constant $a_o > 0$, called the *area constant*, such that

$$\|\mathbf{A}\| = \tfrac{1}{2}a_o, \qquad \dot{\varphi} = \frac{a_o}{\rho^2}. \tag{2.3}$$

The second equality implies the formal operation

$$\frac{d}{dt} = \dot{\varphi}\frac{d}{d\varphi} = \frac{a_o}{\rho^2}\frac{d}{d\varphi}. \tag{2.4}$$

In particular, the radial component of the velocity is

$$\dot{\rho} = \frac{a_o}{\rho^2}\frac{d}{d\varphi}\rho = -a_o\frac{d}{d\varphi}\frac{1}{\rho}.$$

From the second of (2.3), by taking the time derivatives we obtain $\rho\,(2\dot{\rho}\dot{\varphi}+\rho\ddot{\varphi}) = 0$. Therefore in view of (2.1), the transversal component of the acceleration is zero. To compute the radial component, observe that

$$\ddot{\rho} = -a_o\frac{d}{dt}\frac{d}{d\varphi}\frac{1}{\rho} = -\frac{a_o^2}{\rho^2}\frac{d^2}{d\varphi^2}\frac{1}{\rho}, \qquad \rho\dot{\varphi}^2 = \frac{a_o^2}{\rho^3}.$$

Combining these remarks with (2.1) yields the expressions of velocity and acceleration of a central motion in the form

$$\mathbf{v} = -a_o\frac{d}{d\varphi}\frac{1}{\rho}\,\mathbf{u} + \frac{a_o}{\rho}\mathbf{u}^\perp, \qquad \mathbf{a} = -\frac{a_o^2}{\rho^2}\left(\frac{1}{\rho} + \frac{d^2}{d\varphi^2}\frac{1}{\rho}\right)\mathbf{u}. \tag{2.5}$$

These are known as the Binet formulas. Their interest is in that the geometric trajectory alone determines velocity and acceleration. A motion is *circular* if its trajectory is a circle. In such a case the Binet formulas take the form

$$\mathbf{v} = \frac{a_o}{\rho}\mathbf{u}^\perp, \qquad \mathbf{a} = -\frac{a_o^2}{\rho^2}\frac{1}{\rho}\mathbf{u}, \tag{2.6}$$

where ρ is the radius of the trajectory.

3 Geometry of Rigid Motions

A set $E \subset \mathbb{R}^3$ is in *rigid motion* if the mutual distance of any two of its points is constant in time, that is, if

$$\text{for all } P, Q \in E \quad t \to \|P(t) - O(t)\| = (\text{const})_{O,P} \quad \text{for all } t.$$

The rigid motion of E is determined by the motion of any triple $\{P_1, P_2, P_3\}$ of noncollinear points of E. Any other point $Q \in E$ is uniquely determined, along its motion, from the three independent relations $\|Q - P_i\| = (\text{const})_i$, $i = 1, 2, 3$. Having fixed a triple of noncollinear points $\{P_1, P_2, P_3\}$ in E, the three equations

$$\|P_i - P_j\| = (\text{const})_{ij}, \quad i, j = 1, 2, 3, \quad i \neq j$$

are linearly independent. Therefore of the nine coordinates

$$\big(x_1(P_i), x_2(P_i), x_3(P_i)\big), \quad i = 1, 2, 3,$$

only six are linearly independent. The rigid motion of E is uniquely determined by any six, linearly independent, of these functions of t, and therefore a rigid motion has at most *six degrees of freedom*.

The rigid motion of E is described with respect to a triad $\Sigma = \{\Omega; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with origin at Ω and positively oriented unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2\}$. Such a Cartesian reference system will be called *fixed*. Introduce now a triad $S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, clamped to E, with origin at O and positively oriented unit vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. These are uniquely determined by O and three points Q_j, $j = 1, 2, 3$, by the formula

$$\mathbf{u}_i = Q_i - O, \quad (Q_i - O) \cdot (Q_j - O) = \delta_{ij}, \qquad i, j = 1, 2, 3,$$

where δ_{ij} is the Kronecker delta. The $\mathbf{u}_j$ are equivalence classes, and this formula identifies them through representatives. The points O and Q_i might or not belong to E. The triad $\{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is clamped to E, in the sense that every $P \in E$ is required to satisfy

$$\|P - O\| = (\text{const})_o \quad \|P - Q_i\| = (\text{const})_i, \quad i = 1, 2, 3.$$

Therefore the rigid motion of the system E coincides with the rigid motion of the system $E' = E \cup \{O; Q_1, Q_2, Q_3\}$. Thus the rigid motion of E may be regarded as the motion of the triad S with respect to the fixed triad Σ. In this sense, the notion of rigid motion is independent of the presence of a possible material rigid body E and is identified with the geometric positions of the triad S with respect to the fixed triad Σ, following a parameter t. For this reason S is called the *moving triad*. For a point P denote by

$$P = P - O = (x_1, x_2, x_3) = x_i \mathbf{u}_i,$$
$$P = P - \Omega = (y_1, y_2, y_3) = y_i \mathbf{e}_i,$$

its representations with respect to the moving triad S and the fixed triad Σ.

The first three linearly independent functions, determining the rigid motion of S, will be chosen as the functions $t \to y_{o,i}(t)$, $i = 1, 2, 3$, coordinates of $O - \Omega$ with respect to Σ. The remaining three could be, for example, the Euler angles of S with respect to Σ of the three components of the instantaneous angular velocity vector $\boldsymbol{\omega}$. In the next sections we introduce these choices and trace their connection.

4 The Euler Angles

Modulo a translation, assume that $\Omega = O$. If $\mathbf{e}_3$ and $\mathbf{u}_3$ are not parallel, the fixed plane $\{\mathbf{e}_1; \mathbf{e}_2\}$ and the moving plane $\{\mathbf{u}_1; \mathbf{u}_2\}$ intersect along a line called *line of the nodes*. Denote by $\mathbf{n}$ the unit vector along the line of the nodes oriented so that the triple $\{\mathbf{n}, \mathbf{e}_3, \mathbf{u}_3\}$ is positively oriented. The Euler angles are defined as follows [54], [46, pp. 99–125].

Angle φ (of Precession): The angle $\varphi \in [0, 2\pi)$ between $\mathbf{e}_1$ and $\mathbf{n}$ counted from $\mathbf{e}_1$, counterclockwise with respect to $\mathbf{e}_3$.

Angle ψ (of Proper Rotation): The angle $\psi \in [0, 2\pi)$ between $\mathbf{n}$ and $\mathbf{u}_1$, counted from $\mathbf{n}$ counterclockwise with respect to $\mathbf{u}_3$.

Angle θ (of Nutation): The angle $\theta \in [0, \pi]$ between $\mathbf{e}_3$ and $\mathbf{u}_3$ counted from $\mathbf{e}_3$ counterclockwise with respect to $\mathbf{n}$.

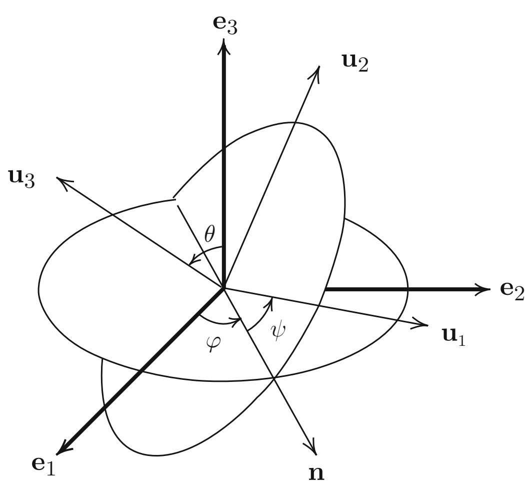

Fig. 4.1.

By keeping $\Omega = O$ fixed, the triad Σ can be moved to coincide with S by effecting sequentially the following three rotations:

1. Rotation of Σ about $\mathbf{e}_3$ of an angle φ to bring $\mathbf{e}_1$ to coincide with $\mathbf{n}$. The new position of Σ is the positive triad $\Sigma_1 = \{O; \mathbf{n}, \mathbf{e}_2', \mathbf{e}_3\}$, and it is realized by the rotation matrix

$$A_1 = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Rotation of Σ_1 about $\mathbf{n}$ of an angle θ to bring $\mathbf{e}_3$ to coincide with $\mathbf{u}_3$. The new position of Σ_1 is the positive triad $\Sigma_2 = \{O; \mathbf{n}, \mathbf{e}_2'', \mathbf{u}_3\}$, and it is realized by the rotation matrix

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

3. Rotation of Σ_2 about $\mathbf{u}_3$ of an angle ψ to bring $\mathbf{n}$ to coincide with $\mathbf{u}_1$. This last rotation takes Σ_2 into S, and it is realized by the rotation matrix

$$A_3 = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The composition of these matrices, in the indicated order, is $A = A_3 A_2 A_1$,

$$A = \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix}. \tag{4.1}$$

The matrix A carries points $y = (y_1, y_2, y_3)$ in Σ into points $x = (x_1, x_2, x_3)$ in S by the formula $x = Ay$. One verifies that A is unitary and that $A^{-1} = A^t$. The composition $A_3 A_2 A_1$ is not commutative, since interchanging the order of any two of these matrices does not carry Σ into S.

5 Rotation Matrices and Angular Velocity

For a vector $\mathbf{w}$ denote by $\mathbf{w}_S$ and $\mathbf{w}_\Sigma$ its representations in $S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\Sigma = \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, i.e.,

$$\begin{aligned} \mathbf{w}_S &= x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3, &\quad \text{where} \quad x_i &= \mathbf{w}\cdot\mathbf{u}_i, \\ \mathbf{w}_\Sigma &= y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + y_3\mathbf{e}_3, &\quad \text{where} \quad y_j &= \mathbf{w}\cdot\mathbf{e}_j. \end{aligned}$$

The transformation from $\mathbf{w}_S$ to $\mathbf{w}_\Sigma$ is realized by the matrix $(\alpha_{ij}) = (\mathbf{e}_i \cdot \mathbf{u}_j)$. This is unitary and $(\alpha_{ij})^{-1} = (\alpha_{ji})$. Therefore

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = (\alpha_{ij}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (\alpha_{ji}) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{5.1}$$

If $\mathbf{w}$ is fixed in S and moves with it, the components x_j of $\mathbf{w}_S$ are constant in time, whereas the components y_i of $\mathbf{w}_\Sigma$ are, in general, nonconstant functions of the parameter t. The velocity $\dot{\mathbf{u}}_i$ of the unit vector $\mathbf{u}_i$ is a free vector in space, representable in S and Σ. Representing it in S,

$$\dot{\mathbf{u}}_{i,S} = (\dot{\mathbf{u}}_i \cdot \mathbf{u}_\ell)\,\mathbf{u}_\ell.$$

The coefficient $(\dot{\mathbf{u}}_i \cdot \mathbf{u}_\ell)$ is zero for $\ell = i$. Let j, k be the values of the index ℓ for which $\ell \neq i$. If the permutation $\{i, j, k\}$ is even, then

$$\begin{aligned}\dot{\mathbf{u}}_{i,S} &= (\dot{\mathbf{u}}_i \cdot \mathbf{u}_j)\mathbf{u}_j + (\dot{\mathbf{u}}_i \cdot \mathbf{u}_k)\mathbf{u}_k \\ &= (\dot{\mathbf{u}}_i \cdot \mathbf{u}_j)\mathbf{u}_k \wedge \mathbf{u}_i - (\dot{\mathbf{u}}_i \cdot \mathbf{u}_k)\mathbf{u}_j \wedge \mathbf{u}_i \\ &= [-(\dot{\mathbf{u}}_i \cdot \mathbf{u}_k)\mathbf{u}_j + (\dot{\mathbf{u}}_i \cdot \mathbf{u}_j)\mathbf{u}_k] \wedge \mathbf{u}_i.\end{aligned}$$

If the permutation $\{i, j, k\}$ is odd, the same conclusion holds up to a sign change on the right-hand side. Let $\boldsymbol{\omega}$ be the free vector whose components in S are given by

$$\omega_{k,S} = (\dot{\mathbf{u}}_i \cdot \mathbf{u}_j)\epsilon_{ijk}, \qquad \text{where} \qquad \epsilon_{ijk} = (\mathbf{u}_i \wedge \mathbf{u}_j) \cdot \mathbf{u}_k. \tag{5.2}$$

The symbol ϵ_{ijk}, called *Ricci alternator*. It is zero if any two of the indices i, j, k are equal and equals ± 1 according to the parity of the permutation $\{i, j, k\}$ [134, 135]. With this notation,

$$\dot{\mathbf{u}}_{i,S} = \boldsymbol{\omega} \wedge \mathbf{u}_i \qquad \text{for} \qquad i = 1, 2, 3.$$

If $\mathbf{w} = x_i \mathbf{u}_i$ is fixed with S, then

$$\dot{\mathbf{w}}_S = x_i \dot{\mathbf{u}}_{i,S} = \boldsymbol{\omega} \wedge x_i \mathbf{u}_i = \boldsymbol{\omega} \wedge \mathbf{w}_S.$$

Since the exterior and scalar product of two vectors are independent of their representation,

$$\dot{\mathbf{w}} = \boldsymbol{\omega} \wedge \mathbf{w} \qquad \text{for every vector } \mathbf{w} \text{ fixed with } S. \tag{5.3}$$

Although defined in terms of its components in S, the vector $\boldsymbol{\omega}$ is intrinsic and it can be equivalently represented in the coordinates of Σ. The vector $\boldsymbol{\omega}$ is an intrinsic characteristic of the rigid motion of S with respect to Σ and, as such, is independent of the representation of S. Indeed, let $S' = \{O'; \mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\}$ be a new triad, clamped to S and following the same rigid motion of S. Denoting by $\boldsymbol{\omega}'$ the vector defined as in (5.2) with $\mathbf{u}_i$ replaced with $\mathbf{u}'_i$,

$$\boldsymbol{\omega}' \wedge \mathbf{u}'_i = \dot{\mathbf{u}}'_i = (\mathbf{u}'_i \cdot \mathbf{u}_j)\dot{\mathbf{u}}_j = (\mathbf{u}'_i \cdot \mathbf{u}_j)\boldsymbol{\omega} \wedge \mathbf{u}_j = \boldsymbol{\omega} \wedge (\mathbf{u}'_i \cdot \mathbf{u}_j)\mathbf{u}_j = \boldsymbol{\omega} \wedge \mathbf{u}'_i.$$

From this, $(\boldsymbol{\omega} - \boldsymbol{\omega}') \wedge \mathbf{u}'_i = 0$ for $i = 1, 2, 3$. Therefore $\boldsymbol{\omega} - \boldsymbol{\omega}'$ is parallel to three nondegenerate mutually orthonormal vectors, and hence it must be zero. The vector $\boldsymbol{\omega}$ is the *angular velocity* of the rigid motion of S, or the vector of *instantaneous rotation*.

The same vector calculus can be effected starting from the transformation formulas (5.1). Given a vector $\mathbf{w}$, identify $\mathbf{w}_S$ and respectively $\mathbf{w}_\Sigma$ with the column vector of its components in S and Σ. Let $\mathbf{w}$ be fixed with S. Then by taking the time derivative of the second equality of (5.1),

$$(\alpha_{ji})\dot{\mathbf{w}}_\Sigma + (\dot{\alpha}_{ji})\,\mathbf{w}_\Sigma = 0. \tag{5.4}$$

Using the first equality of (5.1),

$$\dot{\mathbf{w}}_S = -\left[(\dot{\alpha}_{ji})(\alpha_{ij})\right](\alpha_{ji})\mathbf{w}_\Sigma = -\left[(\dot{\alpha}_{ji})(\alpha_{ij})\right]\mathbf{w}_S.$$

The element of position ij of the product matrix $(\dot{\alpha}_{ji})(\alpha_{ij})$ is computed from

$$\left[(\dot{\alpha}_{ji})(\alpha_{ij})\right]_{ij} = \dot{\alpha}_{\ell i}\alpha_{\ell j} = \dot{\mathbf{u}}_i \cdot \mathbf{u}_j, \qquad \mathbf{u}_i = (\alpha_{1,i}, \alpha_{2,i}, \alpha_{3,i})^t.$$

From this,

$$\begin{aligned}\dot{\mathbf{w}}_S &= -\begin{pmatrix} 0 & \dot{\mathbf{u}}_1 \cdot \mathbf{u}_2 & \dot{\mathbf{u}}_1 \cdot \mathbf{u}_3 \\ \dot{\mathbf{u}}_2 \cdot \mathbf{u}_1 & 0 & \dot{\mathbf{u}}_2 \cdot \mathbf{u}_3 \\ \dot{\mathbf{u}}_3 \cdot \mathbf{u}_1 & \dot{\mathbf{u}}_3 \cdot \mathbf{u}_2 & 0 \end{pmatrix} \mathbf{w}_S \\ &= -\begin{pmatrix} 0 & \omega_{3,S} & -\omega_{2,S} \\ -\omega_{3,S} & 0 & \omega_{1,S} \\ \omega_{2,S} & -\omega_{1,S} & 0 \end{pmatrix} \mathbf{w}_S = (\boldsymbol{\omega} \wedge \mathbf{w})_S.\end{aligned}$$

One arrives at the same formula in terms of the components in Σ. From (5.4),

$$\dot{\mathbf{w}}_\Sigma = -(\alpha_{ij})(\dot{\alpha}_{ji})\mathbf{w}_\Sigma.$$

The element of position ij in the product matrix is

$$\begin{aligned}\left[(\alpha_{ij})(\dot{\alpha}_{ji})\right]_{ij} &= \alpha_{i\ell}\dot{\alpha}_{j\ell} = (\mathbf{e}_i \cdot \mathbf{u}_\ell)(\mathbf{e}_j \cdot \dot{\mathbf{u}}_\ell) \\ &= (\mathbf{e}_i \cdot \mathbf{u}_\ell)[\mathbf{e}_j \cdot (\dot{\mathbf{u}}_\ell \cdot \mathbf{u}_h)\mathbf{u}_h] = (\dot{\mathbf{u}}_\ell \cdot \mathbf{u}_h)(\mathbf{e}_i \cdot \mathbf{u}_\ell)(\mathbf{e}_j \cdot \mathbf{u}_h) \\ &= \sum_{\ell<h} (\dot{\mathbf{u}}_\ell \cdot \mathbf{u}_h)\left[(\mathbf{e}_i \cdot \mathbf{u}_\ell)(\mathbf{e}_j \cdot \mathbf{u}_h) - (\mathbf{e}_i \cdot \mathbf{u}_h)(\mathbf{e}_j \cdot \mathbf{u}_\ell)\right] \\ &= \omega_{n,S}(\mathbf{e}_i \wedge \mathbf{e}_j) \cdot \mathbf{u}_n = \alpha_{kn}\omega_{n,S}\epsilon_{ijk} = \omega_{k,\Sigma}\epsilon_{ijk}.\end{aligned}$$

Therefore

$$\dot{\mathbf{w}}_\Sigma = -\begin{pmatrix} 0 & \omega_{3,\Sigma} & -\omega_{2,\Sigma} \\ -\omega_{3,\Sigma} & 0 & \omega_{1,\Sigma} \\ \omega_{2,\Sigma} & -\omega_{1,\Sigma} & 0 \end{pmatrix} \mathbf{w}_\Sigma = (\boldsymbol{\omega} \wedge \mathbf{w})_\Sigma.$$

While written in terms of the coordinates of Σ or S, these formulas are intrinsic and are equivalent to (5.3).

6 Velocity and Acceleration

For a moving point P denote by $\mathbf{v}(P) = \dot{P}$ its velocity and by $\mathbf{a}(P) = \ddot{P}$ its acceleration. If $P - O$ is fixed to S, its derivatives are computed from (5.3) as

$$\mathbf{v}(P) = \mathbf{v}(O) + \boldsymbol{\omega} \wedge (P - O), \tag{6.1}$$

$$\begin{aligned} \mathbf{a}(P) &= \mathbf{a}(O) + \dot{\boldsymbol{\omega}} \wedge (P - O) + \boldsymbol{\omega} \wedge (P - O)' \\ &= \mathbf{a}(O) + \dot{\boldsymbol{\omega}} \wedge (P - O) + \boldsymbol{\omega} \wedge [\boldsymbol{\omega} \wedge (P - O)]. \end{aligned} \tag{6.2}$$

These are the Poisson formulas of the rigid motion of S [132]. Velocities and accelerations are those recorded by an observer on the fixed triad Σ. However, these vectors are independent of reference frames and as such, could be represented equivalently in S or Σ. All kinematic information on the motion of any point P fixed with S are included in the two vector-valued functions of time $\mathbf{v}(O)$ and $\boldsymbol{\omega}$, which are called the *characteristics* of the rigid motion. Examples of rigid motions are obtained by specifying the form of these two functions.

6.1 Translations

A rigid motion is a translation if $\boldsymbol{\omega} \equiv 0$. The degrees of freedom reduce to three. From (6.1)–(6.2), it follows that all points in S have the same velocity and acceleration of O, and the system moves with the axes of the triad $\{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ remaining parallel. If $\mathbf{v}(O) = \text{const}$, the translation occurs along a straight line.

6.2 Precessions

A rigid motion is a *precession* if S has a fixed point O, called a *pole* of the precession. The degrees of freedom are three, and (6.1) takes the form

$$\mathbf{v}(P) = \boldsymbol{\omega} \wedge (P - O) \qquad \text{for all } \; P \in S.$$

Therefore at every instant t, all the points of the straight line

$$\ell(t) = \{P - O = \lambda \boldsymbol{\omega}(t), \quad \lambda \; \text{real parameter}\}$$

have zero velocity. Such a straight line is called the *axis of instantaneous rotation*, and it varies with t. If the direction of $\boldsymbol{\omega}$ is constant, the line ℓ is constant in time, and all points of S rotate about ℓ with the same angular velocity. If the direction of $\boldsymbol{\omega}$ is constant, the precession is a *rotation*. If $\boldsymbol{\omega}$ is constant, the precession is *uniform*. The name *precession* given to these rigid motions will be justified in §1.5c of Chapter 7.

6.3 Rototranslations

A rigid motion is a rototranslation if the directions of $\mathbf{v}(O)$ and $\boldsymbol{\omega}$ are constant in time. The degrees of freedom are two. If $\mathbf{v}(O) \wedge \boldsymbol{\omega} \equiv 0$, the system translates along the axis through O and parallel to $\boldsymbol{\omega}$ with velocity $\mathbf{v}(O)$ and rotates about the same axis with angular velocity $\boldsymbol{\omega}$. If $\mathbf{v}(O) \cdot \boldsymbol{\omega} \equiv 0$, the system translates along the constant direction of $\mathbf{v}(O)$ and rotates about the variable axis through O and directed as $\boldsymbol{\omega}$. Instantaneously, i.e., during an infinitesimal variation dt of the parameter t, any rigid motion can be regarded as an *instantaneous translation* followed by an *instantaneous rotation*.

7 The Axis of Motion

For a point P fixed with S, we wish to compute the component of its velocity $\mathbf{v}(P)$ along $\boldsymbol{\omega}$. Assuming $\boldsymbol{\omega} \neq 0$ and using the Poisson formula (6.1),

$$\mathbf{v}(P)^{\|} = (\mathbf{v}(P) \cdot \boldsymbol{\omega}) \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|^2} = (\mathbf{v}(O) \cdot \boldsymbol{\omega}) \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|^2} \stackrel{\text{def}}{=} I(\boldsymbol{\omega}), \tag{7.1}$$

where O is any point fixed in S. Therefore $\mathbf{v}(P)^{\|} = I(\boldsymbol{\omega})$ for all $P \in S$. Equivalently, the velocity of every point $P \in S$ has the same component along $\boldsymbol{\omega}$. Since $I(\boldsymbol{\omega})$ is independent of $P \in S$, it is called the *vectorial invariant* of the rigid motion. We decompose the velocity of a point $P \in S$ as

$$\begin{aligned} \mathbf{v}(P) &= \mathbf{v}(P)^{\|} + \mathbf{v}(P)^{\perp} = I(\boldsymbol{\omega}) + [\mathbf{v}(P) - I(\boldsymbol{\omega})] \\ &= I(\boldsymbol{\omega}) + \left[\mathbf{v}(O)^{\perp} - (P - O) \wedge \boldsymbol{\omega}\right], \end{aligned}$$

where we have used the Poisson formula (6.1). The first vector is parallel to $\boldsymbol{\omega}$ and is independent of P. The vector in brackets is perpendicular to $\boldsymbol{\omega}$ and depends on P. Since these two vectors are orthogonal,

$$\|\mathbf{v}(P)\|^2 = \|I(\boldsymbol{\omega})\|^2 + \|\mathbf{v}(O)^{\perp} - (P - O) \wedge \boldsymbol{\omega}\|^2.$$

For a fixed t, we seek the set of all points $P \in S$ for which

$$\mathbf{v}(O)^{\perp} - (P - O) \wedge \boldsymbol{\omega} = 0. \tag{7.2}$$

Equivalently, we look for the geometric locus $\ell(t)$ of those points $P \in S$ whose velocity, at time t, is parallel to $\boldsymbol{\omega}(t)$ and has least modulus. Indeed, a point P fixed with S satisfies (7.2) if and only if

$$\mathbf{v}(P) \wedge \boldsymbol{\omega} = 0 \quad \text{and} \quad \|\mathbf{v}(P)\| = \inf_{Q \in S} \|\mathbf{v}(Q)\|. \tag{7.3}$$

Let P and P' be any two distinct points of $\ell(t)$. Writing (7.2) for each of them and subtracting gives $(P - P') \wedge \boldsymbol{\omega} = 0$. Therefore the locus $\ell(t)$ is a line parallel to $\boldsymbol{\omega}(t)$. Such a line is denoted by $\mu(\boldsymbol{\omega})$ and is called *instantaneous* *axis of motion*, or *axis of Mozzi* [122]. If P_o is a point in $\ell(t)$, the parametric equation of $\mu(\boldsymbol{\omega})$ is $P = P_o + \lambda\boldsymbol{\omega}$ for $\lambda \in \mathbb{R}$. Therefore determining the equation $\mu(\boldsymbol{\omega})$ reduces to finding one of its points P_o. For example, P_o might be the projection on $\mu(\boldsymbol{\omega})$ of a point $O \in S$. To identify such a projection, write (7.2) for such a P_o and form the exterior product by $\boldsymbol{\omega}$ to obtain

$$(P_o - O)\|\boldsymbol{\omega}\|^2 = \boldsymbol{\omega} \wedge \mathbf{v}(O) \implies P_o = O + \frac{\boldsymbol{\omega} \wedge \mathbf{v}(O)}{\|\boldsymbol{\omega}\|^2}, \qquad \boldsymbol{\omega} \neq 0.$$

Therefore the parametric equation of $\mu(\boldsymbol{\omega})$ is [22, pages 92–120],

$$P - O = \frac{\boldsymbol{\omega} \wedge \mathbf{v}(O)}{\|\boldsymbol{\omega}\|^2} + \lambda\boldsymbol{\omega}, \qquad \boldsymbol{\omega} \neq 0, \quad \lambda \in \mathbb{R}. \tag{7.4}$$

The axis of motion $\mu(\boldsymbol{\omega})$ is independent of the point O. That is, if (7.4) were written with O replaced by any other point $Q \in S$, it would describe the same set of points $\ell(t)$. Moreover, (7.4) is written in intrinsic vectorial form and as such is independent of any reference system. Having identified it, one could write it alternatively in the coordinates of Σ or S.

The velocity of every point $P \in \mu(\boldsymbol{\omega})$ is parallel to $\boldsymbol{\omega}$. If $P \in S$ has zero velocity, then $P \in \mu(\boldsymbol{\omega})$. The vectorial invariant $I(\boldsymbol{\omega})$ vanishes if and only if all points of the axis of motion $\mu(\boldsymbol{\omega})$ have instantaneous zero velocity. In a precession of pole O, the axis of motion is a line through O and direction $\boldsymbol{\omega}$; moreover, $I(\boldsymbol{\omega}) = 0$. In a translation, $\boldsymbol{\omega} \equiv 0$ and the axis of motion is not defined. More generally, $\mu(\boldsymbol{\omega})$ is not defined for those values of the parameter t for which $\boldsymbol{\omega}(t) = 0$.

8 Relative Rigid Motions and Coriolis's Theorem

Let S be in rigid motion with respect to Σ with characteristics $\mathbf{v}(O)$ and $\boldsymbol{\omega}$. If P is a point moving with respect to S,

$$P(t) - O = x_i(t)\mathbf{u}_i, \tag{8.1}$$

the velocity and acceleration of P with respect to S are $\mathbf{v}_S(P) = \dot{x}_i\mathbf{u}_i$ and $\mathbf{a}_S(P) = \ddot{x}_i\mathbf{u}_i$. Regard now P and O as a pair of points moving with respect to Σ. Taking the time derivative of (8.1) and using the differentiation formula (5.3) for vectors fixed with S gives

$$\dot{P} = \dot{O} + \dot{x}_i\mathbf{u}_i + x_i\dot{\mathbf{u}}_i = \mathbf{v}_S(P) + [\mathbf{v}(O) + \boldsymbol{\omega} \wedge (P - O)]. \tag{8.2}$$

By the Poisson formula (6.1), the vector in brackets is the velocity of P as if it were fixed with S and moving following the same rigid motion of S; it is called the *transport velocity* of P and is denoted by

$$\mathbf{v}_T(P) = \mathbf{v}(O) + \boldsymbol{\omega} \wedge (P - O). \tag{8.3}$$

Combining these remarks, one obtains the expression of the velocity $\mathbf{v}_\Sigma(P)$ with respect to Σ in the form

$$\mathbf{v}_\Sigma(P) = \mathbf{v}_S(P) + \mathbf{v}_T(P). \tag{8.4}$$

From (8.1) by double differentiation,

$$\ddot{P} - \ddot{O} = \ddot{x}_i \mathbf{u}_i + 2\dot{x}_i \dot{\mathbf{u}}_i + x_i \ddot{\mathbf{u}}_i.$$

By the differentiation formula (5.3),

$$\begin{aligned} x_i \ddot{\mathbf{u}}_i = x_i(\boldsymbol{\omega} \wedge \mathbf{u}_i)' &= x_i \dot{\boldsymbol{\omega}} \wedge \mathbf{u}_i + x_i \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{u}_i) \\ &= \dot{\boldsymbol{\omega}} \wedge (P - O) + \boldsymbol{\omega} \wedge [\boldsymbol{\omega} \wedge (P - O)]. \end{aligned}$$

Therefore the acceleration $\mathbf{a}_\Sigma(P)$ of P with respect to Σ is given by

$$\begin{aligned} \mathbf{a}_\Sigma(P) =& \mathbf{a}(O) + \dot{\boldsymbol{\omega}} \wedge (P - O) + \boldsymbol{\omega} \wedge [\boldsymbol{\omega} \wedge (P - O] \\ &+ \mathbf{a}_S(P) + 2\boldsymbol{\omega} \wedge \mathbf{v}_S(P). \end{aligned}$$

By the second Poisson formula (6.2) the sum of the first three terms on the right-hand side is the acceleration of P as if it were fixed with S and moving following the same rigid motion of S; it is called the *transport acceleration* of P, and it is denoted by

$$\mathbf{a}_T(P) = \mathbf{a}(O) + \dot{\boldsymbol{\omega}} \wedge (P - O) + \boldsymbol{\omega} \wedge [\boldsymbol{\omega} \wedge (P - O)]. \tag{8.5}$$

The last term is the *deflection*, or *Coriolis acceleration*, denoted by

$$\mathbf{a}_C(P) = 2\boldsymbol{\omega} \wedge \mathbf{v}_S(P). \tag{8.6}$$

Theorem 8.1 (Coriolis [34]). *The acceleration $\mathbf{a}_\Sigma(P)$ of P with respect to Σ is the sum of the relative acceleration $\mathbf{a}_S(P)$ of P with respect to S, the transport acceleration $\mathbf{a}_T(P)$ of P, regarded as instantaneously fixed with S, and the deflection acceleration $\mathbf{a}_C(P)$, i.e.,*

$$\mathbf{a}_\Sigma(P) = \mathbf{a}_S(P) + \mathbf{a}_T(P) + \mathbf{a}_C(P). \tag{8.7}$$

9 Composing Rigid Motions

Let Σ, S, and S' be triads in mutual rigid motion, i.e.,

$$\Sigma = \{\Omega; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}, \quad S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}, \quad S' = \{O'; \mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\},$$

where S is in rigid motion with respect to Σ with characteristics $\mathbf{v}_\Sigma(O)$ and $\boldsymbol{\omega}$, and S' is in rigid motion with respect to S with characteristics $\mathbf{v}_S(O')$

and $\boldsymbol{\omega}'$. By (8.4), the velocity $\mathbf{v}_\Sigma(P)$ with respect to Σ of a point $P \in S'$ is the sum of $\mathbf{v}_T(P)$ given by (8.3) and $\mathbf{v}_S(P)$ given by

$$\mathbf{v}_S(P) = \mathbf{v}_S(O') + \boldsymbol{\omega}' \wedge (P - O').$$

Therefore

$$\mathbf{v}_\Sigma(P) = \mathbf{v}_S(O') + \big[\mathbf{v}_\Sigma(O) + \boldsymbol{\omega} \wedge (O' - O)\big] + (\boldsymbol{\omega} + \boldsymbol{\omega}') \wedge (P - O').$$

The first term $\mathbf{v}_S(O')$ is the velocity of O' relative to S, whereas the vector in brackets is the transport velocity of O', regarded as instantaneously fixed with S. By (8.2) the sum of these first two vectors on the right-hand side is the velocity of O' with respect to Σ. Combining these remarks yields

$$\mathbf{v}_\Sigma(P) = \mathbf{v}_\Sigma(O') + (\boldsymbol{\omega} + \boldsymbol{\omega}') \wedge (P - O') \tag{9.1}$$

for all pairs of points $P, O' \in S'$. The motion of S' with respect to Σ results from the *composition* of the motion of S with respect to Σ and the motion of S' with respect to S.

Proposition 9.1 *The triad S' moves with rigid motion with respect to Σ with characteristics $\mathbf{v}_\Sigma(O')$ and $\boldsymbol{\omega} + \boldsymbol{\omega}'$.*

Proof. Write (9.1) in the form

$$\dot{P} - \dot{O}' = (\boldsymbol{\omega} + \boldsymbol{\omega}') \wedge (P - O').$$

Taking the scalar product of this by $P - O'$ gives

$$\|P - O'\| = (\text{const})_{(P,O')} \qquad \text{for every pair of points } \; P, O' \in S'.$$

Therefore the motion of S' with respect to Σ is rigid and has characteristics $\mathbf{v}_\Sigma(O')$ and $\boldsymbol{\omega}^*$ given by (5.2) with S replaced by S' and $\mathbf{u}_j$ replaced by $\mathbf{u}'_j$. Since $P - O'$ is fixed with S', by the differentiation formula (5.3),

$$[\boldsymbol{\omega}^* - (\boldsymbol{\omega} + \boldsymbol{\omega}')] \wedge (P - O') = 0$$

for all pairs of points $P, O' \in S'$. Thus $\boldsymbol{\omega}^* = \boldsymbol{\omega} + \boldsymbol{\omega}'$. ■

Proposition 9.2 *Let $S_o = \Sigma$ be a fixed triad and let S_i, $i = 1, \ldots, n$, be triads such that S_i is in rigid motion with respect to S_{i-1} with characteristics $\mathbf{v}_{S_{i-1}}(O_i)$ and $\boldsymbol{\omega}_i$. Then the motion of S_n with respect to Σ is rigid with characteristics $\mathbf{v}_\Sigma(O_n)$ and $\boldsymbol{\omega} = \sum_{i=1}^{n} \boldsymbol{\omega}_i$.*

Therefore the composition of finitely many rigid motions is a rigid motion. Conversely, any rigid motion can be decomposed into one or several rigid motions. The decomposition, however, in general is not unique.

The composition of n translations is a translation. The composition of n precessions with the same pole is a precession with the same pole. If the precessions are uniform, the composite precession is uniform. The composition of two precessions of poles O_1 and O_2 and angular characteristics $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ is a rigid motion of characteristics

$$\mathbf{v}_\Sigma(O_2) = \boldsymbol{\omega}_1 \wedge (O_2 - O_1), \qquad \boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2.$$

This is not a precession unless $\mathbf{v}_\Sigma(O_2) \equiv 0$.

As an example consider a horizontal platform rotating about a fixed axis with angular velocity $\boldsymbol{\omega} = \omega\mathbf{u}_3$ for a scalar ω. A cylinder is installed on the platform with axis directed as $\mathbf{u}_2$, and at a distance $d > 0$ from the axis of motion of the platform. The cylinder rotates about its axis with angular velocity $\boldsymbol{\omega}' = \omega'\mathbf{u}_2$. Determine the characteristics of the composite rigid motion, identifying the triads Σ, S, and S'. Write down the expression of the velocity of the generic point P of the cylinder with respect to the fixed triad Σ.

9.1 Connecting the Euler Angles and $\boldsymbol{\omega}$

Let Σ and S be two triads with the same origin $\Omega = O$ and assume that S moves by a precession of pole O with respect to Σ, with angular characteristic $\boldsymbol{\omega}$. Let φ, ψ, and θ be the instantaneous Euler angles relative to the two triads. The *nodal line* has direction $\mathbf{n}$ and is the intersection of the planes through the pole O and normals $\mathbf{e}_3$ and $\mathbf{u}_3$ respectively. The precession of S can be regarded as the composition of three precessions with the same pole O as follows:

- **(a)** Precession about $\mathbf{e}_3$ with angular velocity $\boldsymbol{\omega}_1 = \dot\varphi\mathbf{e}_3$;
- **(b)** Precession about $\mathbf{n}$ with angular velocity $\boldsymbol{\omega}_2 = \dot\theta\mathbf{n}$;
- **(c)** Precession about $\mathbf{u}_3$ with angular velocity $\boldsymbol{\omega}_3 = \dot\psi\mathbf{u}_3$.

The rigid motion of S with respect to Σ is the precession of pole O and angular characteristic

$$\boldsymbol{\omega} = \dot\varphi\mathbf{e}_3 + \dot\theta\mathbf{n} + \dot\psi\mathbf{u}_3. \tag{9.2}$$

From the definition of the line of the nodes we obtain

$$\mathbf{n} = \cos\psi\mathbf{u}_1 - \sin\psi\mathbf{u}_2, \qquad \mathbf{n} = \cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2. \tag{9.3}$$

Moreover, from the transformation matrix from S to Σ, introduced in (4.1),

$$\mathbf{e}_3 = \sin\psi\sin\theta\mathbf{u}_1 + \cos\psi\sin\theta\mathbf{u}_2 + \cos\theta\mathbf{u}_3.$$

Putting this in (9.2) gives the expression of $\boldsymbol{\omega}$ in the coordinates of the moving triad S in terms of the Euler angles

$$\begin{aligned}\boldsymbol{\omega} &= (\dot\varphi\sin\psi\sin\theta + \dot\theta\cos\psi)\mathbf{u}_1 \\ &\quad + (\dot\varphi\cos\psi\sin\theta - \dot\theta\sin\psi)\mathbf{u}_2 + (\dot\varphi\cos\theta + \dot\psi)\mathbf{u}_3.\end{aligned} \tag{9.4}$$

The expression of $\boldsymbol{\omega}$ in the coordinates of the fixed triad Σ is

$$\begin{aligned}\boldsymbol{\omega} &= (\dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi)\mathbf{e}_1 \\ &\quad - (\dot\psi\sin\theta\cos\varphi - \dot\theta\sin\varphi)\mathbf{e}_2 + (\dot\psi\cos\theta + \dot\varphi)\mathbf{e}_3.\end{aligned} \tag{9.5}$$

10 Fixed and Moving Axodes

Equation (7.4) of the axis of motion is defined for a fixed value of the parameter t. As t ranges over its domain of definition, (7.4) may be regarded as a map

$$(t, \lambda) \to P(t; \lambda) = O(t) + \frac{\boldsymbol{\omega}(t) \wedge \mathbf{v}(O(t))}{\|\boldsymbol{\omega}(t)\|^2} + \lambda \boldsymbol{\omega}(t) \tag{10.1}$$

of the pair of parameters (t, λ). With this interpretation, (10.1) defines a surface, which we denote by $\mathbb{G}$. Since for every t fixed (10.1) represents a line, such a surface is a *ruled surface*, called an *axode*. The equation (10.1) of the axode is intrinsic, i.e., independent of any reference system. When written in coordinates of Σ it represents a surface fixed with Σ called a *fixed axode* and denoted by $\mathbb{G}_\Sigma$. When it is written in the coordinates of S, it represents a

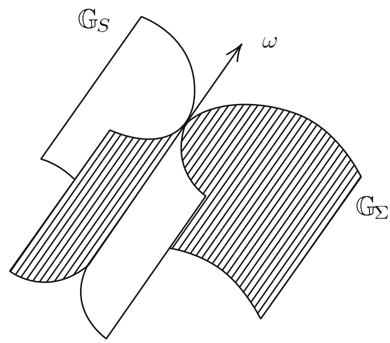

Fig. 10.1.

surface fixed with S and moving with it, called a *moving axode* and denoted by $\mathbb{G}_S$. The surfaces $\mathbb{G}_\Sigma$ and $\mathbb{G}_S$ are geometric representations of the positions of the axis of motion $\mu(\boldsymbol{\omega})$ along its motion relative to the triads Σ and S. Therefore $\mu(\boldsymbol{\omega}) = \mathbb{G}_\Sigma \cap \mathbb{G}_S$. A geometric point $P \in \mu(\boldsymbol{\omega})$ can be regarded as the instantaneous position of a point moving on $\mathbb{G}_\Sigma$. As such, it has velocity

$$\mathbf{v}_\Sigma(P) = \begin{cases} \text{time derivative of the right-hand side} \\ \text{of (10.1) written in the coordinates of } \Sigma. \end{cases}$$

Similarly, the same point $P \in \mu(\boldsymbol{\omega})$ can also be regarded as the instantaneous position of a point moving in $\mathbb{G}_S$, and as such it has velocity

$$\mathbf{v}_S(P) = \begin{cases} \text{time derivative of the right-hand side} \\ \text{of (10.1) written in the coordinates of } S. \end{cases}$$

Finally, the very same $P \in \mu(\boldsymbol{\omega})$, regarded as instantaneously fixed with S, and thus instantaneously following the rigid motion of S, has a velocity given by the vectorial invariant $I(\boldsymbol{\omega})$. Thus the same geometric point $P \in \mu(\boldsymbol{\omega})$ can be interpreted in three different ways as the instantaneous position of

$$\begin{aligned} t \to P_\Sigma(\lambda; t), &\quad \text{motion of } P(\lambda;\cdot) \text{ on the surface } \mathbb{G}_\Sigma; \\ t \to P_S(\lambda; t), &\quad \text{motion of } P(\lambda;\cdot) \text{ on the surface } \mathbb{G}_S; \\ t \to P(t; \lambda), &\quad \text{motion of } P(\lambda;\cdot) \text{ transported by } S. \end{aligned}$$

Theorem 10.1. *The two surfaces* $\mathbb{G}_\Sigma$ *and* $\mathbb{G}_S$ *are mutually tangent at points* $P \in \mu(\boldsymbol{\omega})$ *such that* $\mathbf{v}_\Sigma(P) \wedge \boldsymbol{\omega} \neq 0$.

Proof. By definition of the axis of motion, the transport velocity of a point $P \in \mu(\boldsymbol{\omega})$ is the vectorial invariant $I(\boldsymbol{\omega})$. From (8.4),

$$\mathbf{v}_\Sigma(P) - \mathbf{v}_S(P) = \mathbf{v}_T(P) = I(\boldsymbol{\omega}).$$

Since $\mathbf{v}_\Sigma(P)$ is tangent to $\mathbb{G}_\Sigma$ at P and $\mu(\boldsymbol{\omega})$ is entirely contained in σ_Σ, the tangent plane to $\mathbb{G}_\Sigma$ at a point $P \in \mu(\boldsymbol{\omega})$ has normal $\mathbf{v}_\Sigma(P) \wedge \boldsymbol{\omega}$. Similarly, $\mathbf{v}_S(P)$ is tangent to $\mathbb{G}_S$ at P, and the tangent plane to $\mathbb{G}_S$ at such a point has normal $\mathbf{v}_S(P) \wedge \boldsymbol{\omega}$. Taking the exterior product of the previous relation by $\boldsymbol{\omega}$ yields

$$\mathbf{v}_\Sigma(P) \wedge \boldsymbol{\omega} = \mathbf{v}_S(P) \wedge \boldsymbol{\omega}.$$

Therefore at every point of the axis of motion for which $\mathbf{v}_S(P) \wedge \boldsymbol{\omega} \neq 0$, the two surfaces $\mathbb{G}_\Sigma$ and $\mathbb{G}_S$ have the same tangent plane. ■

The surface $\mathbb{G}_S$ is attached to S, and as such it shares the same rigid motion of S. Moreover, for each value of the parameter t it must be tangent to $\mathbb{G}_\Sigma$ along the axis of rotation. Therefore $\mathbb{G}_S$ rolls over $\mathbb{G}_\Sigma$ and slides along $\mu(\boldsymbol{\omega})$. The sliding velocity is $I(\boldsymbol{\omega})$. If the vectorial invariant is zero, then $\mathbb{G}_S$ rolls without sliding over $\mathbb{G}_\Sigma$. The relevance of these ruled surfaces is in that a rigid motion can be realized as the motion of rolling and sliding of a ruled surface $\mathbb{G}_S$ over another ruled surface $\mathbb{G}_\Sigma$ along their common generators $\mu(\boldsymbol{\omega})$. This correspondence between rigid motions and mutual rolling and sliding of two ruled surfaces with the same generators is at the foundation of the theory of mechanical *gears*. If $\boldsymbol{\omega}$, although variable, has constant direction, then the axodes $\mathbb{G}_S$ and $\mathbb{G}_\Sigma$ are both cylindrical surfaces.

Using (10.1), we write down the equations of these cylindrical surfaces and examine the case that there is a point $O \in S$ such that $\mathbf{v}(O) \wedge \boldsymbol{\omega} \equiv 0$.

10.1 Precessions: Fixed and Moving Cones

In a precession of pole O and characteristic $\boldsymbol{\omega}$ the axodes are cones with vertex at O, called *Poinsot cones* [129]. If $\boldsymbol{\omega}$, although variable, forms a constant angle $\theta \in (0, \frac{\pi}{2})$ with $\mathbf{e}_3$, then $\mathbb{G}_\Sigma$ is a right circular cone. In general, however, $\mathbb{G}_S$, while a cone, need not be circular. If $\boldsymbol{\omega}$ has constant direction, one might take $\mathbf{e}_3 = \boldsymbol{\omega}/\|\boldsymbol{\omega}\|$. Then the fixed axode $\mathbb{G}_\Sigma$ degenerates into an axis through the

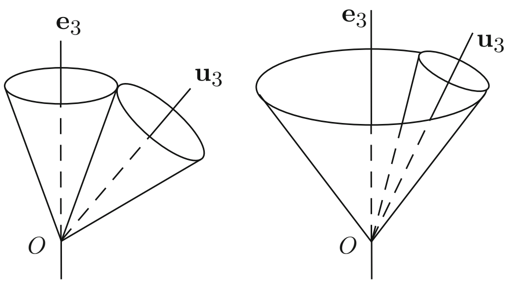

Fig. 10.2.

pole of the precession and directed like $\boldsymbol{\omega}$. In this case the precession is a *rotation*. The moving axode $\mathbb{G}_S$ is a right circular cone, possibly degenerate. For this it suffices to show that the angle θ between $\mathbf{e}_3$ and $\mathbf{u}_3$ is constant. Since $\mathbf{u}_3$ is fixed with S, by the differentiation formula (5.3),

$$\frac{d}{dt}(\mathbf{e}_3 \cdot \mathbf{u}_3) = -\dot{\theta}\,\sin\theta = \mathbf{e}_3 \cdot (\boldsymbol{\omega} \wedge \mathbf{u}_3) = 0.$$

Let Σ, S_1, S_2 be triads with the same origin O. Assume that S_2 precesses with respect to S_1 with characteristic $\omega_2\mathbf{u}_3$. Assume, moreover, that S_1 precesses with respect to Σ with characteristic $\omega_1\mathbf{e}_3$. The composition is a precession of pole O and characteristic $\boldsymbol{\omega} = \omega_2\mathbf{u}_3 + \omega_1\mathbf{e}_3$. If ω_1 and ω_2 are constant, then the ruled surfaces $\mathbb{G}_\Sigma$ and $\mathbb{G}_{S_2}$ relative to the resulting precession are both right circular cones. Indeed, the parallelogram of the two vectors $\omega_2\mathbf{u}_3$ and $\omega_1\mathbf{e}_3$ keeps a constant configuration along the rigid motion. Therefore its diagonal that is directed as $\boldsymbol{\omega}$ forms a constant angle with the axes $\mathbf{e}_3$ and $\mathbf{u}_3$.

11 Plane Rigid Motions

Let $\Sigma = \{\Omega; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be two triads and denote by π the coordinate plane $y_3 = 0$ in Σ and by p the coordinate plane $x_3 = 0$ in S. If S moves with rigid motion with respect to Σ in such a way that $\pi = p$ at all times, the rigid motion of S is said to be *planar*. In such a case $\mathbf{e}_3 \equiv \mathbf{u}_3$ and the trajectory of O is in π, i.e., $(\Omega - O) \cdot \mathbf{u}_3 \equiv 0$. Since $\dot{\mathbf{u}}_3 = \boldsymbol{\omega} \wedge \mathbf{u}_3 = \dot{\mathbf{e}}_3 \equiv 0$, the vector $\boldsymbol{\omega}$ is always parallel to $\mathbf{u}_3$ and the plane p slides over π in the sense that the trajectory of every point in p is in π. Let $P \in S$ and denote by P_p its projection on p. By the Poisson formula (6.1),

$$\mathbf{v}(P) = \mathbf{v}(O) + \boldsymbol{\omega} \wedge (P - O) = \mathbf{v}(O) + \boldsymbol{\omega} \wedge (P_p - O) = \mathbf{v}(P_p).$$

Therefore the velocity of any point $P \in S$ is uniquely determined by its projection on p. In particular, $\mathbf{v}(P)$ is normal to $\boldsymbol{\omega}$. In this sense the plane p is called the *representative* plane of the rigid motion.

11.1 Center of Instantaneous Rotation

The axis of motion is parallel to $\mathbf{u}_3$ at all times. Its trace C on the representative plane p is called the *center* of instantaneous rotation. By equation (7.4) of the axis of motion such a trace may be realized, with no loss of generality, for $\lambda = 0$. Therefore

$$C = O + \frac{\boldsymbol{\omega} \wedge \mathbf{v}(O)}{\|\boldsymbol{\omega}\|^2}, \qquad \text{provided } \boldsymbol{\omega} \neq 0. \tag{11.1}$$

Since the vectorial invariant $I(\boldsymbol{\omega})$ is zero, the velocity $\mathbf{v}(C)$ of the center of instantaneous rotation is identically zero. This permits an entirely geometrical determination of C.

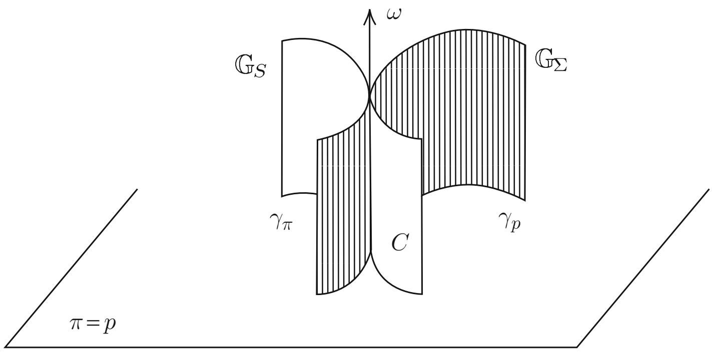

Fig. 11.1.

Theorem 11.1 (Chasles [27]). *If $\boldsymbol{\omega} \neq 0$, the center C is on the normal line to the trajectory of any point P of the representative plane p, drawn through P.*

Proof. Write the Poisson formula (6.1) for $O = C$, and equation (11.1) with O replaced by the generic point P. This gives

$$\mathbf{v}(P) = \boldsymbol{\omega} \wedge (P - C), \qquad C = P + \frac{\boldsymbol{\omega} \wedge \mathbf{v}(P)}{\|\boldsymbol{\omega}\|^2}. \qquad \blacksquare$$

Chasles's theorem implies that C is geometrically determined by the trajectories of any two points P_1 and P_2 of the representative plane p. The normal lines to these trajectories drawn through $P_1(t)$ and $P_2(t)$ intersect at $C(t)$. If $\boldsymbol{\omega}(t) = 0$, the center of instantaneous rotation $C(t)$ is not defined. If $\boldsymbol{\omega}$ is identically zero, the motion is a translation and the trajectories of any two points are parallel lines. In such a case C might be defined as the point at infinity of the normal bundle of such parallel lines.

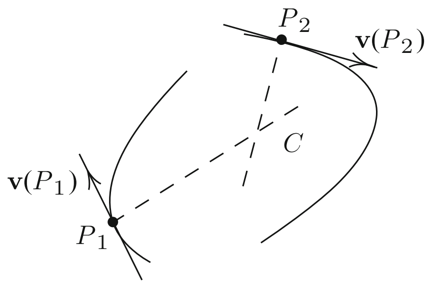

Fig. 11.2.

11.2 Centrodes

The traces of $\mathbb{G}_\Sigma$ and $\mathbb{G}_S$ on the fixed plane π and the representative plane p are two curves $\varGamma_\pi$ and $\varGamma_p$, called *centrodes*. Since $I(\boldsymbol{\omega}) \equiv 0$, the moving centrode $\varGamma_p$ rolls without slipping over the fixed centrode $\varGamma_\pi$, and their tangency point is the center of instantaneous rotation.

Problems and Complements

2c Areolar Velocity and Central Motions

2.1c Cycloidal Trajectories

A circle of center O and radius R rolls without slipping on a line, as in **Figure 2.1c**. A point P of the rolling circle traces a curve called a *cycloid*. Assume that P starts from the origin $\varOmega$. Denote by C the contact point of the rolling circle with the line and by φ the angle formed by OC and OP, measured counterclockwise from OC. With this notation, $\stackrel{\frown}{PC} = \overline{\varOmega C} = R\varphi$, and the parametric equations of the cycloid are

$$x_1 = R(\varphi - \sin\varphi), \qquad x_2 = R(1 - \cos\varphi),$$

and the corresponding Cartesian form is

$$x_1 = R \arccos\left(\frac{R - x_2}{R}\right) - \sqrt{2Rx_2 - x_2^2}.$$

Compute $\mathbf{t}(\varphi)$, $\mathbf{n}(\varphi)$, and the curvature $\kappa(\varphi)$.

2.2c The Brachistochrone and Tautochrone

Consider an arc of a cycloid, inverted and translated as in **Figure 2.2c**. A point "falls" from A along a curve γ_{AB} of extremities A and B, subject only to gravity. The time it takes the point to reach B depends on the curve γ_{AB}. The problem of the *brachistochrone* is that of finding the curve γ_{AB} that minimizes this time. The arc of the cycloid through A and B, as in **Figure 2.2c**, is the curve of least time (§1.3c of the Complements of Chapter 9). Points starting to "fall" from $A, M_1, M_2, \ldots \neq B$ at the same instant all reach B simultaneously. In this sense the brachistochrone is also the *tautochrone* through A and B.

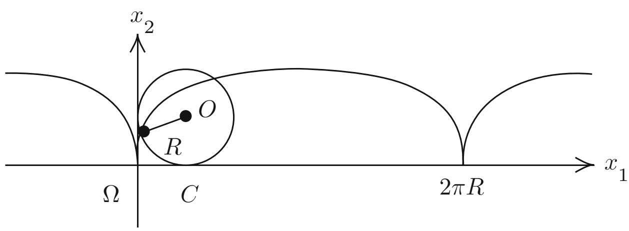

Fig. 2.1c.

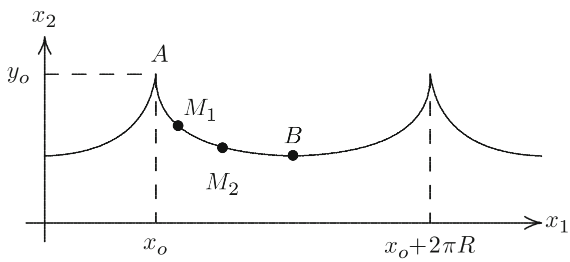

Fig. 2.2c.

2.3c Hypocycloidal and Epicycloidal Trajectories

A circle of center O and radius ρ rolls without slipping in the interior of a fixed circle of center Ω and radius $R > \rho$ (left of **Figure 2.3c**). A point P of the rolling circle traces a curve called a *hypocycloid*. If P starts from $P_o = (R, 0)$ and $\varphi = \widehat{P_o \Omega O}$, the parametric equations of the hypocycloid are

$$x_1 = (R - \rho)\cos\varphi + \rho\cos\frac{R-\rho}{\rho}\varphi,$$
$$x_2 = (R - \rho)\sin\varphi - \rho\sin\frac{R-\rho}{\rho}\varphi.$$

For $R = 4\rho$, the Cartesian form of the hypocycloid is the *astroid*

$$x_1^{2/3} + x_2^{2/3} = R^{2/3}.$$

A circle of center O and radius ρ rolls without slipping on the exterior of a fixed circle of center Ω and radius R (right of **Figure 2.3c**). A point P of the rolling circle traces a curve called an *epicycloid*. If P starts from $P_o = (R, 0)$ and $\varphi = \widehat{P_o \Omega O}$, the parametric equations of the epicycloid are

$$x_1 = (R+\rho)\cos\varphi - \rho\cos\frac{R+\rho}{\rho}\varphi,$$
$$x_2 = (R+\rho)\sin\varphi - \rho\sin\frac{R+\rho}{\rho}\varphi.$$

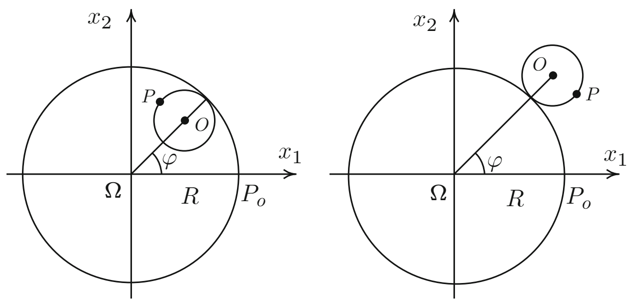

Fig. 2.3c.

2.4c Kepler's Gravitational Laws

Planets move about the Sun by Kepler's laws ([92, 93]):

1. The orbits of the planets are ellipses and the Sun occupies one of its foci.
2. The vector radius from the Sun to one of the planets sweeps equal areas in equal intervals of time (**law of areas**).
3. The square of the period of revolution of a planet is proportional to the cube of the semimajor axis of the planet's orbit (**harmonic law**).

It follows that the motion of the planets is central. Assume that a planet revolves along an ellipse of semiaxes $a > b$. Denote by S the focus on the semiaxis $(0, a)$ and by φ the angle between the generic position $P - S$ of the planet and the positive direction of the major axis of the ellipse. Using the second of the Binet formulas (2.5), prove that $\mathbf{a} = -a\, a_o^2/b^2\rho^2\mathbf{u}$. By the **harmonic law**, the proportionality factor $a\, a_o^2/b^2$ is the same for all planets (§4.1 and §5 of Chapter 3).

2.5c Apsidal Points

Given a central motion $t \to P(t)$ of center O, decompose $\dot{P}$ and $\ddot{P}$ in their radial and transversal components as in (2.1). A point P_o, on the trajectory of P is an *apsis* if the radial velocity at P_o is zero. If $\ddot{\rho}(P_o) \neq 0$, an apsis is an extremal point for the function $t \to \rho(t)$. A maximum is an *apocenter*, whereas a minimum is a *pericenter*. If P_o is an apsis, the line trough the center of motion O and P_o is an *apsidal axis* and $\rho_o = \|P_o - O\|$ is an *apsidal radius*. In a circular motion all points of the trajectory are apsidal, and there is only one apsidal radius. Assume now that the acceleration $\mathbf{a}(\rho)$ is a known function

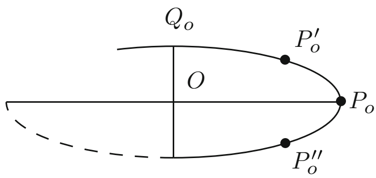

Fig. 2.4c.

of ρ, and is explicitly independent of t. An example is the gravitational motion of a planet about the Sun. In such a case the apocenter and pericenter are called *aphelion* and *perihelion* respectively. Then the motion is governed by the system of differential equations

$$\ddot{\rho} - \rho\dot{\varphi}^2 = \mathbf{a}(\rho) \cdot \mathbf{u}, \qquad 2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi} = \mathbf{a}(\rho) \cdot \mathbf{u}^{\perp}, \tag{2.1c}$$

complemented by some initial data that we choose at an apsis P_o. Denoting by a_o the area constant and using (2.3), such initial data take the form

$$\rho(0) = \rho_o = \|P_o - O\|, \quad \dot{\rho}(0) = 0, \quad \dot{\varphi}(0) = \frac{a_o}{\rho_o^2}. \tag{2.2c}$$

The system (2.1c) does not change by changing t into $-t$. Therefore the solution of (2.1c)–(2.2c) is symmetric with respect to the apsidal axis through P_o.

Proposition 2.1c *The trajectory of a central motion whose acceleration depends only on ρ is symmetric with respect to any of its apsidal axes. Moreover, there exist at most two apsidal radii.*

Proof. Let P_o be an apsis. If there are no other apsides, the statement is trivial. This occurs, for example, if the trajectory is a parabola (§5 of Chapter 3).

If P_o' is the next apsis, its symmetric P_o'' with respect the apsidal axis through P_o is also an apsis. By symmetry, $\|P_o' - O\| = \|P_o'' - O\|$. The same argument, starting from P_o', shows that the apsis next to P_o' must have a distance from O equal to $\|P_o - O\|$. Iteration of the same argument shows that $\|P_o - O\|$ and $\|P_o' - O\|$ are the only possible apsidal radii. ∎

2.6c Elliptic Trajectories of Some Central Motions

If a point P moves with acceleration $-k(P - O)$, for a given positive constant k and a fixed point O, then its trajectory is an ellipse, possibly degenerate.

The motion is central and thus planar. In a suitable coordinate system $\ddot{x} = -kx$ and $\ddot{y} = -ky$, from which

$$x = a\cos(\omega t + \varphi) \qquad y = b\sin(\omega t + \psi), \qquad \text{where } \omega^2 = k$$

for nonnegative constants a, b and real constants φ, ψ. Therefore

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\frac{xy}{ab}\sin(\psi - \varphi) = \cos^2(\psi - \varphi).$$

8c Relative Rigid Motions and Coriolis's Theorem

A point P moves with constant velocity $\mathbf{v}$ along a straight line ℓ, which in turn spins about one of its points O, on a horizontal plane, with constant angular velocity $\omega\mathbf{e}_3$. Take a fixed system with origin at O and coordinate plane $y_3 = 0$ coincident with the horizontal plane where ℓ spins. Choose a moving triad $S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ with $\mathbf{u}_3 = \mathbf{e}_3$ and $\mathbf{u}_1$ as the unit direction of ℓ. Then S precesses with respect to Σ, with characteristic $\boldsymbol{\omega} = \omega\mathbf{e}_3$. Assume that initially $\mathbf{u}_1 = \mathbf{e}_1$ and set $v = \|\mathbf{v}\|$. Then

$$(P - O)_S = vt\mathbf{u}_1, \qquad (P - O)_\Sigma = vt(\cos\omega t\mathbf{e}_1 + \sin\omega t\mathbf{e}_2).$$

From these and the formulas (8.5)–(8.7) of relative kinematics one computes

$$\begin{aligned}
\mathbf{v}_S(P) &= v(\cos\omega t\mathbf{e}_1 + \sin\omega t\mathbf{e}_2)\\
\mathbf{v}_T(P) &= \omega vt\,\mathbf{e}_3 \wedge (\cos\omega t\mathbf{e}_1 + \sin\omega t\mathbf{e}_2)\\
&= \omega vt(-\sin\omega t\mathbf{e}_1 + \cos\omega t\mathbf{e}_2),\\
\mathbf{v}_\Sigma(P) &= \mathbf{v}_S(P) + \mathbf{v}_T(P) = v(\cos\omega t\mathbf{e}_1 + \sin\omega t\mathbf{e}_2)\\
&\qquad + \omega vt(-\sin\omega t\mathbf{e}_1 + \cos\omega t\mathbf{e}_2) = (P - O)'_\Sigma,\\
\mathbf{a}_T(P) &= \dot{\boldsymbol{\omega}} \wedge (P - O) + \boldsymbol{\omega} \wedge [\boldsymbol{\omega} \wedge (P - O)]\\
&= \omega^2 vt\mathbf{e}_3 \wedge (\mathbf{e}_3 \wedge \mathbf{u}_1) = -\omega^2 vt\mathbf{u}_1\\
&= -\omega^2 vt(\cos\omega t\mathbf{e}_1 + \sin\omega t\mathbf{e}_2),\\
\mathbf{a}_C(P) &= 2\boldsymbol{\omega} \wedge \mathbf{v}_S(P) = 2\omega v\mathbf{e}_3 \wedge \mathbf{u}_1 = 2v\omega\mathbf{u}_2\\
&= 2\omega v(-\sin\omega t\mathbf{e}_1 + \cos\omega t\mathbf{e}_2),\\
\mathbf{a}_\Sigma(P) &= (-\omega^2 vt\cos\omega t - 2\omega v\sin\omega t)\mathbf{e}_1\\
&\qquad + (-\omega^2 vt\sin\omega t + 2\omega v\cos\omega t)\mathbf{e}_2 = (P - O)''_\Sigma.
\end{aligned}$$

Making use of the expression of $(P - O)_\Sigma$, the parametric equations of the trajectory of P with respect to Σ are

$$\begin{cases} y_1 = vt\cos\omega t \\ y_2 = vt\sin\omega t \end{cases} \implies \sqrt{y_1^2 + y_2^2} = \rho = \|\mathbf{v}\|t.$$

Setting $\theta = \|\boldsymbol{\omega}\|t$, the polar form of the trajectory is $\rho = (\|\mathbf{v}\|/\|\boldsymbol{\omega}\|)\,\theta$, which is the Archimedean spiral.

9c Composing Rigid Motions

9.1c Connecting the Euler Angles and $\boldsymbol{\omega}$

A moving point in $\mathbb{R}^3$ is represented by its Cartesian coordinates, as functions of time, and the Cartesian components of its velocity are the derivatives of such coordinates. The Cartesian representation (9.5) might suggest that $\boldsymbol{\omega}$ might be represented in a similar way. The issue is then whether there exist scalar functions $f_i(\theta, \varphi, \psi)$, $i = 1, 2, 3$, of the Euler angles such that setting

$$\mathbf{w} = f_1(\theta, \varphi, \psi)\mathbf{e}_1 + f_2(\theta, \varphi, \psi)\mathbf{e}_2 + f_3(\theta, \varphi, \psi)\mathbf{e}_3,$$

one might compute $\boldsymbol{\omega} = \dot{\mathbf{w}}$, i.e.,

$$\omega_i = \dot{f}_i(\theta, \varphi, \psi) = \frac{\partial f_i}{\partial \theta}\dot{\theta} + \frac{\partial f_i}{\partial \varphi}\dot{\varphi} + \frac{\partial f_i}{\partial \psi}\dot{\psi}, \qquad i = 1, 2, 3.$$

From the first component of $\boldsymbol{\omega}$ given by (9.5), it follows that

$$\frac{\partial f_1}{\partial \psi} = \sin\theta \sin\varphi, \qquad \frac{\partial f_1}{\partial \theta} = \cos\varphi.$$

Take the θ-derivative of the first and the ψ-derivative of the second to get

$$\frac{\partial^2 f_1}{\partial \psi \partial \theta} = \cos\theta \sin\varphi, \qquad \frac{\partial^2 f_1}{\partial \psi \partial \theta} = 0.$$

Therefore no such functions f_i exist. The vector $\boldsymbol{\omega}$ is said to be a *nonintegrable* combination of the rotation parameters, and the components of $\boldsymbol{\omega}$ in (9.5) are called *pseudocoordinates.*

10c Fixed and Moving Axodes

10.1c Cone with Fixed Vertex and Rolling without Slipping

The vertex O of a right circular cone of height h, opening 2α, and radius R is fixed on a vertical axis at distance $0 \le d \le \sqrt{R^2 + h^2}$ from a horizontal plane π, whereas the base of the cone rolls without slipping on π as in **Figure 10.1c**, with instantaneous contact point C. Let Ω be the projection of O on π, and denote by φ the angle between a fixed direction on π and $C - \Omega$.

Identify the axodes and write down the expression of $\boldsymbol{\omega}$ in terms of $\dot{\varphi}$.

Choose a fixed triad S with origin at the vertex of the cone and $\mathbf{u}_3$ directed as $Q - O$, and denote by θ the angle between $(C - O)$ and $-\mathbf{e}_3$. The instantaneous velocity of C as transported by the rigid motion of S is zero. Therefore C is on the axis of instantaneous rotation. Since O is fixed, it also is on such an axis. Thus the axodes are cones with vertex at O generated by

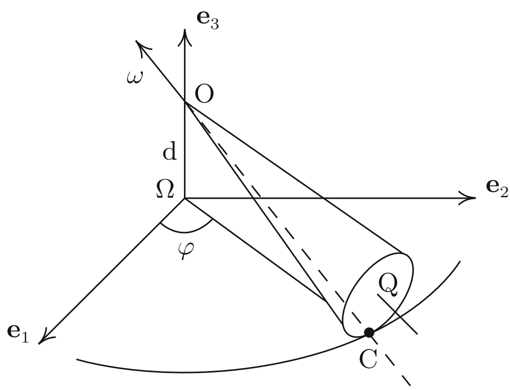

Fig. 10.1c.

the lines through O and C. For every value of d, the moving axode $\mathbb{G}_S$ is the cone with vertex at O and aperture 2α. The fixed axode $\mathbb{G}_\Sigma$ is the cone with vertex at O and aperture 2θ. If $d = 0$, the axode $\mathbb{G}_\Sigma$ degenerates into the plane π through $\Omega = O$, and if $d = \sqrt{R^2 + h^2}$, it degenerates into the vertical axis through O.

The vector $\boldsymbol{\omega}$ is directed as $O - C$, and by (6.1), $\dot{Q} = \boldsymbol{\omega} \wedge (Q - O)$. The trajectory of Q is the circle of radius $h \sin(\alpha + \theta)$ on the fixed plane $y_3 = d - h\cos(\alpha + \theta)$, and its angular velocity is $\dot{\varphi}\mathbf{e}_3$. Therefore

$$\begin{aligned} \dot{Q} &= \dot{\varphi} h \sin(\alpha + \theta)(-\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2) \\ &= \|\boldsymbol{\omega}\| \mathrm{sign}\{\dot{\varphi}\} h \sin\alpha(-\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2), \\ \|\boldsymbol{\omega}\| &= |\dot{\varphi}|(\cos\theta + \sin\theta\cot\alpha). \end{aligned}$$

From these,

$$\begin{aligned} \boldsymbol{\omega} &= -\|\boldsymbol{\omega}\| \,\mathrm{sign}\{\dot{\varphi}\} \left[\sin\theta(\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2) - \cos\theta\mathbf{e}_3\right] \\ &= -\dot{\varphi}(\sin\theta + \cos\theta\cot\alpha)\left[\sin\theta(\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2) - \cos\theta\mathbf{e}_3\right]. \end{aligned}$$

If $d = 0$, then $\theta = \frac{1}{2}\pi$ and $\boldsymbol{\omega} = -\dot{\varphi}\frac{C - \Omega}{|C - \Omega|}$. If $d = \sqrt{R^2 + h^2}$, then $\theta = 0$ and $\boldsymbol{\omega} = \dot{\varphi}\cot\alpha\mathbf{e}_3$.

10.2c Cylinder Rolling without Slipping

Consider the rigid motion of characteristics $\mathbf{v}(O) = v\mathbf{e}_1$ and $\boldsymbol{\omega} = (v/\rho)\mathbf{e}_2$, where v and ρ are given positive constants. If initially $O = (0, 0, \rho)$, one has $O(t) = (vt, 0, \rho)$. From (7.4), or equivalently (10.1), written in Σ, a point P on the axis of motion has coordinates

$$y_1(t; \lambda) = vt, \quad y_2(t; \lambda) = \lambda\frac{v}{\rho}, \quad y_3(t; \lambda) = 0.$$

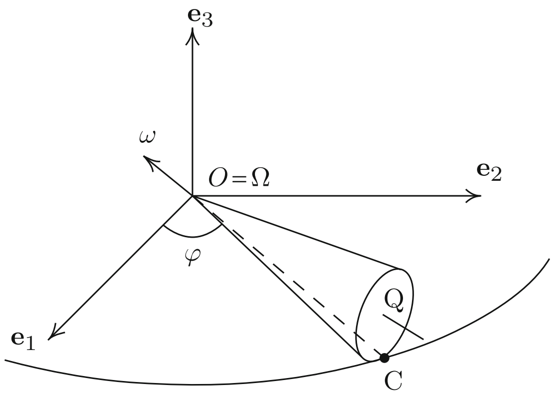

Fig. 10.2c.

Therefore $\mu(\boldsymbol{\omega})$ is represented in Σ as the line through $(vt, 0, 0)$, in the plane $y_3 = 0$, and parallel to the coordinate axis y_2. The fixed axode $\mathbb{G}_\Sigma$ is the plane $y_3 = 0$ regarded as the union of lines parallel to $\mathbf{e}_2$. The moving triad is taken with origin at O and

$$\mathbf{u}_2 = \mathbf{e}_2, \quad \mathbf{u}_3 = \sin\omega t\mathbf{e}_1 + \cos\omega t\mathbf{e}_3, \quad \mathbf{u}_1 = \mathbf{u}_2 \wedge \mathbf{u}_3, \quad \omega = \frac{v}{\rho}.$$

Expressing (7.4), or equivalently (10.1), in S gives

$$P(t;\lambda) = \rho(\mathbf{e}_2 \wedge \mathbf{e}_1) + \lambda\omega\mathbf{e}_2 = \rho\sin\omega t\mathbf{u}_1 + \lambda\omega\mathbf{u}_2 - \rho\cos\omega t\mathbf{u}_3.$$

The axis $\mu(\boldsymbol{\omega})$ is parallel to $\mathbf{u}_2$ and goes through $\rho(\sin\omega t, 0, -\cos\omega t)$. Therefore the moving axode $\mathbb{G}_S$ is the cylinder $x_1^2 + x_3^2 = \rho^2$. For $P \in \mu(\boldsymbol{\omega})$,

$$\mathbf{v}_S(P) = v(\cos\omega t\mathbf{u}_1 + \sin\omega t\mathbf{u}_3) = v\mathbf{e}_1.$$

Therefore the vectorial invariant $I(\boldsymbol{\omega})$ is zeroand the cylinder $x_1^2 + x_3^2 = \rho^2$ rolls without sliding on the plane $y_3 = 0$.

11c Plane Rigid Motions

11.1c Parametric Equations of Fixed and Moving Centrodes

The triad $S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ moves with plane rigid motion with respect to $\Sigma = \{\Omega; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_3 = \mathbf{u}_3$. Denote by θ the angle between $\mathbf{e}_1$ and $\mathbf{u}_1$ and set $\boldsymbol{\omega} = \dot\theta\mathbf{e}_3$. Then

$$\begin{aligned} \mathbf{u}_1 &= \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, & \mathbf{e}_1 &= \cos\theta\mathbf{u}_1 - \sin\theta\mathbf{u}_2, \\ \mathbf{u}_2 &= -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2, & \mathbf{e}_2 &= \sin\theta\mathbf{u}_1 + \cos\theta\mathbf{u}_2. \end{aligned}$$

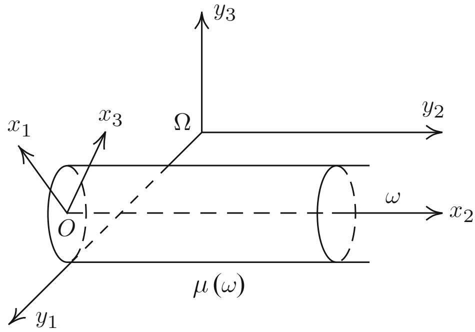

Fig. 10.3c.

Denote by

$$O = y_{o,1}\mathbf{e}_1 + y_{o,2}\mathbf{e}_2, \quad \mathbf{v}(O) = \dot{y}_{o,1}\mathbf{e}_1 + \dot{y}_{o,2}\mathbf{e}_2,$$

the positions and velocity of O in Σ. Writing (11.1) in Σ yields

$$C = (y_{o,1}, y_{o,2}) + \frac{1}{\dot{\theta}}(-\dot{y}_{o,2}, \dot{y}_{o,1}). \tag{11.1c}$$

These are the time-parametric equations of the fixed centrode Γ_π. If the trajectory of O is known in terms of θ, i.e., $\theta \to \big(\eta_{o,1}(\theta), \eta_{o,2}(\theta)\big)$, then

$$\frac{1}{\dot{\theta}}\frac{d}{dt}\big(y_{o,1}(t), y_{o,2}(t)\big) = \frac{d}{d\theta}\big(\eta_{o,1}(\theta),\, \eta_{o,2}(\theta)\big).$$

Substituting these in (11.1c) gives the equations of Γ_π in terms of θ:

$$\begin{aligned} \eta_1(\theta) &= \eta_{o,1}(\theta) - \eta'_{o,2}(\theta), \\ \eta_2(\theta) &= \eta_{o,2}(\theta) + \eta'_{o,1}(\theta). \end{aligned} \tag{11.2c}$$

For the equation of the moving centrode Γ_p compute

$$\dot{O} = \dot{y}_{o,1}(\cos\theta\mathbf{u}_1 - \sin\theta\mathbf{u}_2) + \dot{y}_{o,2}(\sin\theta\mathbf{u}_1 + \cos\theta\mathbf{u}_2).$$

Writing (11.1) in S and taking into account that $O \equiv 0$ in S, one gets

$$C = \frac{1}{\dot{\theta}}(\dot{y}_{o,1}\sin\theta - \dot{y}_{o,2}\cos\theta)\mathbf{u}_1 + \frac{1}{\dot{\theta}}(\dot{y}_{o,1}\cos\theta + \dot{y}_{o,2}\sin\theta)\mathbf{u}_2. \tag{11.3c}$$

These are the time-parametric equations of the moving centrode. If the trajectory of the origin of S is known as a function of the parameter θ, one obtains from (11.3c) the parametric equations of Γ_p in terms of θ only:

$$\begin{aligned} \xi_1(\theta) &= \sin\theta\,\eta'_{o,1}(\theta) - \cos\theta\,\eta'_{o,2}(\theta), \\ \xi_2(\theta) &= \cos\theta\,\eta'_{o,1}(\theta) + \sin\theta\,\eta'_{o,2}(\theta). \end{aligned} \tag{11.4c}$$

Thus if the trajectory of O is known in Σ as a function of θ, then fixed and moving centrodes can be regarded as geometric curves independent of motion.

11.2c Centrodes for Hypocycloidal Motions

A right circular cylinder of center O and radius ρ rolls without slipping in the cavity of a right circular cylinder of center Ω and radius $R > 2\rho$. A normal cross section is as in **Figure 11.1c**. Denote by P a point fixed on the moving circle, by P_o a point fixed on the fixed circle, and by C the contact point between the two circles. Set also $\varphi = \widehat{P_o \Omega C}$ and $\theta = \widehat{COP}$. Find a relation between $\dot{\varphi}$ and $\dot{\theta}$ and write down the parametric equations of fixed and moving centrodes in terms of t and in terms of θ. Compute the acceleration of P.

For a hypocycloidal motion, compute geometrically and analytically fixed and moving centrodes.

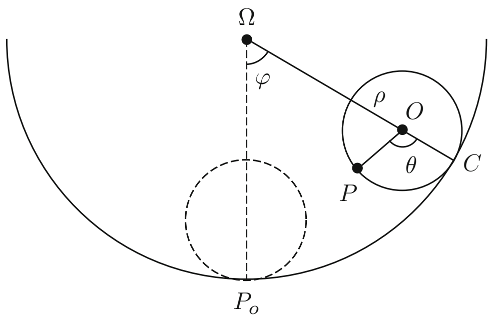

Fig. 11.1c.

11.3c The Cardano Device

A rigid rod of length 2ρ moves with its extremities A and B constrained on the axes of a Cartesian system with origin in Ω, as in **Figure 11.2c**. Compute fixed and moving centrodes, geometrically and analytically.

The trajectories of A and B are the coordinate axes $y_2 = 0$ and $y_1 = 0$. Therefore by Chasles's theorem, the center C is at the intersections of the normals to the coordinate axes through A and B. The angle $\widehat{ACB}$ is a right angle and the triangle ABC can be inscribed in a semicircumference of radius ρ. It follows that the moving centrode is the circle of diameter $\overline{AB}$. Moreover, the distance $\overline{\Omega C}$ equals 2ρ for all positions of C. Thus the fixed centrode is the circle centered at Ω and radius 2ρ.

Introduce fixed and moving triads Σ and S as in **Figure 11.2c**. Then

$$\eta_{o,1} = 2\rho \sin\theta, \qquad \eta_{o,2} \equiv 0.$$

Therefore from (11.2c), the parametric equations of Γ_π in terms of θ are

$$\eta_1 = 2\rho \sin\theta, \qquad \eta_2 = 2\rho \cos\theta.$$

From (11.4c) one obtains the parametric equations of the moving centrode Γ_p in terms of θ,

$$\xi_1 = 2\rho \sin\theta \cos\theta, \qquad \xi_2 = 2\rho \cos^2\theta,$$

whose Cartesian form is

$$\xi_1^2 + (\xi_2 - \rho)^2 = \rho^2.$$

By construction, the circle of radius ρ, fixed with S, rolls without slipping in the interior of the fixed circle centered at Ω and of radius 2ρ. Therefore the Cardano device generates and is generated by a hypocycloidal motion.

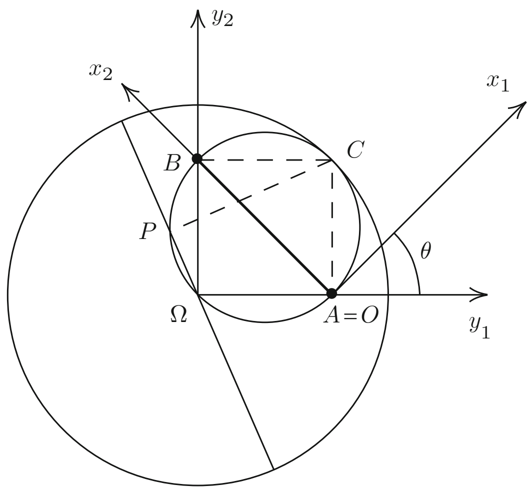

Fig. 11.2c.

Theorem 11.1c (Cardano [18, 19]). *Every point of the moving centrode traces a diameter of the fixed centrode.*

Proof. The velocity of a point P on the moving centrode is by Chasles's theorem normal to $P - C$. Since the triangle ΩPC is inscribed in a semicircle, the direction of the velocity of P goes through Ω at all times. Thus the trajectory of P is a curve whose tangent at each of its points goes through Ω. ■

11.4c More on the Cardano Device

Every hypocycloidal motion can be realized as the motion of a rigid rod with extremities constrained on two intersecting, not necessarily orthogonal, guides. As an example consider a rigid rod of length 2ρ moving with its extremities A and B constrained on two axes forming an angle $\alpha \in (0, \frac{1}{2}\pi]$. Prove, geometrically and analytically, that (a) the moving centrode is the circle through Ω,

A, and B, and radius $\ell/2\sin\alpha$; (b) the fixed centrode is the circle of center Ω and radius $\ell/\sin\alpha$; (c) every point of the moving centrode traces a diameter of the fixed centrode (see also §3.4.1c of the Complements of Chapter 5).

For the Cardano device ($\alpha = \frac{1}{2}\pi$) prove that every point of the rod traces an ellipse, possibly degenerate. Prove that the same conclusion holds for $\alpha \in (0, \frac{1}{2}\pi)$.

2

CONSTRAINTS AND LAGRANGIAN COORDINATES

1 Constrained Trajectories

Constraints are limitations imposed on the geometrical or kinematic configuration of a mechanical system. For example, in a rigid motion any two points are required to be at constant mutual distance. This is a *rigidity constraint*. A system with one of its points constrained on a surface is an example of a constrained mechanical system. Assume that a point P moves, being constrained to a surface $\mathcal{S} \subset \mathbb{R}^3$. Such a surface can be represented, at least locally, as the level set of a regular function f defined in a domain $G \subset \mathbb{R}^3$, i.e.,

$$\mathcal{S} = \{P \bigm| [f(P) = 0] \text{ and } \|\nabla f(P)\| > 0\}, \quad P \in G. \tag{1.1}$$

By the implicit function theorem, one of the coordinates, say for example x_3,

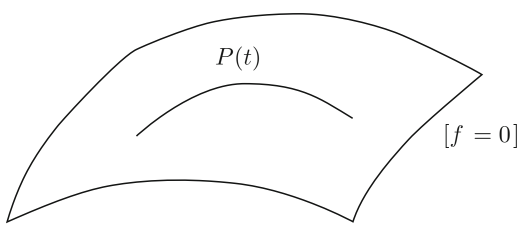

Fig. 1.1.

may be represented explicitly in terms of the remaining two. This provides a local parameterization of $\mathcal{S}$ in terms of (x_1, x_2). Such a parameterization is not unique. Indeed, choosing any pair of parameters $q = (q_1, q_2)$, the surface $\mathcal{S}$ can be represented, at least locally, by

$$\mathcal{S} = \begin{cases} x_1 = x_1(q_1, q_2), \\ x_2 = x_2(q_1, q_2), \\ x_3 = x_3\big(x_1(q), x_2(q)\big), \end{cases} \qquad \text{provided} \qquad \det\left(\frac{\partial(x_1, x_2)}{\partial(q_1, q_2)}\right) \neq 0.$$

E. DiBenedetto, *Classical Mechanics: Theory and Mathematical Modeling*, Cornerstones, DOI 10.1007/978-0-8176-4648-6_2,

If P moves on $\mathcal{S}$, the Cartesian representation of its motion is determined by the function of time $t \to q(t) = \big(q_1(t), q_2(t)\big)$, through the composition

$$P(q) = \big(x_1(q), x_2(q), x_3(q)\big).$$

Conversely, the Cartesian representation $t \to P(t)$ of the motion of P permits one, by inversion, to determine $t \to q(t)$. The velocity of P is given by

$$\dot{P} = \frac{dP}{\partial q_h}\dot{q}_h = \nabla_q P \cdot \dot{q}.$$

The parameters $q = (q_1, q_2)$ are the *Lagrangian coordinates*, whereas $\dot{q} = (\dot{q}_1, \dot{q}_2)$ are the *Lagrangian velocities* of P [101]. The system has two degrees of freedom. Assume now that P is constrained on a regular curve $\gamma \subset \mathbb{R}^3$.

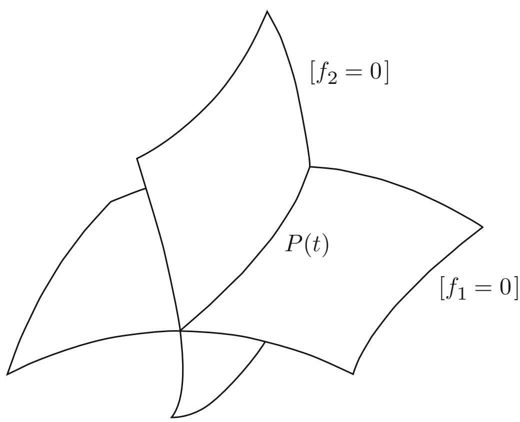

Fig. 1.2.

Such a curve can be represented, at least locally, as the intersection of the level sets of two smooth functions f_1 and f_2 defined in a domain $G \subset \mathbb{R}^3$, i.e.,

$$\gamma = \left\{ \begin{array}{c} P \mid [f_1(P) = 0] \cap [f_2(P) = 0] \\ (\nabla f_1, \nabla f_2) \text{ of rank } 2 \end{array} \right\}, \qquad P \in G. \tag{1.2}$$

Then, at least locally, two of the coordinate variables, say for example x_2 and x_3, can be represented explicitly in terms of x_1. This provides a local parametric representation of γ in terms of the parameter x_1. At times it might be more convenient to introduce directly a parameter q and parameterize γ as $q \to P(q)$. Such a parameterization can be recast, at least locally, in terms of x_1, or any other parameter, provided $\|P'(q)\| > 0$, which we assume. The geometric trajectory of P is γ, and the motion is determined by the function of time $t \to q(t)$. The velocity is $\dot{P} = (dP/dq)\dot{q}$. The parameter q is the

Lagrangian coordinate of P, whereas $\dot{q}$ is its Lagrangian velocity. A moving point P constrained on γ has one degree of freedom.

The choice of the Lagrangian coordinates is not unique. In the applications it often occurs that one may introduce them directly, as suggested by the mechanical problem at hand, with no reference to the Cartesian representations of the constraints. The constraints in (1.1)–(1.2) are independent of time and are called *fixed* or *workless*. It is conceivable that P might move on a surface, or a curve, itself depending on time. As an example consider a point P moving in a horizontal plane $x_3 = 0$ and constrained by

$$x_1 \sin\omega t - x_2 \cos\omega t = 0, \qquad \omega \in \mathbb{R}.$$

This is a time-dependent restriction on the configurations of P. For a fixed t, the constraint is a straight line. As t changes, the constraint contributes to the determination of the geometric trajectory and the time-law of motion. Taking $q = \|P - O\|$ as Lagrangian coordinate, we have

$$P(q;t) = (q\cos\omega t, q\sin\omega t).$$

Therefore the position of P depends on q, and *explicitly* also on time. By differentiation,

$$\dot{P} = \frac{dP}{dq}\dot{q} + \frac{\partial P}{\partial t}.$$

The first of these vectors is the velocity of P on the constraint as if the constraint were independent of time. The second is the transport velocity due to the movement of the constraint. Constraints of this kind are *moving*.

2 Constrained Mechanical Systems

Consider n points $P_\ell = (x_{\ell,1}, x_{\ell,2}, x_{\ell,3})$, $\ell = 1, \dots, n$, subject to m constraints

$$f_j(P_1, \dots, P_n; t) = 0, \qquad j = 1, \dots, m, \tag{2.1}$$

where f_j are m smooth functions of their arguments. They are defined in $G \times I$, where G is an open subset of $\mathbb{R}^{3n}$ and I is an interval of $\mathbb{R}$. We assume that

$$\mathcal{S}(t) = \bigcap_{j=1}^{m} [f_j(\cdot;t) = 0] \neq \emptyset \quad \text{for all } \ t \in I$$

and that for all $P \in S(t)$, the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial f_j(P;t)}{\partial x_{\ell,1}} & \dfrac{\partial f_j(P;t)}{\partial x_{\ell,2}} & \dfrac{\partial f_j(P;t)}{\partial x_{\ell,3}} \end{pmatrix} \tag{2.2}$$

has maximum rank for all $t \in I$. For example, if $m < 3n$, such a matrix has rank m and the system has $3n - m$ degrees of freedom. This defines, at least locally, a $(3n-m)$-dimensional moving manifold $\mathcal{S}(t)$. Such a manifold can be parameterized, for all $t \in I$, in terms of $3n-m$ Lagrangian coordinates $q \in \mathbb{R}^{3n-m}$, i.e.,

$$\mathcal{S}(t) \ni P_\ell = P_\ell(q;t) \qquad \text{rank of} \left(\frac{\partial P_\ell}{\partial q}\right) = 3n-m \quad \forall t \in I.$$

If the points P_ℓ move on their constraints, their motion is determined by the $3n-m$ functions $t \to q(t)$. Their velocity is

$$\dot{P}_\ell = \frac{\partial P_\ell}{\partial q_h}\dot{q}_h + \frac{\partial P_\ell}{\partial t}.$$

The first is the instantaneous velocity of P_ℓ as moving on $\mathcal{S}(t)$, as if this surface were instantaneously fixed. The second is the velocity of transport of the constraint $\mathcal{S}(t)$. The constraints in (2.1) are in general moving constraints. If they do not depend on time, they are fixed, or workless. In such a case $\mathcal{S}$ and its parameterization, in terms of Lagrangian coordinates, are independent of t. The points $P_\ell = P_\ell(q)$ are represented only in terms of q and have no explicit dependence on time. Therefore

$$\frac{\partial P_\ell}{\partial t} = 0 \qquad \text{and} \qquad \dot{P}_\ell = \frac{\partial P_\ell}{\partial q_h}\dot{q}_h \qquad \text{(fixed constraints)}.$$

2.1 Actual and Virtual Displacements

An elemental displacement of the n points P_ℓ,

$$\big(P_1,\dots,P_n;t\big) \longrightarrow \big(P_1+dP_1,\dots,P_n+dP_n;t+dt\big),$$

is said to be *actual* or *admissible* if it is compatible with the constraints in (2.1) along their time evolutions, i.e.,

$$\begin{aligned} f_j(P_1,\dots,P_n;t) &= 0,\\ f_j\big(P_1+dP_1,\dots,P_n+dP_n;t+dt\big) &= 0, \end{aligned} \qquad j=1,\dots,m.$$

From these we obtain

$$df_j = \frac{\partial f_j}{\partial P_\ell}dP_\ell + \frac{\partial f_j}{\partial t}dt = 0, \qquad j=1,\dots,m. \tag{2.3}$$

An elemental displacement of the n points P_ℓ of the form

$$(P_1,\dots,P_n;t) \longrightarrow (P_1+\delta P_1,\dots,P_n+\delta P_n;t)$$

is said to be *virtual* if it is compatible with the constraints (2.1) regarded as fixed at time t, i.e.,

$$\begin{aligned} f_j(P_1,\dots,P_n;t) &= 0,\\ f_j(P_1+\delta P_1,\dots,P_n+\delta P_n;t) &= 0, \end{aligned} \qquad j=1,\dots,m.$$

These imply

$$\frac{\partial f_j}{\partial P_\ell} \cdot \delta P_\ell = \nabla_{P_\ell} f_j \cdot \delta P_\ell = 0, \qquad j = 1, \ldots, m, \tag{2.4}$$

where the symbol δ denotes an elemental *virtual differential*. If the constraints in (2.1) are fixed, then virtual and actual displacements coincide.

2.2 Holonomic Constraints

A constraint, fixed or moving, is *holonomic* if it imposes restrictions only on the geometrical configuration of the points P_ℓ, and imposes no restriction on their time variations $\dot{P}_\ell$, $\ddot{P}_\ell$, etc. The constraints in (2.1) are holonomic. Consider two configurations $\mathcal{E} = (P_1, \ldots, P_n; t)$ and $\mathcal{E}' = (P_1', \ldots, P_n'; t')$ of the n points P_ℓ. These are compatible with the constraints (2.1) if they both satisfy the equations of the constraints. However, no restriction is placed on the displacements of the system needed to move $\mathcal{E}$ into $\mathcal{E}'$. For this reason, constraints of the type of (2.1), moving or fixed, are called also *configurational* constraints. A constraint that would impose restrictions on how $\mathcal{E}$ has to move into $\mathcal{E}'$ is not holonomic. For example, a constraint that would impose restrictions of the curvature of the trajectories of the points P_ℓ is not holonomic.

2.3 Unilateral Constraints

A point $P = (x_1, x_2, x_3)$ subject to the limitation $x_3 \ge x_1^2 + x_2^2$ is constrained to move within a paraboloid, possibly up to its boundary. Similarly, the constraint $\|P\| \le 1$ forces P to move within the unit ball about the origin of $\mathbb{R}^3$, possibly up to its boundary.

Let $f \in C^1(G)$ be such that $\|\nabla f\| > 0$ in G. A point P is said to be subject to a *unilateral* constraint if it is required to satisfy $f(P) \le 0$. If P is in the open set $[f < 0]$, its elemental displacements δP, starting at P, are unrestricted. Suppose now that $P \in [f = 0]$ and undergoes an elemental displacement δP, starting from this configuration. Since $\nabla f(P)$ points outside the set $[f \le 0]$, the displacement δP will be compatible with the constraint only if the angle between ∇f and δP is right or obtuse. Therefore elemental displacements δP from boundary configurations $P \in [f = 0]$ are admissible only if $\nabla f(P) \cdot \delta P \le 0$. It follows that elemental displacements of a point P subject to a unilateral constraint are, in general, not reversible.

A system of n points P_ℓ is subject to a unilateral constraint if

$$\{P_1, \ldots, P_n\} \in \bigcap_{j=1}^{m} \big[f_j \le 0\big], \qquad f_j \in C^1(G \times I).$$

It is assumed that such an intersection is not empty and that (2.2) is in force. If a point P_ℓ is in the interior of its constraint then its elemental displacements δP_ℓ are unrestricted. If P_ℓ belongs to one of the surfaces $[f_j = 0]$, its virtual displacements δP_ℓ must satisfy $\nabla f_j \cdot \delta P_\ell \le 0$.

3 Intrinsic Metrics and First Fundamental Form

A surface $\mathcal{S} \subset \mathbb{R}^3$ is the image of a smooth vector-valued function

$$G \ni (u,v) \longrightarrow P(u,v) = \big(x_1(u,v), x_2(u,v), x_3(u,v)\big),$$

defined in a connected open set $G \subset \mathbb{R}^2$, such that the matrix

$$\begin{pmatrix} \dfrac{\partial P}{\partial u} \\ \\ \dfrac{\partial P}{\partial v} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_3}{\partial u} \\ \\ \dfrac{\partial x_1}{\partial v} & \dfrac{\partial x_2}{\partial v} & \dfrac{\partial x_3}{\partial v} \end{pmatrix}$$

has maximum rank. Set

$$\begin{aligned} A &= \left(\frac{\partial P}{\partial u}\right)^2 = \frac{\partial P}{\partial u} \cdot \frac{\partial P}{\partial u} = \sum_{i=1}^{3} \left(\frac{\partial x_i}{\partial u}\right)^2, \\ B &= \frac{\partial P}{\partial u} \cdot \frac{\partial P}{\partial v} = \sum_{i=1}^{3} \frac{\partial x_i}{\partial u}\frac{\partial x_i}{\partial v}, \\ C &= \left(\frac{\partial P}{\partial v}\right)^2 = \frac{\partial P}{\partial v} \cdot \frac{\partial P}{\partial v} = \sum_{i=1}^{3} \left(\frac{\partial x_i}{\partial v}\right)^2, \end{aligned}$$

and consider the quadratic form

$$\begin{aligned} (\xi\ \eta) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= A\xi^2 + 2B\xi\eta + C\eta^2 \\ &= \frac{1}{A}\left[(A\xi + B\eta)^2 + (AC - B^2)\eta^2\right], \end{aligned} \tag{3.1}$$

where $(\xi, \eta) \in \mathbb{R}^2$ is arbitrary. By the Cauchy inequality, $AC - B^2 \geq 0$. Therefore the quadratic form in (3.1) is positive definite. It is called the *first fundamental form* of the surface $\mathcal{S}$. An elemental variation (du, dv) of the parameters (u, v) induces an infinitesimal variation dP on $\mathcal{S}$, whose modulus is

$$\begin{aligned} (ds)^2 = dP \cdot dP &= \left(\frac{\partial P}{\partial u} du + \frac{\partial P}{\partial v} dv\right)^2 \\ &= A(du)^2 + 2Bdudv + C(dv)^2. \end{aligned} \tag{3.2}$$

This is the *intrinsic metric* on $\mathcal{S}$. It is intrinsic, since it depends only on the geometry of $\mathcal{S}$, and it is independent of its parameterization (see §3c of the Complements). To a regular curve $\gamma \subset G$, parameterized by t, there corresponds a curve $\Gamma_\gamma \subset \mathcal{S}$ by the correspondence

$$\gamma = \big\{t \to \big(u(t), v(t)\big)\big\} \quad \Longleftrightarrow \quad \Gamma_\gamma = \big\{t \to P\big(u(t), v(t)\big)\big\}. \tag{3.3}$$

If the elemental variation (du, dv) occurs along γ, (3.2) gives an elemental arc length on the corresponding Γ_γ. If γ is parameterized by v or respectively by u, the elemental arc length of Γ_γ is computed from (3.2) as

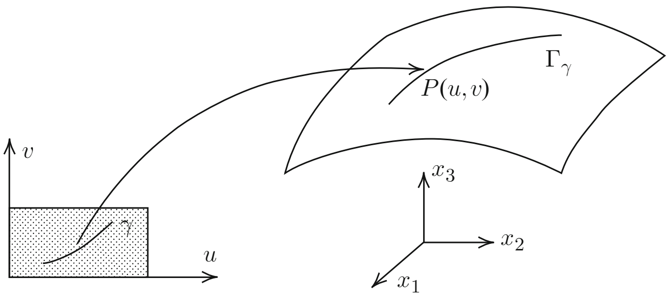

Fig. 3.1.

$$\begin{aligned} ds &= \sqrt{A(du)^2 + 2Bdudv + C(dv)^2} \\ &= \sqrt{A\left(\frac{du}{dv}\right)^2 + 2B\left(\frac{du}{dv}\right) + C}\ dv \\ &= \sqrt{A + 2B\left(\frac{dv}{du}\right) + C\left(\frac{dv}{du}\right)^2}\ du. \end{aligned} \tag{3.4}$$

4 Geodesics

Consider now those curves in G for which one of the two parameters is fixed:

$$\begin{aligned} \gamma_u &= \{u \to (u, v_o),\ v_o = \text{const}\} \quad \Longleftrightarrow \quad \Gamma_u = \{u \to P(u, v_o)\}\,, \\ \gamma_v &= \{v \to (u_o, v),\ u_o = \text{const}\} \quad \Longleftrightarrow \quad \Gamma_v = \{v \to P(u_o, v)\}\,. \end{aligned}$$

The vector $\partial P/\partial u$ is tangent to Γ_u and $\partial P/\partial v$ is tangent to Γ_v. These two vectors are linearly independent, since the matrix $(\partial P/\partial u, \partial P/\partial v)$ has maximum rank. They permit one to compute the elemental area $d\sigma$ on $\mathcal{S}$ and its normal unit vector ν by the formulas

$$d\sigma = \left\| \frac{\partial P}{\partial u} \wedge \frac{\partial P}{\partial v} \right\| du\,dv, \qquad \nu = \frac{\dfrac{\partial P}{\partial u} \wedge \dfrac{\partial P}{\partial v}}{\left\| \dfrac{\partial P}{\partial u} \wedge \dfrac{\partial P}{\partial v} \right\|}.$$

A curve $\Gamma_\gamma \subset \mathcal{S}$ is a *geodesic* if its normal $\mathbf{n}$ is parallel to ν at any of its points, i.e., if $\mathbf{n} \wedge \nu = 0$. This occurs if

$$\mathbf{n} \cdot \frac{\partial P}{\partial u} = 0 \quad \text{and} \quad \mathbf{n} \cdot \frac{\partial P}{\partial v} = 0.$$

If $\varGamma_\gamma$ is parameterized as in (3.3), set

$$\varDelta = A\dot{u}^2 + 2B\dot{u}\dot{v} + C\dot{v}^2 = \dot{s}^2.$$

Then the unit tangent $\mathbf{t}$ and the unit normal $\mathbf{n}$ to $\varGamma_\gamma$ are

$$\mathbf{t} = \frac{1}{\sqrt{\varDelta}}\left(\frac{\partial P}{\partial u}\dot{u} + \frac{\partial P}{\partial v}\dot{v}\right), \qquad \kappa\mathbf{n} = \frac{d}{dt}\left[\frac{1}{\sqrt{\varDelta}}\left(\frac{\partial P}{\partial u}\dot{u} + \frac{\partial P}{\partial v}\dot{v}\right)\right]\frac{dt}{ds}.$$

Imposing now that $\mathbf{n}$ be normal to $\partial P/\partial u$ and discarding the factor dt/ds gives

$$\begin{aligned}
0 &= \frac{\partial P}{\partial u}\frac{d}{dt}\left[\frac{1}{\sqrt{\varDelta}}\left(\frac{\partial P}{\partial u}\dot{u} + \frac{\partial P}{\partial v}\dot{v}\right)\right] \\
&= \frac{d}{dt}\left[\frac{1}{\sqrt{\varDelta}}\left(\frac{\partial P}{\partial u}\right)^2\dot{u} + \frac{\partial P}{\partial u}\frac{\partial P}{\partial v}\dot{v}\right] - \frac{1}{\sqrt{\varDelta}}\left(\frac{\partial P}{\partial u}\dot{u} + \frac{\partial P}{\partial v}\dot{v}\right)\frac{d}{dt}\frac{\partial P}{\partial u} \\
&= \frac{d}{dt}\left[\frac{1}{\sqrt{\varDelta}}(A\dot{u} + B\dot{v})\right] - \frac{1}{\sqrt{\varDelta}}\left(\frac{\partial P}{\partial u}\frac{\partial^2 P}{\partial u^2}\dot{u}^2 + \frac{\partial P}{\partial v}\frac{\partial^2 P}{\partial u^2}\dot{u}\dot{v}\right) \\
&\quad - \frac{1}{\sqrt{\varDelta}}\left(\frac{\partial P}{\partial u}\frac{\partial^2 P}{\partial u\partial v}\dot{u}\dot{v} + \frac{\partial P}{\partial v}\frac{\partial^2 P}{\partial u\partial v}\dot{v}^2\right) \\
&= \frac{d}{dt}\frac{\partial}{\partial\dot{u}}\sqrt{\varDelta} - \frac{1}{2\sqrt{\varDelta}}\left(\frac{\partial A}{\partial u}\dot{u}^2 + 2\frac{\partial B}{\partial u}\dot{u}\dot{v} + \frac{\partial C}{\partial u}\dot{v}^2\right) \\
&= \frac{d}{dt}\frac{\partial}{\partial\dot{u}}\sqrt{\varDelta} - \frac{\partial}{\partial u}\sqrt{\varDelta}.
\end{aligned}$$

Proposition 4.1 *A curve $\varGamma_\gamma \subset \mathcal{S}$ parameterized as in (3.3) is a geodesic if and only if the functions $t \to u(t), v(t)$ are solutions of the system of second-order differential equations*

$$\frac{d}{dt}\frac{\partial}{\partial\dot{u}}\sqrt{\varDelta} - \frac{\partial}{\partial u}\sqrt{\varDelta} = 0, \qquad \frac{d}{dt}\frac{\partial}{\partial\dot{v}}\sqrt{\varDelta} - \frac{\partial}{\partial v}\sqrt{\varDelta} = 0. \tag{4.1}$$

These may be rewritten in terms of u or v taken as local parameters. For example, taking u as local parameter, one may express locally $v = v(u)$ as a function of u. Then the second equality of (4.1) takes the form

$$\frac{d}{du}\frac{\partial}{\partial v'}\sqrt{\varDelta} - \frac{\partial}{\partial v}\sqrt{\varDelta} = 0. \tag{4.2}$$

Remark 4.1 The same equations arise by regarding the geodesics as the curves of least path on $\mathcal{S}$ between any two of its points (§1.4c of Chapter 9).

5 Examples of Geodesics

5.1 Geodesics in a Plane

Let $\mathcal{S}$ be the plane of equation $a_i x_i = b$. Assuming $a_3 \neq 0$, we may take $u = x_1$ and $v = x_2$ and compute

$$A = \frac{a_1^2 + a_3^2}{a_3^2}, \qquad B = \frac{a_1 a_2}{a_3^2}, \qquad C = \frac{a_2^2 + a_3^2}{a_3^2}.$$

In view of (4.2), the problem reduces to solving

$$\frac{d}{du} \frac{Cv' + B}{\sqrt{A + 2Bv' + Cv'^2}} = 0,$$

which implies $v'' = 0$. Therefore the geodesics in a plane are line segments.

5.2 Geodesics on a Sphere

From the parameterization of the sphere of radius R in terms of polar coordinates, we have

$$A = R^2 \sin^2 v, \quad B = 0, \quad C = R^2; \qquad \Delta = R^2 \left(\sin^2 v + v'^2 \right).$$

Therefore, by (4.2), the geodesics $v = v(u)$ on a sphere are solutions of

$$\frac{d}{du} \frac{v'}{\sqrt{\sin^2 v + v'^2}} - \frac{\sin v \cos v}{\sqrt{\sin^2 v + v'^2}} = 0.$$

Multiplying by v' and performing elementary manipulations yields

$$\frac{d}{du} \left(\frac{v'^2}{\sqrt{\sin^2 v + v'^2}} - \sqrt{\sin^2 v + v'^2} \right) = 0.$$

It follows that for a constant $c \in (0, 1)$,

$$\begin{aligned} u(v) &= c \int \frac{dv}{\sqrt{\sin^4 v - c^2 \sin^2 v}} = \int \frac{1}{\sin^2 v} \frac{dv}{\sqrt{\left(\frac{1-c^2}{c^2}\right) - \cot^2 v}} \\ &= -\arcsin \left(\frac{c}{\sqrt{1 - c^2}} \cot v \right) + \bar{c}. \end{aligned}$$

This finally implies

$$(\sin \bar{c}) R \sin v \cos u - (\cos \bar{c}) R \sin v \sin u = \frac{c}{\sqrt{1 - c^2}} R \cos v.$$

Therefore the geodesics are the intersection of the sphere with the planes through the origin,

$$(\sin \bar{c}) x_1 - (\cos \bar{c}) x_2 = \frac{c}{\sqrt{1 - c^2}} x_3.$$

The meridians on a sphere are geodesics, whereas none of the parallels, except the equator, is a geodesic.

5.3 Geodesics on Surfaces of Revolution

If $\mathcal{S}$ is a surface of revolution, after a possible rotation and relabeling of the coordinate axes, it can be parameterized as

$$x_1 = u\cos v, \quad x_2 = u\sin v, \quad x_3 = f(u). \tag{5.1}$$

Here f is a smooth function of the variable u, and $\mathcal{S}$ is interpreted as a surface obtained by rotating the graph of $x_3 = f(x_1)$ about the x_3 axis. From such a

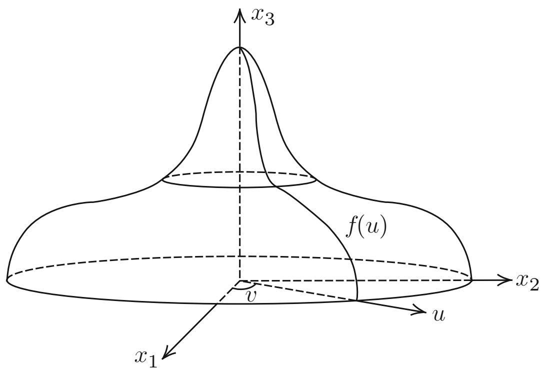

Fig. 5.1.

parametric representation, we compute

$$A = 1 + f'^2(u), \quad B = 0, \quad C = u^2; \qquad \varDelta = 1 + f'^2(u) + u^2v'^2.$$

If $\varGamma_\gamma \subset \mathcal{S}$ is a geodesic parameterized by u, then by (4.2),

$$\frac{d}{du}\frac{\partial}{\partial v'}\sqrt{1 + f'^2(u) + u^2v'^2} = \frac{d}{du}\frac{u^2v'}{\sqrt{1 + f'^2(u) + u^2v'^2}} = 0.$$

Therefore, for a constant $c \in \mathbb{R}$ and $u > c$,

$$u^4v'^2 = c^2\left[1 + f'^2(u) + u^2v'^2\right] \implies v' = \pm c\frac{\sqrt{1 + f'^2(u)}}{u\sqrt{u^2 - c^2}}.$$

From these one finds the implicit equation of the geodesics in the form

$$v - v_o = \pm c\int_{u_o}^{u}\frac{\sqrt{1 + f'^2(\eta)}}{\eta\sqrt{\eta^2 - c^2}}d\eta.$$

Recalling the geometric meaning of $\varDelta$, we obtain

$$\frac{du}{ds} = \frac{1}{\sqrt{1 + f'^2(u) + u^2v'^2}},$$

where ds is the elemental arc length on the geodesic. With this symbolism, the previous differential equations of a geodesic can be rewritten in the form

$$\frac{d}{ds}\left(u^2\frac{dv}{ds}\right) = 0, \qquad \frac{dv}{ds} = \pm c\frac{\sqrt{1+f'^2(u)}}{u\sqrt{u^2-c^2}}\frac{du}{ds}. \tag{5.2}$$

If $c = 0$, then $dv/ds = 0$ and therefore the curves $v = \text{const}$ are geodesics. These are the meridians traced on $\mathcal{S}$. The parallels, i.e., the curves $u = \text{const}$ on $\mathcal{S}$, in general are not geodesics. From (5.2) it follows that for a parallel to be a geodesic, we must have $dv/ds = \text{const} \neq 0$, and moreover,

$$c\frac{du}{ds} = \pm\frac{u\sqrt{u^2-c^2}}{\sqrt{1+f'^2(u)}}\frac{dv}{ds}, \qquad \text{i.e.,} \qquad f'^2(u) = \infty.$$

Geometrically, a parallel $\mathcal{P} \subset \mathcal{S}$ is a geodesic if $\mathcal{S}$ is tangent along $\mathcal{P}$ to a right circular cylinder with vertical axis. Thus for a sphere, only the equatorial parallel is a geodesic. For a cylinder the geodesics are curves normal to a generator at each of their points.

Problems and Complements

1c Constrained Trajectories

A class of Lagrangian coordinates arises by a change of variables in $\mathbb{R}^3$ as indicated by the following examples.

1.1c Elliptic Coordinates

Let ℓ be a fixed positive parameter and define

$$\begin{cases} x_1 = \ell\sinh u\cos\varphi\sin\theta, & u \in \mathbb{R}^+, \\ x_2 = \ell\sinh u\sin\varphi\sin\theta, & \varphi \in [0, 2\pi), \\ x_3 = \ell\cosh u\cos\theta, & \theta \in [0, \pi]. \end{cases} \tag{1.1c}$$

From these we obtain

$$\frac{x_1^2+x_2^2}{\ell^2\sinh^2 u} + \frac{x_3^2}{\ell^2\cosh^2 u} = 1, \qquad \frac{x_3^2}{\ell^2\cos^2\theta} - \frac{x_1^2+x_2^2}{\ell^2\sin^2\theta} = 1.$$

Therefore the surfaces $u = \text{const} > 0$ are ellipsoids of revolution about the x_2-axis. The semiaxes are $a_1 = a_2 = \ell\sinh u$ and $a_3 = \ell\cosh u$. The surfaces $\theta = \text{const}$ are hyperboloids of two sheets. They are of revolution about the x_3-axis. Determine the "coordinate planes" $\varphi = \text{const}$.

The velocity of a moving point P expressed in elliptic coordinates is

$$\dot{P} = \ell \begin{pmatrix} \dot{u}\cosh u\cos\varphi\sin\theta - \dot{\varphi}\sinh u\sin\varphi\sin\theta + \dot{\theta}\sinh u\cos\varphi\cos\theta \\ \dot{u}\cosh u\sin\varphi\sin\theta + \dot{\varphi}\sinh u\cos\varphi\sin\theta + \dot{\theta}\sinh u\sin\varphi\cos\theta \\ \dot{u}\sinh u\cos\theta - \dot{\theta}\cosh u\sin\theta \end{pmatrix},$$

and its modulus squared is

$$\|\dot{P}\|^2 = \ell^2\big(\dot{u}^2 + \dot{\theta}^2\big)\big(\sinh^2 u + \sin^2\theta\big) + 2\ell^2\dot{\varphi}^2\sinh^2 u\sin^2\theta.$$

If $u = \text{const} > 0$, one obtains from these the expressions of the velocity and its modulus for a point constrained on an ellipsoid of revolution about the x_3-axis. Set

$$\mathcal{A}(u,\theta) = \sinh^2 u + \sin^2\theta$$

and compute from (1.1c)

$$u_{x_1} = \frac{\cosh u\cos\varphi\sin\theta}{\ell\,\mathcal{A}(u,\theta)}, \qquad \theta_{x_1} = \frac{\sinh u\cos\varphi\cos\theta}{\ell\,\mathcal{A}(u,\theta)},$$

$$u_{x_2} = \frac{\cosh u\sin\varphi\sin\theta}{\ell\,\mathcal{A}(u,\theta)}, \qquad \theta_{x_2} = \frac{\sinh u\sin\varphi\cos\theta}{\ell\,\mathcal{A}(u,\theta)},$$

$$u_{x_3} = \frac{\sinh u\cos\theta}{\ell\,\mathcal{A}(u,\theta)}, \qquad \theta_{x_3} = \frac{-\cosh u\sin\theta}{\ell\,\mathcal{A}(u,\theta)},$$

$$\varphi_{x_1} = \frac{-\sin\varphi}{\ell\sinh u\sin\theta}, \qquad \varphi_{x_2} = \frac{\cos\varphi}{\ell\sinh u\sin\theta}, \qquad \varphi_{x_3} = 0.$$

From these, the Jacobian of the transformation from Cartesian coordinates into elliptic coordinates is

$$J_{\text{Cart}\to\text{ell}} = \ell^3\mathcal{A}(u,\theta)\sinh u\sin\theta.$$

Let $x \to f(x)$ be a smooth function in a domain $G \subset \mathbb{R}^3$. Set

$$F(u,\varphi,\theta) = f\big(\ell\sinh u\cos\varphi\sin\theta, \ell\sinh u\sin\varphi\sin\theta, \ell\cosh u\cos\theta\big)$$

and verify that

$$\|\nabla_x f\|^2 = \frac{F_u^2}{\ell^2\mathcal{A}(u,\theta)} + \frac{F_\theta^2}{\ell^2\mathcal{A}(u,\theta)} + \frac{F_\varphi^2}{\ell^2\sinh^2 u\sin^2\theta}.$$

1.2c Parabolic Coordinates

Set

$$\begin{cases} x_1 = \sqrt{uv}\cos\varphi, & u,v \ge 0, \\ x_2 = \sqrt{uv}\sin\varphi, & \varphi \in [0,2\pi), \\ x_3 = \dfrac{u-v}{2}. \end{cases} \tag{1.2c}$$

For $u > 0$ fixed, compute v from the first two equalities and put it in the last to get

$$x_3 = \frac{u^2 - r^2}{2u}, \qquad r = \sqrt{x_1^2 + x_2^2}.$$

For $u = \text{const} > 0$ these are paraboloids with vertex at $(0, 0, \frac{1}{2}u)$. Analogously, keeping v constant, a similar calculation gives the paraboloids

$$x_3 = \frac{r^2 - v^2}{2v}, \qquad v > 0.$$

Therefore the generalized coordinate surfaces $u = \text{const}$ or $v = \text{const}$ are paraboloids. For this reason the variables (u, v, φ) are called *parabolic* coordinates. Describe the surfaces $\varphi = \text{const}$.

The velocity of a point P in terms of parabolic coordinates is

$$\dot{P} = \begin{pmatrix} \frac{1}{2}\dot{u}\sqrt{\frac{v}{u}}\cos\varphi + \frac{1}{2}\dot{v}\sqrt{\frac{u}{v}}\cos\varphi - \dot{\varphi}\sqrt{uv}\sin\varphi \\ \\ \frac{1}{2}\dot{u}\sqrt{\frac{v}{u}}\sin\varphi + \frac{1}{2}\dot{v}\sqrt{\frac{u}{v}}\sin\varphi + \dot{\varphi}\sqrt{uv}\cos\varphi \\ \\ \frac{1}{2}(\dot{u} - \dot{v}) \end{pmatrix},$$

and its modulus squared is

$$\|\dot{P}\|^2 = \frac{u+v}{4}\left(\frac{\dot{u}^2}{u} + \frac{\dot{v}^2}{v}\right) + \dot{\varphi}^2 uv.$$

From the first two equalities of (1.2c) we have $uv = r^2$. From this and the third equality of (1.2c),

$$u_{x_i} = \frac{2x_i}{u+v}, \qquad v_{x_i} = \frac{2x_i}{u+v}, \qquad i = 1, 2,$$

$$u_{x_3} = \frac{2u}{u+v}, \qquad v_{x_3} = \frac{-2v}{u+v},$$

$$\varphi_{x_1} = -\frac{x_2}{x_1^2}\cos^2\varphi, \quad \varphi_{x_2} = \frac{1}{x_1}\cos^2\varphi.$$

From these the Jacobian of the transformation from Cartesian coordinates to parabolic coordinates is

$$J_{\text{Cart}\to\text{parab}} = \tfrac{1}{4}(u+v).$$

Let $x \to f(x)$ be a smooth function in a domain $G \subset \mathbb{R}^3$. Set

$$F(u, v, \varphi) = f\big(\sqrt{uv}\cos\varphi, \sqrt{uv}\sin\varphi, \tfrac{1}{2}uv\big)$$

and verify that

$$\|\nabla_x f\|^2 = \frac{4}{u+v}\left(F_u^2 u + F_v^2 v\right) + \frac{1}{uv}F_\varphi^2.$$

1.3c Spherical Coordinates

Compute the velocity of a point $P \in \mathbb{R}^3$ in terms of spherical coordinates

$$\begin{cases} x_1 = \rho \cos\varphi \sin\theta, & \rho \geq 0, \\ x_2 = \rho \sin\varphi \sin\theta, & \varphi \in [0, 2\pi), \\ x_3 = \rho \cos\theta, & \theta \in [0, \pi]. \end{cases} \tag{1.3c}$$

Verify that

$$\dot{P} = \begin{pmatrix} \dot{\rho}\cos\varphi\sin\theta - \dot{\varphi}\rho\sin\varphi\sin\theta + \dot{\theta}\rho\cos\varphi\cos\theta \\ \dot{\rho}\sin\varphi\sin\theta + \dot{\varphi}\rho\cos\varphi\sin\theta + \dot{\theta}\rho\sin\varphi\cos\theta \\ \dot{\rho}\cos\theta - \dot{\theta}\rho\sin\theta \end{pmatrix}$$

and

$$\|\dot{P}\|^2 = \dot{\rho}^2 + \dot{\varphi}^2\rho^2\sin^2\theta + \dot{\theta}^2\rho^2.$$

Compute the expressions of the velocity of a point constrained to move in the cavity of a sphere (spherical pendulum).

1.4c Cylindrical Coordinates

Compute the velocity of a moving point $P \in \mathbb{R}^3$ in terms of cylindrical coordinates

$$\begin{cases} x_1 = r\cos\varphi, & r \geq 0, \\ x_2 = r\sin\varphi, & \varphi \in [0, 2\pi), \\ x_3 = x_3, & y_3 \in \mathbb{R}. \end{cases} \tag{1.4c}$$

Verify that

$$\dot{P} = \begin{pmatrix} \dot{r}\cos\varphi - \dot{\varphi}r\sin\varphi \\ \dot{r}\sin\varphi + \dot{\varphi}r\cos\varphi \\ \dot{x}_3 \end{pmatrix}, \qquad \|\dot{P}\|^2 = \dot{r}^2 + \dot{\varphi}^2 r^2 + \dot{x}_3^2.$$

Compute the expression of the velocity of a point moving on a right circular cylinder with vertical axis.

2c Constrained Mechanical Systems

2.1c Holonomic and Nonholonomic Constraints

Let a mechanical system be described by N independent Lagrangian parameters $q = (q_1, \ldots, q_N)$. A holonomic constraint on the system is of the form $f(q;t) = \text{const}$, or by taking derivatives in t,

$$f_{q_h}\dot{q}_h + f_t = 0 \quad \text{along the motion.}$$

On the other hand, a constraint of the type

$$A_h(q;t)\dot{q}_h + A_o(q;t) = 0$$

is in general not holonomic, since it imposes limitations on the Lagrangian configurations q and the Lagrangian velocities $\dot{q}$. However, if there exists a smooth function $f(q;t)$ such that

$$A_h(q;t) = f_{q_h}(q;t), \quad i = 1, \ldots, N, \quad A_o(q;t) = f_t(q;t),$$

then such a constraint can be rewritten as $\dot{f}(q;t) = 0$ or $f(q;t) = \text{const}$ and is therefore holonomic. As an example consider a point P constrained by $\dot{P} = \mathbf{u}$, where $\mathbf{u}$ is a fixed vector. Such a constraint is holonomic, since it requires only that the trajectory be a straight line.

The constraint $\|\dot{P}\| = c$, where c is a given positive constant, restricts the modulus of the velocity, and it cannot be reduced to a holonomic constraint by integration. Notice that, in contrast to the previous example, such a constraint does not restrict the configurations of P. In particular, P might go from P_1 to P_2 along an arbitrary path, provided the motion occurs at constant speed.

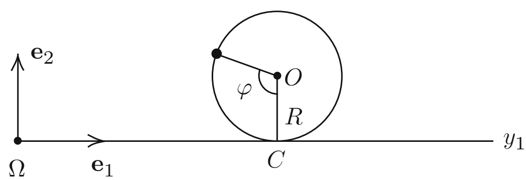

Fig. 2.1c.

2.2c Disk Rolling without Slipping on a Line

A disk of center O and radius R is constrained to move on a linear horizontal guide while remaining in a fixed vertical plane, as in **Figure 2.1c**. The system has two degrees of freedom, and we may choose as Lagrangian coordinates the angle φ between $C-O$ and a fixed radius. Requiring that the disk roll without slipping means to impose on the contact point C, regarded as part of the rigid motion of the disk, to have zero velocity,

$$\dot{C} = \dot{O} - \dot{\varphi}\mathbf{e}_3 \wedge (C - O) = 0.$$

This can be written in the form $\dot{y}_1 - R\dot{\varphi} = 0$, which is equivalent to

$$y_1 - R\varphi = \text{const}.$$

Therefore for a disk on a guide, the constraint of "rolling without slipping" is holonomic. Assume that the disk moves on a parabola, an ellipse, or a cycloid, remaining on a fixed vertical plane. Write down the analytical expression of the constraint “rolling without slipping” and conclude that in all cases, the constraint is holonomic.

2.3c Sphere Rolling without Slipping in a Plane

A sphere of center O and radius R is required to roll without slipping in a horizontal plane, as in **Figure 2.2c**. As Lagrangian coordinates take the Cartesian coordinates y_1, y_2 of the center O and the Euler angles φ, ψ, θ, formed by a moving triad S with origin O and fixed with the sphere, with a fixed triad Σ. The constraint of “rolling without slipping” translates into

$$\dot{C} = \dot{y}_1\mathbf{e}_1 + \dot{y}_2\mathbf{e}_2 + \boldsymbol{\omega} \wedge (C - O) = 0.$$

Using the expression of the vector $\boldsymbol{\omega}$ in terms of the Euler angles (formula

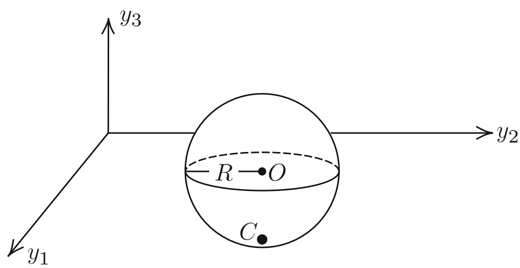

Fig. 2.2c.

(9.5) of Chapter 1), this can be rewritten as

$$\begin{aligned}\dot{y}_1 + R(\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi) = 0,\\ \dot{y}_2 + R(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi) = 0.\end{aligned}$$

Such a constraint cannot be integrated, that is, cannot be expressed as

$$f\big(y_1, y_2, \theta, \varphi, \psi; t\big) = \text{const}$$

for some smooth function f. If such an f were to exist, it would have to satisfy

$$f_{y_1}\dot{y}_1 + f_{y_2}\dot{y}_2 + f_\theta\dot{\theta} + f_\varphi\dot{\varphi} + f_\psi\dot{\psi} + f_t = 0. \qquad (*)$$

Put into this the previous expressions of $\dot{y}_1$ and $\dot{y}_2$ to get

$$\begin{aligned}-f_{y_1}R(\dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi) - f_{y_2}R(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi)&\\ + f_\theta\dot{\theta} + f_\varphi\dot{\varphi} + f_\psi\dot{\psi} + f_t = 0.&\end{aligned}$$

Taking the derivative with respect to $\dot{\varphi}$ gives $\partial f/\partial\varphi = 0$. Therefore f is independent of φ. Taking now the derivative with respect to φ, and keeping in mind that f is independent of φ, yields

$$\sin\theta(f_{y_1}\sin\varphi - f_{y_2}\cos\varphi)\dot{\psi} + (f_{y_1}\cos\varphi + f_{y_2}\sin\varphi)\dot{\theta} = 0.$$

Since the displacements $d\theta$ and $d\varphi$ are arbitrary, this generates the algebraic homogeneous linear system

$$f_{y_1}\sin\varphi - f_{y_2}\cos\varphi = 0,$$
$$f_{y_1}\cos\varphi + f_{y_2}\sin\varphi = 0,$$

in the unknowns f_{y_i}. The system admits only the trivial solution $f_{y_i} = 0$, $i = 1, 2$. Therefore f is independent of y_1 and y_2. The independence of φ, y_1, y_2 permits one to rewrite $(*)$ as

$$f_\theta\dot{\theta} + f_\psi\dot{\psi} + f_t = 0.$$

Taking now the derivative with respect to $\dot{\theta}$ gives $f_\theta = 0$, and analogously we also have $f_\psi = 0$. Therefore f is independent of θ and ψ. Finally, it is also independent of t. The contradiction implies that no such f exists. Therefore for a sphere moving in a plane, the constraint of "rolling without slipping" is not holonomic. The nonexistence of f means that the Lagrangian parameters $(y_1, y_2, \theta, \varphi, \psi)$ are not restricted, i.e., the sphere might take any configuration in the plane. Thus the constraints must act by limiting the Lagrangian velocities.

2.4c Rigid Rod with Constrained Extremities

The extremities A and B of a rigid rod of length h are constrained to move in two orthogonal planes π_1 and π_2 as in **Figure 2.3c**. One of the extremities, say B, is connected to a point $C \in \pi_2$ through a rigid rod BC of length ℓ. The other extremity, A, is connected to a point $O \in \pi_1 \cap \pi_2$ by a rigid rod OA, of length ℓ. The point C is at distance ℓ from $\pi_1 \cap \pi_2$. Take a Cartesian system with origin in O, and x-axis as $\pi_1 \cap \pi_2$, oriented so that $C = (-\ell, 0, \ell)$.

(a). Determine the number of degrees of freedom of the system. Write down the equations of the constraints and form their Jacobian matrix.

(b). Compute the determinant of all minors of maximum rank and find conditions on h and ℓ for the Jacobian matrix to have maximum rank.

The system has one degree of freedom and the constraints are

$$z_A = 0, \quad y_B = 0, \quad x_A^2 + y_A^2 = \ell^2, \quad (x_B + \ell)^2 + (z_B - \ell)^2 = \ell^2,$$

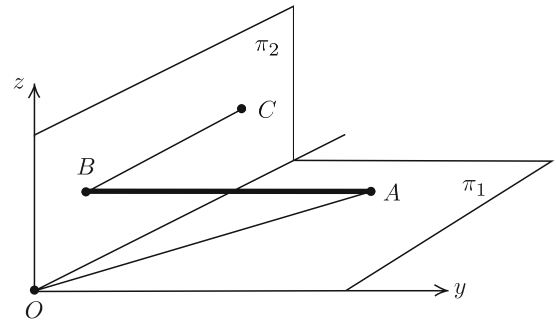

Fig. 2.3c.

and in addition, $|B-A|=h$. Using the third and fourth equations of the constraints, this can be rewritten as

$$2x_A x_B + 2\ell\,(x_B - z_B) + h^2 = 0.$$

From these one computes the Jacobian matrix

$$J = \begin{pmatrix} 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ x_A & y_A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_B+\ell & 0 & z_B-\ell \\ x_B & 0 & 0 & x_A+\ell & 0 & -\ell \end{pmatrix}.$$

The minors of order 5 with nonzero determinant must contain the third and fifth columns. Therefore the problem reduces to extracting the nontrivial non minors of order three out of the last three rows. These are

$$D_1 = \begin{pmatrix} x_A & y_A & 0 \\ 0 & 0 & x_A+\ell \\ x_B & 0 & x_A+\ell \end{pmatrix}, \qquad D_2 = \begin{pmatrix} x_A & y_A & 0 \\ 0 & 0 & z_B-\ell \\ x_B & 0 & -\ell \end{pmatrix},$$

$$D_3 = \begin{pmatrix} x_A & 0 & 0 \\ 0 & x_B+\ell & z_B-\ell \\ x_B & x_A+\ell & -\ell \end{pmatrix}, \qquad D_4 = \begin{pmatrix} y_A & 0 & 0 \\ 0 & x_B+\ell & z_B-\ell \\ 0 & x_A+\ell & -\ell \end{pmatrix}.$$

By direct calculation,

$$\begin{aligned} \det D_1 &= y_A x_B (x_B+\ell), \\ \det D_2 &= y_A x_B (z_B-\ell), \\ \det D_3 &= -x_A[(x_B+\ell)\ell + (z_B-\ell)(x_A+\ell)], \\ \det D_4 &= -y_A\,[(x_B+\ell)\ell + (z_B-\ell)(x_A+\ell)]. \end{aligned}$$

For J to have maximum rank we must have $\sum_{i=1}^4 \det^2 D_i > 0$. Using the last two equations of the constraints, we obtain

$$\sum_{i=1}^{4} \det^2 D_i = y_A^2 x_B^2 \ell^2 + \left[(x_B+\ell)\ell + (z_B-\ell)(x_A+\ell)\right]^2 \ell^2.$$

Therefore J is not of maximum rank if

$$y_A x_B = 0 \quad \text{and} \quad (x_B + \ell)\ell = -(z_B - \ell)(x_A + \ell). \qquad (*)$$

If $x_B = 0$, then $z_B = \ell = 0$ and $h^2 = 2\ell^2$. If $x_B \neq 0$ and $y_A = 0$, then $x_A = \pm\ell$. If $x_A = -\ell$, then also $x_B = -\ell$ and $h^2 = 2\ell^2$. Examine the remaining cases.

3c Intrinsic Metrics and First Fundamental Form

A new parameterization of $\mathcal{S}$ is a smooth invertible transformation

$$G \ni (u,v) \quad \left\{ \begin{array}{ll} u = u(u',v') & v = v(u',v') \\ u' = u'(u,v) & v' = v'(u,v) \end{array} \right\}, \qquad (u',v') \in G',$$

from G into a domain $G' \subset \mathbb{R}^2$. The matrix is invertible if the Jacobian determinant is nonzero, i.e., if

$$J = \begin{pmatrix} \dfrac{\partial u}{\partial u'} & \dfrac{\partial u}{\partial v'} \\[2ex] \dfrac{\partial v}{\partial u'} & \dfrac{\partial v}{\partial v'} \end{pmatrix}, \qquad \det J \neq 0.$$

The surface may be then parameterized by

$$G' \ni (u',v') \longrightarrow Q(u',v') = P\big(u(u',v'), v(u',v')\big),$$

and one computes $(du, dv) = (du', dv')J^t$. From these,

$$\begin{aligned} ds^2 &= (du, dv)\begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix} = (du', dv')J^t \begin{pmatrix} A & B \\ B & C \end{pmatrix} J \begin{pmatrix} du' \\ dv' \end{pmatrix} \\ &= (du', dv')\begin{pmatrix} A' & B' \\ B' & C' \end{pmatrix}\begin{pmatrix} du' \\ dv' \end{pmatrix} = ds'^2, \end{aligned}$$

where A', B', C' are the coefficients of the first fundamental form, relative to the new parameterization of $\mathcal{S}$.

3.1c A Parameterization of the Torus

A torus is the surface obtained by a rigid revolution about the x_3-axis of a circumference of center $(x_o, 0, 0)$ and radius $R \in (0, x_o)$. A parameterization of the torus is

$$\begin{aligned} x_1(u,v) &= (x_o + R\cos u)\cos v, \quad && u \in [0, 2\pi], \\ x_2(u,v) &= (x_o + R\cos u)\sin v, \quad && v \in [0, 2\pi], \\ x_3(u,v) &= R\sin u. \end{aligned}$$

Prove that the surface is nondegenerate, i.e., its first fundamental form is positive definite at each of its points.

4c Geodesics

Let $\varGamma_\gamma \subset \mathcal{S}$ be as in (3.3). The condition for $\varGamma_\gamma$ to be a geodesic may be expressed using the intrinsic parameterization in terms of the arc length. Denoting by κ the curvature of $\varGamma_\gamma$, we have

$$\begin{aligned}\mathbf{t} &= \frac{dP}{ds} = \frac{\partial P}{\partial u}\frac{du}{ds} + \frac{\partial P}{\partial v}\frac{dv}{ds},\\ \kappa\mathbf{n} &= \frac{d^2P}{ds^2} = \frac{\partial^2 P}{\partial u^2}\left(\frac{du}{ds}\right)^2 + 2\frac{\partial^2 P}{\partial u\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial^2 P}{\partial v^2}\left(\frac{dv}{ds}\right)^2\\ &\qquad + \frac{\partial P}{\partial u}\frac{d^2u}{ds^2} + \frac{\partial P}{\partial v}\frac{d^2v}{ds^2}.\end{aligned}$$

Imposing the condition that $\varGamma_\gamma$ be a geodesic yields the differential system

$$\frac{1}{2}\frac{\partial A}{\partial u}\left(\frac{du}{ds}\right)^2 + \frac{\partial A}{\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial P}{\partial u}\frac{\partial^2 P}{\partial v^2}\left(\frac{dv}{ds}\right)^2 + A\frac{d^2u}{ds^2} + B\frac{d^2v}{ds^2} = 0,$$

$$\frac{\partial P}{\partial v}\frac{\partial^2 P}{\partial u^2}\left(\frac{du}{ds}\right)^2 + \frac{\partial A}{\partial u}\frac{du}{ds}\frac{dv}{ds} + \frac{1}{2}\frac{\partial C}{\partial v}\left(\frac{dv}{ds}\right)^2 + B\frac{d^2u}{ds^2} + C\frac{d^2v}{ds^2} = 0.$$

Observing that

$$\frac{\partial P}{\partial u}\frac{\partial^2 P}{\partial v^2} = \frac{\partial B}{\partial v} - \frac{1}{2}\frac{\partial C}{\partial u}, \qquad \frac{\partial P}{\partial v}\frac{\partial^2 P}{\partial u^2} = \frac{\partial B}{\partial u} - \frac{1}{2}\frac{\partial A}{\partial v},$$

this system can be rewritten as

$$\begin{aligned}Au'' + Bv'' &= -\tfrac{1}{2}[A_u u'^2 + 2A_v u'v' + (2B_v - C_u)v'^2],\\ Bu'' + Cv'' &= -\tfrac{1}{2}[(2B_u - A_v)u'^2 + 2A_u u'v' + C_v v'^2].\end{aligned} \tag{4.1c}$$

Solving it, we arrive at the system in normal form:

$$\begin{aligned}u'' &= -\left(c^1_{11}u'^2 + 2c^1_{12}u'v' + c^1_{22}v'^2\right),\\ v'' &= -\left(c^2_{11}u'^2 + 2c^2_{12}u'v' + c^2_{22}v'^2\right).\end{aligned} \tag{4.2c}$$

The coefficients c^k_{ij}, $i, j, k = 1, 2$, are called the *Christoffel symbols*, and can be computed explicitly from (4.1c).

Prove that (4.1c) is equivalent to (4.1). Compute the Christoffel symbols in the case that $\mathcal{S}$ is a plane, a sphere, or a surface of revolution.

5c Examples of Geodesics

5.1c The Clairaut Theorem [29]

Assume that the constant c in (5.2) is not zero, thereby excluding that the geodesic is a meridian. From the parametric representation (5.1) it follows that the unit vector tangent to a parallel ($u = \text{const}$) is $\mathbf{u} = (-\sin v, \cos v, 0)$. Given now a geodesic that is not a meridian, compute its unit tangent at the generic point of curvilinear coordinate s. From (5.1), written in terms of the parameter s,

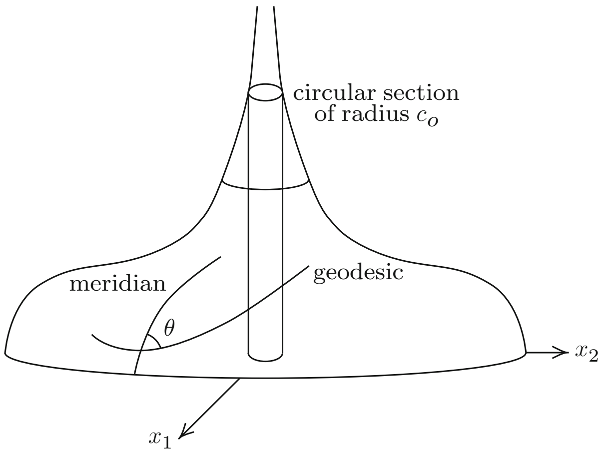

Fig. 5.1c.

$$\mathbf{t} = \left(\cos v \frac{du}{ds} - u \sin v \frac{dv}{ds}, \sin v \frac{du}{ds} + u \cos v \frac{dv}{ds}, f'(u) \frac{du}{ds} \right).$$

Let $\theta(s)$ be the angle formed by the geodesic at s, with the meridian passing through the same point. From the expression of $\mathbf{u}$ and $\mathbf{t}$,

$$\mathbf{u} \cdot \mathbf{t} = \sin\theta = u \frac{dv}{ds}.$$

Combining this with the first equality of (5.2) gives the Clairaut theorem [29]

$$u \sin\theta = \text{const},$$

along geodesics that are not meridians. Give a geometric interpretation of this fact in the particular case when $f(u) \nearrow \infty$ as $u \searrow 0$.

3

DYNAMICS OF A POINT MASS

1 Newton's Laws and Inertial Systems

A point mass $\{P; m\}$ is in a uniform mechanical state if its velocity is constant. Departures from a uniform state occur only by variations of velocity caused by solicitations external to $\{P; m\}$ and acting on it. Such external solicitations are called *forces*. The vector equation

$$\mathbf{F} = m\mathbf{a}, \qquad m \in \mathbb{R}^+, \tag{1.1}$$

encompasses the first and second Newton's laws and it describes how an external force causes its variation from a uniform mechanical state [123]. The first law asserts that in absence of external solicitations ($\mathbf{F} = 0$), a uniform mechanical state remains uniform ($\mathbf{a} = 0$).[1] The second law asserts that variations from a uniform mechanical state ($\mathbf{a} \not\equiv 0$) are proportional to the acting solicitation ($\mathbf{F} \neq 0$).[2] The proportionality factor m in (1.1) is the *inertial*

[1]Lex I: Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum nisi quatenus illud a viribus impressis cogitur statum suum mutare, [123, §13, page 54]. The first law was perceived by Leonardo da Vinci although in a non-mathematical formalism: "*…ogni moto attende al suo mantenimento, ovvero ogni corpo mosso sempre si muove, in mentre che la potenzia del motore in lui si rinserra, … ogni corpo seguirà tanto la via del suo corso per linea retta quanto durerà in esso la natura della violenza fatta…*" Codex Atlanticus (1478–1518). A physical notion of the first law appears in G. Galilei [61] *… Una nave che vadìa movendosi per la bonaccia del mare … è disposta, quando le fusser rimossi tutti gli ostacoli accidentarii ed esterni, a muoversi, con l'inpulso concepito una volta, incessabilmente e uniformemente… .*

[2]Lex II: Mutationem motus proportionalem esse vi motrici impressae et fieri secundum lineam rectam qua vis illa imprimitur..., [123, §13, page 54]. ... Vis impressa est actio in corpus exercita, ad mutandum ejus statum vel quiescendi vel movendi uniformiter in directum..., [123, §2, page 40]; this is one of the *Definitiones* preceding the *Axiomata*, (Def. III). The *vis impress* is also called by Newton *vis motrix* in Def. VIII, book I of [123], page 44.

E. DiBenedetto, *Classical Mechanics: Theory and Mathematical Modeling*, Cornerstones, DOI 10.1007/978-0-8176-4648-6_3,

mass of P, and it is positive, since it expresses that accelerations resulting from external forces are directed as such acting forces.[3] These laws and their mathematical formulation in (1.1) are formulated with respect to a reference system, termed *inertial*, whose existence is postulated.[4] We define as *inertial* any system Σ within which (1.1) holds.

Let S be a triad in rigid motion with respect to Σ, with characteristics $\mathbf{v}_\Sigma(O)$ and $\boldsymbol{\omega}$. If S translates with respect to Σ with $\mathbf{v}_\Sigma(O) = \text{const}$ (e.g., $\boldsymbol{\omega} = 0$ and $\dot{\mathbf{v}}_\Sigma(O) = 0$), by the Coriolis theorem, an observer in S detects the same acceleration as an observer in Σ, and (1.1) continues to hold in S. Therefore, if Σ is inertial, along with Σ are inertial those and only those systems in uniform, straight-line translation with respect to Σ. More generally, multiplying (8.7) of Chapter 1 by m and using (1.1) gives

$$m\mathbf{a}_S(P) = \mathbf{F} + \mathbf{F}_T + \mathbf{F}_C, \tag{1.2}$$

where

$$\mathbf{F}_T = -m\mathbf{a}_T(P) \quad \text{and} \quad \mathbf{F}_C = -m\mathbf{a}_C(P)$$

are the forces due to transport and Coriolis acceleration respectively. Thus the inertial law (1.1) continues to hold in S, provided that in the account of the external forces one includes the forces due to transport of S and the ones due to the Coriolis acceleration.

The third law asserts that if a point mass $\{P_1; m_1\}$ exerts a force $\mathbf{F}$ on another point mass $\{P_2; m_2\}$, then the latter exerts a force $-\mathbf{F}$ on the former, so that the applied system of vectors $\{(\mathbf{F}; P_1), (-\mathbf{F}; P_2)\}$ forms a couple of zero moment.[5]

For the third law to hold it is not required that the two point mass be in contact. Actions and reactions are postulated to occur simultaneously even at distance. This is equivalent to postulating that mechanical effects between material points propagate at infinite speed.

A further assumption in these laws is the existence of an absolute time, independent of any reference system.[6] The infinite speed of propagation of mechanical effects is a consequence of the postulate of an absolute time.

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[3] Materiae vis insita et potentia resistendi qua corporis unumquodque, quantum in se est, perserverat in statu suo vel quiescendi vel movendi uniformiter in directum. Hanc autem quantitatem (materiae) sub nomine corporis vel massae in sequentibus passim intelligo... [123, §1, page 39].

[4] Newton set such an inertial system in the *fixed stars*, e.g., those stars whose relative position and configuration had not significantly changed up to the eighteenth century since the astronomical observations of Ptolemy, about 130 CE.

[5] Lex III: Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi, [123, §14, page 55].

[6] Tempus absolutum, verum, et mathematicum, in se et natura sua, sine relatione ad esternum quodvis aequabiliter fluit, alioque nomine dicitur duratio..., [123, §11, page 52].

2 Mathematical Formulations of (1.1)–(1.2)

A force $\mathbf{F}$ acting on $\{P; m\}$ is given through a smooth vector-valued function

$$(P, \dot{P}; t) \longrightarrow \mathbf{F}(P, \dot{P}; t) = (F_1(x, \dot{x}; t), F_2(x, \dot{x}; t), F_3(x, \dot{x}; t))$$

defined in a region of $\mathbb{R}^7$ with values in $\mathbb{R}^3$. With this symbolism, (1.1) is a vector differential equation of the second order, or equivalently a system of three scalar differential equations, e.g.,

$$m\,\ddot{P} = \mathbf{F}(P, \dot{P}; t) \qquad \text{or} \qquad m\,\ddot{x}_j = F_j(x, \dot{x}; t), \quad j = 1, 2, 3. \tag{2.1}$$

Either one of these describes the evolution of $t \to P(t)$ starting from some position P_o and velocity $\dot{P}_o$ at some prescribed time t_o. The typical problem of the dynamics of a point mass $\{P; m\}$ consists in integrating the system (2.1) starting from such “initial data.” The system (2.1) may be rewritten as a system of six differential equations of the first order:

$$\begin{cases} m\dot{Q} = F(P, Q; t), \\ \dot{P} = Q, \end{cases} \qquad \text{or} \qquad \begin{cases} m\dot{y}_j = F_j(x, y; t), \\ \dot{x}_j = y_j, \quad j = 1, 2, 3. \end{cases}$$

If P moves on a constraint one has to add the equation of the constraint, yielding a problem in the dynamics of a constrained point mass. If the trajectory of P is known, one might write (2.1) in terms of its intrinsic triad, e.g.,

$$F_{\mathbf{t}}(s, \dot{s}; t) = m\ddot{s}, \qquad F_{\mathbf{n}}(s, \dot{s}; t) = m\kappa\dot{s}^2, \qquad F_{\mathbf{b}}(s, \dot{s}; t) = 0, \tag{2.2}$$

where $\mathbf{t}(s)$, $\mathbf{n}(s)$, and $\mathbf{b}(s)$ are respectively the tangent, normal, and binormal unit vectors to the trajectory for the value s of the parameter. The third of these further signifies the parallelism of external solicitation and resulting acceleration. In particular, the acting force is always on the osculating plane to the trajectory.

3 General Theorems of Point-Mass Dynamics

The elemental work done by $\mathbf{F}$ for an elemental displacement dP is $dL = \mathbf{F} \cdot dP$. The work done by $\mathbf{F}$ in displacing $\{P; m\}$ along a smooth curve

$$\gamma = \{t \to P(t),\, t \in [t_o, t_1]\}, \quad P(t_o) = P_o, \quad P(t_1) = P_1, \tag{3.1}$$

is given by

$$L = \int_\gamma \mathbf{F} \cdot dP = \int_{t_o}^{t_1} \mathbf{F} \cdot \dot{P}\, dt.$$

One defines *momentum* $\mathbf{Q}$ and kinetic energy T of $\{P;m\}$ as[7]

$$\mathbf{Q} = m\dot{P}, \qquad T = \tfrac{1}{2}\, m\, \dot{P}^2.$$

From these and (1.1), by taking derivatives, we obtain $\dot{Q} = \mathbf{F}$ and $\dot{T} = \mathbf{F}\cdot\dot{P}$. Therefore

$$T(t_1) - T(t_o) = L, \qquad \text{or in differential form,} \qquad dT = dL. \tag{3.2}$$

Theorem 3.1. **(i)** *The time derivative of the momentum of $\{P;m\}$ equals the external force acting on it.*
(ii) *In the absence of external solicitations, the momentum remains constant.*[8]

(iii) *The variation of kinetic energy in some time interval equals the work done by the external forces in the same time interval.*

3.1 Positional and Central Forces

Forces $P \to \mathbf{F}(P)$ dependent only on P and independent of $\dot{P}$ and t, are *positional*. If $\mathbf{F}(\cdot)$ is defined in a region $G \subset \mathbb{R}^3$, the pair $\{G;\mathbf{F}\}$ defines a *positional field*. A field is uniform if $\mathbf{F}$ is constant; an example is the gravitational field $\mathbf{F}(P) = -mg\mathbf{e}_3$.[9] A positional field is *central* if there exists a point O, called the *center* of the field, such that

$$(P - O) \wedge \mathbf{F}(P) = 0 \qquad \forall P \in G.$$

An example is the gravitational field generated by a point mass $\{O;m_o\}$. Any other material point $\{P;m\}$ in the field is subject to the force[10]

$$\mathbf{F}(P) = -\gamma\, \frac{mm_o}{\|P-O\|^2}\, \frac{P-O}{\|P-O\|}. \tag{3.3}$$

Elastic forces provide a further example of central fields. A spring of fixed endpoint O and mobile endpoint P is extended from its position of rest $P = O$. Then P is acted on by a force given by Hooke's law,

$$\mathbf{F}(P) = -k(P - O), \qquad \text{where } k > 0 \text{ is Hooke's constant.} \tag{3.4}$$

A more general example of a central field is

$$\mathbf{F}(P) = f(P)(P - O), \qquad O \in \mathbb{R}^3 \ \text{ fixed,} \tag{3.5}$$

where f is a smooth function defined in G.

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[7] The original terminology for momentum was *quantitas motus*, literally *quantity of motion* [123, Liber I, Def. II, page 40]. The kinetic energy was initially called *vis viva*, i.e., "living force," by G. W. Leibniz, in his *Theoria Motus Abstracti*. Leibniz conceived an elemental motion as an instantaneous elemental insurgence of the *vis mortua* into *vis viva*.

[8] Observed first by Newton in [123, Corollarium III, § II, page 59].

[9] At sea level $g = 9.8066\,\mathrm{m/s^2}$. It is, however, a function of altitude [76, F–158].

[10] Here $\gamma = 6.7 \cdot 10^{-11}\,\mathrm{m^3\,Kg/s^2}$ is the gravitational constant [76, F–87].

3.2 Conservative Forces

A positional force $\mathbf{F}(P)$ defined in G is conservative if there exists a function $U \in C^1(G)$, called a *potential*, uniquely determined up to a constant, such that

$$\mathbf{F}(P) = \nabla U(P) \quad \text{for all} \;\; P \in G.$$

Denote by dP an elemental displacement of P along a curve γ as in (3.1) and contained in G. If $\mathbf{F}$ is conservative, then

$$dU = \nabla U \cdot dP = \mathbf{F} \cdot dP = dL.$$

By integration,

$$\int_\gamma \mathbf{F} \cdot dP = \int_\gamma dL = \int_{P_o}^{P_1} dU = U(P_1) - U(P_o)$$

for every curve $\gamma \subset G$ with endpoints P_o and P_1. Therefore the work done by $\mathbf{F}$ in displacing $\{P; m\}$ from P_o to P_1, within G, is independent of γ. By (3.2),

$$d(T - U) = m\ddot{P} \cdot dP - \nabla U \cdot dP = \mathbf{F} \cdot dP - dU = 0, \tag{3.6}$$

or in integral form,

$$T(t) - U\big(P(t)\big) = T(t_o) - U(P_o). \tag{3.7}$$

The quantity $-U(P)$ is the *potential energy* of $\{P; m\}$, whereas $E = T(\dot{P}) - U(P)$, sum of the kinetic and potential energies, is the energy of $\{P; m\}$. The previous relations assert that E is constant along the motion. The conservation of energy expressed formally by (3.6) or equivalently by (3.7) is called the *energy integral* of the motion.[11]

For $k \in \mathbb{R}$ consider the level sets

$$[U = k] = \{P \in G \mid U(P) = k, \; \|\nabla U(P)\| > 0\}.$$

By the implicit function theorem these sets, if nonempty, are at least locally smooth surfaces; they are called *equipotential* surfaces and have unit normal

$$\mathbf{n} = \frac{\nabla U}{\|\nabla U\|} \qquad \text{pointing in the direction of increasing } U.$$

If P_o and P are in $[U = k]$, the work done by $\mathbf{F}$ in displacing $\{P_o; m\}$ into $\{P; m\}$, along any path γ lying or not on such an equipotential surface, is zero. From this, $T(P_o) = T(P_1)$. Therefore equipotential surfaces are also surfaces of constant kinetic energy. Uniform fields $\mathbf{F} = \mathbf{u}$ are conservative, and their potential is

$$U(P) = (P - O) \cdot \mathbf{u} + \text{const} \qquad \text{in} \;\; \mathbb{R}^3.$$

[11] A formal notion of *integral of motion*, is in §6.1 of Chapter 6.

The gravitational force in (3.3) is conservative, and its potential is

$$U(P) = \gamma \frac{m m_o}{\|P - O\|} + \text{const}, \qquad \text{in } \mathbb{R}^3 - \{O\}.$$

The elastic force in (3.4) is conservative, and its potential is

$$U(P) = -\tfrac{1}{2} k \|P - O\|^2 + \text{const}, \qquad \text{in } \mathbb{R}^3.$$

The central force in (3.5) is conservative if f is radial. Indeed,

$$f(P) = f\left(\|P - O\|\right) \quad \Longrightarrow \quad \mathbf{F}(P) = \nabla \int^{\|P-O\|} r f(r) dr.$$

The next proposition characterizes all fields of the form (3.5) that are also conservative.

Proposition 3.1 *The field (3.5) is conservative if and only if f is radial.*

Proof. If $\mathbf{F}$ is conservative with potential U, then

$$\nabla U = \rho f(P) \nabla \rho \quad \text{where} \quad \rho = \|P - O\|. \qquad \blacksquare$$

4 The Two-Body Problem

Assimilate the Sun to a point mass $\{O; m_o\}$ and Earth to a point mass $\{P; m\}$. With respect to an inertial system Σ, Earth is acted upon by the gravitational force $\mathbf{F}$ given by (3.3), which imparts to it an acceleration $\mathbf{a}_\Sigma(P)$. By Newton's third law, the Sun is acted upon by the force $-\mathbf{F}$, which imparts to it an acceleration $\mathbf{a}_\Sigma(O)$. By Newton's first law,

$$\mathbf{a}_\Sigma(P) = \frac{\mathbf{F}}{m} \quad \text{and} \quad \mathbf{a}_\Sigma(O) = -\frac{\mathbf{F}}{m_o}.$$

Choose a triad S with origin in the Sun and axes kept at all times parallel to those of the inertial triad Σ so that S translates with respect to Σ with some velocity $\mathbf{v}(O)$. Since $\mathbf{a}_\Sigma(O) \neq 0$, the translation velocity $\mathbf{v}(O)$ is not constant, e.g., the motion of S with respect to Σ is not a uniform straight-line motion. Therefore S is not inertial.

The *two-body problem* consists in describing the motion of Earth with respect to the Sun, i.e., the motion of $\{P; m\}$ with respect to the triad S. By Coriolis's theorem, since $\boldsymbol{\omega} = 0$,

$$\mathbf{a}_S(P) = \mathbf{a}_\Sigma(P) - \mathbf{a}_\Sigma(O) = \frac{m + m_o}{m\, m_o} \mathbf{F}.$$

Therefore

$$m_S \mathbf{a}_S(P) = \mathbf{F}, \quad \text{where} \quad m_S = \frac{m\, m_o}{m + m_o} \quad \text{is the } \textit{reduced mass}. \tag{4.1}$$

Proposition 4.1 *Earth moves with respect to the Sun as a point mass with reduced mass m_S, acted upon by the gravitational force* $\mathbf{F}$ *in (3.3), as if S were inertial.*

The same arguments continue to hold for the motion of any planet about the Sun, regarded as an isolated system. More generally, one might consider two material bodies, assimilated to point masses $\{O; m_o\}$ and $\{P; m\}$ and ask to describe the motion of one relative to the other. The terminology *two-body problem* originates from such a more general setting.

4.1 Gravitational Trajectories

The first equality of (4.1) with the gravitational force $\mathbf{F}$ given by (3.3) takes the form

$$m\mathbf{a}_S(P) = -\gamma \frac{m(m+m_o)}{\|P-O\|^2}\frac{P-O}{\|P-O\|}. \tag{4.2}$$

It follows that the motion is central and thus planar. Setting $\|P-O\| = \rho$, the second of the Binet formulas in (2.5) of Chapter 1 implies

$$\frac{d^2}{d\varphi^2}\frac{1}{\rho} + \frac{1}{\rho} = \frac{\gamma(m+m_o)}{a_o^2},$$

where a_o is the area constant and the angle φ is measured from a fixed direction in S to $P-O$. The general integral of this differential equation with respect to the variable $1/\rho$ is

$$\frac{1}{\rho} = A\sin(\varphi+\alpha) + \gamma\frac{m+m_o}{a_o^2},$$

where A and α are arbitrary constants. Choosing $\alpha = \frac{1}{2}\pi$ and setting

$$\rho_o = \frac{a_o^2}{\gamma(m+m_o)}, \qquad e = A\rho_o, \tag{4.3}$$

gives the polar equation of the trajectory in the form

$$\rho = \frac{\rho_o}{1+e\cos\varphi}. \tag{4.4}$$

This is the polar equation of a *conic* with one of its *focii* in O, parameter ρ_o, and eccentricity e. If $e<1$, the conic is an ellipse; if $e=1$, it is a parabola; and if $e>1$, it is a hyperbola.[12]

[12] On a plane π fix a line ℓ and a point O not in ℓ. A conic is the geometric locus of all points in π such that the ratio of their distance to O and to ℓ is constant. The constant value of such ratios is denoted by e and is called the *eccentricity* of the conic. The point O is a *focus* and the line ℓ is the *directrix*. The elliptic, parabolic, or hyperbolic nature of these orbits was observed by Newton in [123, Liber I, De Motu Corporum §66, page 134].

The force on the right-hand side of (4.2) is conservative and has potential

$$U = \gamma \frac{m(m + m_o)}{\rho}.$$

Therefore by the energy integral,

$$\frac{1}{2} m \mathbf{v}_S^2 - \gamma \frac{m(m + m_o)}{\rho} = E, \tag{4.5}$$

where E is the total energy of the system, which remains constant along an arbitrary but fixed trajectory. From the first of Binet's formulas (2.5) of Chapter 1, we compute

$$\mathbf{v}_S^2 = a_o^2 \left(\frac{d}{d\varphi} \frac{1}{\rho} \right)^2 + \frac{a_o^2}{\rho^2}.$$

Combining this with (4.3)–(4.5) gives

$$E = a_o^2 m \frac{e^2 - 1}{2\rho_o^2}.$$

It follows that if the total energy of the system is negative, then the trajectory is an ellipse. In the limiting case $e = 0$, the trajectory is a circle of radius ρ_o, as expressed by (4.4). From the second equality of (4.3) it follows that this occurs only if $A = 0$. In such a case,

$$E = -\frac{a_o^2 \, m}{2 \, \rho_o^2} = -\frac{1}{2} m \mathbf{v}_S^2,$$

so that the total energy equals the kinetic energy with opposite sign.

The cases $E \geq 0$ are characterized similarly.

5 Newton's and Kepler's Laws and Inertial Systems

The discussion of the previous section assumes that the gravitational force $\mathbf{F}$ has the form (3.3) stipulated by Newton. The conclusion is that the motion of the planets about the Sun is planar and their trajectories are ellipses.[13] Moreover, the areolar velocity is constant and the vector radius $P - O$ sweeps equal areas in equal times. These conclusions are precisely the first two laws of Kepler. Let now a and b be the major and minor semiaxes of a planetary orbit, and let T be the corresponding period of revolution. From the definition of the area constant and the geometric properties of ellipses,

$$a_o = 2\pi \frac{ab}{T}; \qquad \rho_o = \frac{b^2}{a}.$$

[13] Theoretically they could be ellipses, parabolas, or hyperbolas. Astronomical observations confirm that they are ellipses. Parabolic and hyperbolic orbits are observed in the motion of comets [49, 28–104], [50, 105–251], [52].

Therefore

$$\frac{a^3}{T^2} = \frac{\gamma(m + m_o)}{4\pi^2}.$$

The same formula holds for any other planet of mass m' orbiting along an ellipse of major semiaxis a' with period T', e.g.,

$$\frac{a'^3}{T'^2} = \frac{\gamma(m' + m_o)}{4\pi^2} = \frac{m' + m_o}{m + m_o}\frac{\gamma(m + m_o)}{4\pi^2} = \left(1 + \frac{m' - m}{m + m_o}\right)\frac{a^3}{T^2}.$$

Since the mass of a planet is negligible with respect to the mass of the Sun,[14]

$$1 + \frac{m' - m}{m + m_o} \approx 1 \qquad \Longrightarrow \qquad \frac{a'^3}{T'^2} \approx \frac{a^3}{T^2}.$$

This is Kepler's third law. Thus Newton's gravitational law validates, although approximately, Kepler's third law. Conversely, Kepler's laws validate Newton's gravitational law.

Having then accepted these laws one as a mutual validation of the other, it follows that the plane of the motion of a planet is fixed with respect to the inertial system Σ, whose existence has been postulated. In this sense the orbital planes of the planets are inertial.

It must be stressed, however, that these conclusions follow from having assumed the system Sun–planet to be isolated. In reality, the gravitational contribution of the other celestial bodies is nonzero, and as a consequence, the orbital planes are only approximately inertial.[15]

In what follows, in describing mechanical phenomena on Earth, we will assume that the orbital plane of Earth is inertial, within the indicated approximations. To be specific, we will take to coincide with the coordinate plane of $\{\mathbf{e}_1, \mathbf{e}_2\}$ of the inertial system $\Sigma = \{\Omega; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Since the mass of Earth is negligible with respect to the mass of the Sun, from the definition of reduced mass in (4.1)[16] we obtain

$$m_S = m\left(\frac{m_o}{m + m_o}\right) = m\left(1 - \frac{m}{m + m_o}\right) \approx m.$$

[14]Let m_E be the mass of Earth. The mass m_J of Jupiter, the planet of largest mass in the solar system, is $m_J = 318\, m_E$. The mass m_M of Mercury, the planet of smallest mass in the solar system, is $m_M = .05\, m_E$. The mass of the Sun is $m_o = 331950\, m_E$. For these values,

$$\frac{m_J - m_M}{m_M + m_o} \approx 0.958 \cdot 10^{-3}.$$

The numerical data are taken from [76, F–145 and F–165].

[15]See for example the formulation of the n-body problem in §4.5c of the Complements. Newton was well aware of such a mutual gravitational interaction ... *Coelos nostros infra coelos fixarum in orbem revolvi volunt, et planetas secum deferre; singuale coelorum partes et planetae qui relative quidem in coelis suis proximis quiescunt, moventur vere...*; Newton [123], §11, page 52.

[16]From the previous numerical data, $m_S = m\, 0.99999699$.

This, along with Proposition 4.1, is a further validation that the system centered in the Sun and congruent to $\Sigma = \{\Omega; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ may be assumed to be inertial with respect to gravitational phenomena occurring on Earth.[17]

6 Dynamics of a Point Mass Subject to Gravity [133]

Let $\Sigma = \{\Omega; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an inertial triad with origin in the Sun and such that the plane of $\{\mathbf{e}_1, \mathbf{e}_2\}$ coincides with the orbital plane of Earth. We also assume, still approximately, that the axis of rotation of Earth is normal to its orbital plane and therefore is directed as $\mathbf{e}_3$. Let also $S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a triad fixed with Earth, with origin on its center O, and with $\mathbf{u}_3 = \mathbf{e_3}$ oriented from south to north. The triad S is in rigid rototranslation with respect to Σ with characteristics $\boldsymbol{\omega} = \omega \mathbf{u}_3$ and $\mathbf{v}_\Sigma(O)$, which we assume given.[18] Assume also that the acceleration $\mathbf{a}_\Sigma(O)$ of the center of Earth with respect to Σ is negligible.[19] Therefore the transport acceleration $\mathbf{a}_T(P)$ of a point P transported by the rigid motion of S is (formula (8.5) of Chapter 1)

$$\begin{aligned}\mathbf{a}_T(P) &= \mathbf{a}_\Sigma(O) + \dot{\boldsymbol{\omega}} \wedge (P - O) + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge (P - O)) \\ &= \boldsymbol{\omega} \wedge [\boldsymbol{\omega} \wedge (P - O)] = -\omega^2 (P - Q),\end{aligned}$$

[17] Newton elaborates on the approximate nature of such an inertial system and on the conceptual difficulty of identifying an inertial system other than as mathematical postulate: *. . . Motus quidem veros corporum singulorum cognoscere et ab apparentibus actu discriminare difficillium est; propterea quod partes spatii illius immobilis in quo corpora vere moventur, non incurrunt in sensus*, [123, §11, page 52]. Despite Newton's attempt to ground mechanics on a purely rationalistic basis, mechanical phenomena are based on observations that are true only within some order of approximation. This interplay between rational mechanics and experimental mechanics was clear in Galileo's thinking: *. . . Prendiamo per ora questo come postulato, la verità assoluta del quale ci verrà poi stabilita dal vedere altre conclusioni, fabbricate sopra questa ipotesi, rispondere e puntualmente confrontarsi con l'esperienza. . .* [61].

[18] The average speed of the center of Earth about the Sun is 29.8 km/s, or $2.98 \cdot 10^6$ cm/s [76, F–145]. Therefore $\|\mathbf{v}_\Sigma(O)\| \approx 3 \cdot 10^6$ cm/s. The direction of $\mathbf{v}_\Sigma(O)$ is determined by the trajectory of O according to Kepler's first law. Since Earth completes a self-revolution about $\mathbf{u}_3$ in one *sidereal day*, e.g., 86,164 s [76, F–103–105; F–146], one computes $\omega = 2\pi/86{,}164\,\text{s}^{-1} = 7.292 \cdot 10^{-5}\,\text{s}^{-1}$.

[19] This is the centripetal acceleration of O directed as $\Omega - O$. Let R denote the average distance from Earth to the Sun. Its numerical value is $R = 149{,}5 \cdot 10^6$ km, or $1{,}495 \cdot 10^{13}$ cm [76, F–145]. Therefore, assuming that the trajectory is approximately circular, by (2.6) of Chapter 1, $\|\mathbf{a}_\Sigma(O)\| = \|\mathbf{v}_\Sigma(O)\|^2/R = 0.6\,\text{cm/s}^2$. This value is less than one-thousandth of the mean acceleration of gravity $g = 9.83225\,\text{m/s}^2$ [76, F–147–148].

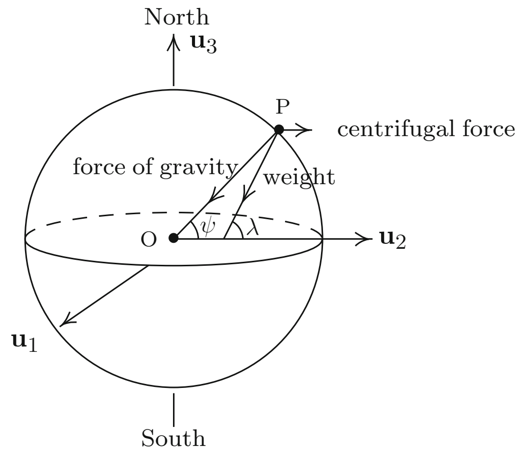

Fig. 6.1.

where Q is the projection of P on the coordinate axis of $\mathbf{u}_3$. It follows, by Coriolis's theorem, that the acceleration of P relative to S is

$$\mathbf{a}_S(P) = \mathbf{a}_\Sigma(P) + \omega^2(P - Q) - 2\boldsymbol{\omega} \wedge \mathbf{v}_S(P).$$

If $\{P; m\}$ is a moving point mass, multiplying this expression by m and taking into account (1.2) gives

$$m\mathbf{a}_S(P) = \mathbf{F}_\Sigma + m\omega^2(P - Q) - 2m\boldsymbol{\omega} \wedge \mathbf{v}_S(P). \tag{6.1}$$

6.1 On the Notion of Weight, Vertical Axis, and Gravity

The point mass $\{P; m\}$ moves by gravity if the only force $\mathbf{F}_\Sigma$ detected by the inertial system Σ is the gravitational force (3.3), where m_o is the mass of Earth regarded as concentrated in its center O. Therefore

$$m\mathbf{a}_S(P) = -\frac{\gamma m m_o}{\|P - O\|^2} \frac{P - O}{\|P - O\|} + m\omega^2(P - Q) - 2m\boldsymbol{\omega} \wedge \mathbf{v}_S(P). \tag{6.2}$$

Weight is measured at rest, e.g., with the point mass $\{P; m\}$ on the surface of Earth and $\mathbf{v}_S(P) = 0$. On $\{P; m\}$ act the gravitational force (3.3) and the centrifugal force $m\omega^2(P - Q)$. What one measures as weight is

$$-mg\,\mathbf{k} = -\gamma \frac{m m_o}{\|P - O\|^2} \frac{P - O}{\|P - O\|} + m\omega^2(P - Q). \tag{6.3}$$

The unit vector $\mathbf{k}$, referred to as *vertical*, does not coincide with the unit vector normal to the surface of Earth, unless P is at one of the poles or at the equator. Assume, without loss of generality, that P is in the plane $\{\mathbf{u}_2, \mathbf{u}_3\}$ as in **Figure 6.1**. The angle ψ formed by $P - O$ with the horizontal unit vector $\mathbf{u}_2$ is called the *geocentric latitude*, whereas the angle λ formed by the vertical unit vector $\mathbf{k}$ with $\mathbf{u}_2$ is the *astronomical latitude*.

If one were to take into account the transport acceleration $\mathbf{a}_\Sigma(O)$ of the center of Earth with respect to the Sun, this would have to be added to the right-hand side of (6.3). In such a case the unit vector $\mathbf{k}$ would not lie in the plane $\{\mathbf{u}_2, \mathbf{u}_3\}$. We estimate the deflection of $\mathbf{k}$ from such a plane and conclude that it is negligible.

6.2 Gravitational Motion near the Surface of Earth

To describe the motion of $\{P; m\}$ with respect to an observer on the surface of Earth, choose a triad $S_o = \{P_o; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, with origin at the initial position of P, with $\mathbf{k}$ vertical as in (6.3), the unit vector $\mathbf{j}$ from west to east and $\mathbf{i}$ chosen so that S_o is positive. Measures of weight leading to (6.3) are carried

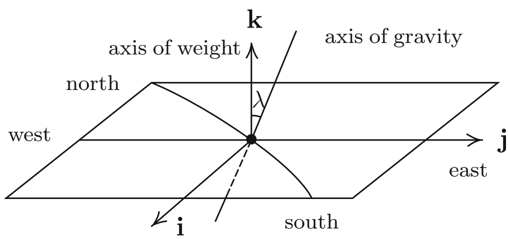

Fig. 6.2.

on a fixed position P_o on the surface of Earth. If P departs from P_o, vertically or longitudinally, both the constant g and the vertical unit vector $\mathbf{k}$ change. We will assume that the motion of $\{P; m\}$ takes place in a sufficiently small region about P_o where both g and $\mathbf{k}$ might be taken as constants. This occurs, for example, if along the motion, $\|P - P_o\|$ is negligible with respect to the radius of Earth.[20] From (6.1)–(6.3),

$$\mathbf{a}_S(P) = -g\mathbf{k} - 2\omega \mathbf{u}_3 \wedge \mathbf{v}_S(P),$$

and by integration,

$$\mathbf{v}_S(P) = -gt\mathbf{k} - 2\omega \mathbf{u}_3 \wedge (P - P_o) + \mathbf{v}_o,$$

where $\mathbf{v}_o$ is the initial velocity of P. Denote by (x, y, z) the coordinates of P in S_o. Moreover, $\mathbf{u}_3 = -\cos\lambda \mathbf{i} + \sin\lambda \mathbf{k}$. Therefore the previous expression of $\mathbf{v}_S(P)$ written in the coordinates of S_o generates the system of first-order differential equations

[20] For a range of values of g from the equator to the poles in terms of latitude, as well as in terms of altitude from sea level, see [76, F–133, F–151, F–158].

$$\begin{aligned} \dot{x} &= \ \ 2\omega y \sin\lambda + \dot{x}_o, \\ \dot{y} &= -2\omega\left(z\cos\lambda + x\sin\lambda\right) + \dot{y}_o, \\ \dot{z} &= -gt + 2\omega y\cos\lambda + \dot{z}_o, \end{aligned} \tag{6.4}$$

with initial conditions $P(0) = P_o$. Taking the derivative of the second equation and putting, in the expression so obtained, $\dot{x}$ and $\dot{z}$ given by the first and third equations, we obtain

$$\ddot{y} = -4\omega^2 y + At - B, \qquad \text{where} \qquad \begin{cases} A = 2\omega g\cos\lambda, \\ B = 2\omega\left(\dot{z}_o\cos\lambda + \dot{x}_o\sin\lambda\right), \end{cases}$$

whose general integral is

$$y = C\sin\left(2\omega t + \alpha\right) + \frac{At - B}{4\omega^2}. \tag{6.5}$$

The first term is the general integral of the associated homogeneous equation, and it depends on two arbitrary parameters C and α. In the case of freefall from rest, $\mathbf{v}_o = 0$ and $B = \alpha = 0$ and $C = -A/8\omega^3$. Therefore

$$y = \frac{g\cos\lambda}{4\omega^2}\left[2\omega t - \sin\left(2\omega t\right)\right]. \tag{6.6}$$

Put now $y(t)$ in the first and third equations of (6.4) and integrate. This gives

$$x = \frac{g\sin 2\lambda}{4\omega^2}\left[\omega^2 t^2 - \sin^2\omega t\right], \tag{6.7}$$

$$z = -\frac{1}{2}gt^2 + \frac{g\cos^2\lambda}{2\omega^2}\left[\omega^2 t^2 - \sin^2\omega t\right]. \tag{6.8}$$

If in (6.6)–(6.8) we let $\omega \to 0$, we recover the classical laws of freefall of a material body. If $\omega \neq 0$, then (6.6) detects an eastward deflection that is zero at the poles ($\lambda = \pm\pi/2$) and largest at the equator. The $x(t)$ in (6.7) is a deflection along a meridian. Such a deflection is toward the south in the northern hemisphere ($\lambda > 0$) and toward the north in the southern hemisphere. An estimate of such deflections is in §6c of the Complements.

7 Motion of a Constrained Point Mass

If $\{P; m\}$ is subject to a single constraint $[f(P;t) = 0]$, then (2.1) are augmented by the equation of the constraint and become

$$\begin{cases} m\ddot{P} = \mathbf{F}(P, \dot{P}; t) + \mathbf{R}(P, \dot{P}; t), \\ f(P; t) = 0, \\ P(t_o) = P_o, \ \dot{P}(t_o) = \dot{P}_o, \end{cases} \tag{7.1}$$

where P_o and $\dot{P}_o$ are given vectors. The force $\mathbf{R}$ is the reaction due to the constraint. The level sets $[f(\cdot;t) = 0]$ are smooth surfaces in $\mathbb{R}^3$, which might be regarded as moving following the parameter t. At each instant t the point P lies on one of them. If f is independent of time, the constraint is fixed, and P moves on the fixed surface $[f = 0]$.

If P is subject to a double constraint, the equations of motions are

$$\begin{cases} m\ddot{P} = \mathbf{F}(P, \dot{P}; t) + \mathbf{R}(P, \dot{P}; t), \\ f_j(P; t) = 0, \quad j = 1, 2, \\ P(t_o) = P_o, \ \dot{P}(t_o) = \dot{P}_o. \end{cases} \tag{7.2}$$

The point $P(t)$ lies at the intersection of the level sets $[f_j(\cdot;t) = 0]$, $j = 1, 2$. If both functions f_j are independent of t, e.g., if the constraints are fixed, their intersection γ is the trajectory of P. The reaction $\mathbf{R}$, while not a priori

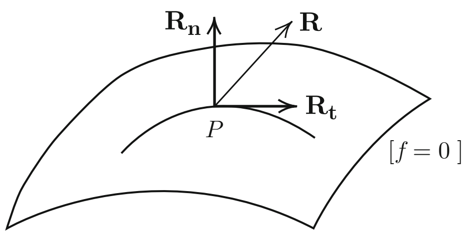

Fig. 7.1.

known, must be included in the account of all external forces, since it arises from enforcing the constraints, which are external to $\{P; m\}$. The system (7.1) consists of four scalar equations, whereas the system (7.2) consists of five scalar equations. In either case the unknowns are the three scalar functions making up $t \to P(t)$ and the three components of $\mathbf{R}$. Therefore the problem of motion of a constrained point mass is underdetermined, and its solvability hinges upon the availability of further information on the nature of the constraints.

7.1 Smooth Constraints and Relative Energy

Assume first that $\{P; m\}$ is subject to a single constraint $[f(\cdot;t) = 0]$. Such a constraint is *smooth* or *frictionless* if at each time t, it only generates reactions normal to the surface $[f(\cdot;t) = 0]$, equivalently if there exists a function $t \to \lambda(t) \in \mathbb{R}$ such that

$$\mathbf{R}(P, \dot{P}; t) = \lambda(t)\nabla f(P; t).$$

In such a case (7.1) becomes

$$\begin{cases} m\ddot{P} = \mathbf{F}(P, \dot{P}; t) + \lambda\nabla f(P; t), \\ f(P; t) = 0, \\ P(t_o) = P_o, \ \dot{P}(t_o) = \dot{P}_o. \end{cases} \tag{7.3}$$

This is a system of four equations in the four scalar unknown functions of time $\{P, \lambda\}$ and therefore, at least in principle, well posed. For fixed constraint, the fixed surface $[f = 0]$ is *smooth* or *frictionless* if it does not oppose the sliding of P on it, and offers a reaction only to motions that would let P leave the surface. Let δP be an elemental virtual displacement and denote by $\delta L_{\mathbf{R}}$ the elemental virtual work done by $\mathbf{R}$ for such a virtual displacement. Then

$$\delta L_{\mathbf{R}} = \mathbf{R} \cdot \delta P = \lambda(t) \nabla f(P; t) \cdot \delta P = 0.$$

Therefore the virtual work done by the reaction offered by the frictionless constraint $[f(\cdot; t) = 0]$ is zero. For an actual displacement dP, one has $df = 0$ and $\nabla f \cdot dP + f_t dt = 0$. Therefore the reaction due to the constraint does the actual elemental work

$$dL_{\mathbf{R}} = \mathbf{R} \cdot dP = -\lambda(t) f_t(P; t) \cdot dt.$$

Multiplying the first equation of (7.3) by $\dot{P}$ gives

$$\dot{T}(t) = \mathbf{F} \cdot \dot{P} - \lambda(t) f_t,$$

or in differential form,

$$d(T - L_{\mathbf{F}}) = -\lambda f_t dt,$$

where $dL_{\mathbf{F}}$ is the elemental work done by the external forces $\mathbf{F}$ applied to $\{P; m\}$. This relation shows how variations of kinetic energy are affected by the motion of the constraint. For a fixed constraint one has $d\,(T - L_{\mathbf{F}}) = 0$. A double constraint $[f_j(\cdot; t) = 0]$, $j = 1, 2$, is *smooth* or *frictionless* if there

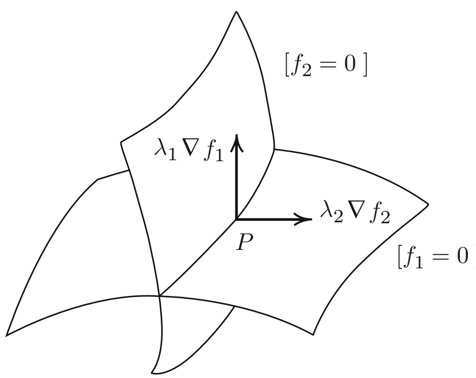

Fig. 7.2.

exist two scalar functions $t \to \lambda_j(t)$ for $j = 1, 2$ such that

$$\mathbf{R}(P, \dot{P}; t) = \lambda_j(t) \nabla f_j(P; t).$$

In such a case (7.2) takes the form

$$\begin{cases} m\ddot{P} = \mathbf{F}(P,\dot{P};t) + \lambda_j \nabla f_j(P;t), \\ f_j(P;t) = 0, \quad j = 1,2, \\ P(t_o) = P_o,\ \dot{P}(t_o) = \dot{P}_o. \end{cases} \tag{7.4}$$

This is a system of five scalar equations in the five scalar unknown functions $\{P, \lambda_1, \lambda_2\}$, which, at least in principle, is well posed. For fixed constraints, the trajectory of P is determined by the constraints. Such a trajectory is *smooth* or *frictionless* if it does not oppose the sliding of P on it and resists only motions that would let P abandon its trajectory. By energetic considerations analogous to that for single constraints,

$$\begin{aligned} \delta L_{\mathbf{R}} &= \mathbf{R} \cdot \delta P = 0, \\ dL_{\mathbf{R}} &= \mathbf{R} \cdot dP = -\lambda_j f_{j,t} dt, \\ d\,(T - L_{\mathbf{F}}) &= -\lambda_j f_{j,t} dt. \end{aligned}$$

For fixed constraints, the last of these reduces to **(iii)** of Theorem 3.1.

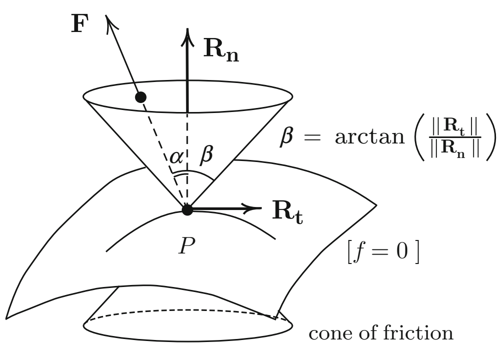

Fig. 7.3.

7.2 Rough Constraints and Relative Energy

A single constraint $[f = 0]$ acting on $\{P; m\}$ generates a reaction

$$\mathbf{R} = \mathbf{R_t} + \mathbf{R_n}, \qquad \mathbf{t} = \frac{\dot{P}}{\|\dot{P}\|}, \qquad \mathbf{n} = \frac{\nabla f}{\|\nabla f\|},$$

where $\mathbf{R_t}$ and $\mathbf{R_n}$ denote the components of $\mathbf{R}$ along $\mathbf{t}$ and $\mathbf{n}$. The constraint is *rough* if there exists a positive constant $\gamma > 0$ such that

$$\|\mathbf{R_t}\| = \gamma \|\mathbf{R_n}\|. \tag{7.5}$$

The constant γ, called the *dynamic friction coefficient*, depends on the nature of the contact and is determined experimentally. In the case of smooth constraints information was provided on the components of the reaction, normal, and tangent to the constraint. Formula (7.5), called *Coulomb's law*, has the same role for rough constraints. Similar considerations hold in the case of rough double constraints, modulo the obvious changes in the meaning of $\mathbf{t}$ and $\mathbf{n}$. The mechanical problems (7.1)–(7.2) augmented by (7.5) are well posed.

The component $\mathbf{R_t}$ opposes the motion of P, so that $\mathbf{R} \cdot dP \le 0$. This in turn implies $d(T - L_\mathbf{F}) \le 0$. Therefore rough constraints dissipate energy.

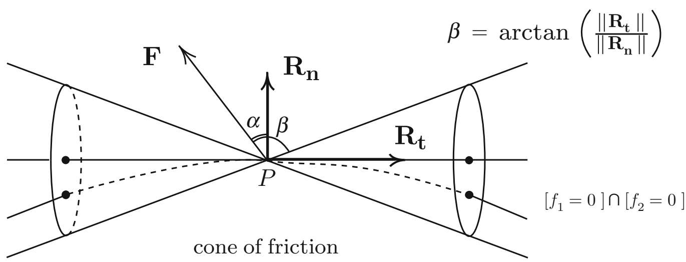

Fig. 7.4.

7.3 Remarks on Fixed Constraints

Assume that $\{P; m\}$ is subject to a single fixed constraint $[f = 0]$, smooth or rough, and rewrite (7.1) in the components

$$\begin{aligned} m\mathbf{a_n} &= \mathbf{F_n}(P, \dot{P}; t) + \mathbf{R_n}, \\ m\mathbf{a_t} &= \mathbf{F_t}(P, \dot{P}; t) + \mathbf{R_t}. \end{aligned} \tag{7.6}$$

By the constraint, $f(P) = 0$ along the motion, so that by differentiation, $\dot{P} \cdot \nabla f = 0$. Differentiating this once again gives

$$\mathbf{a} \cdot \nabla f = -\dot{P}^t \left(f_{x_i, x_j}\right) \dot{P}, \qquad \mathbf{a} = \ddot{P}.$$

Therefore if $\|\nabla f\| > 0$, as we have assumed, then

$$\mathbf{a_n} = \mathbf{a} \cdot \frac{\nabla f}{\|\nabla f\|} = -\frac{\dot{P}^t \left(f_{x_i, x_j}\right) \dot{P}}{\|\nabla f\|}.$$

Thus $\mathbf{a_n}$ can be expressed as an explicit function of P and $\dot{P}$. But then, by virtue of (7.6), also $\mathbf{R_n}$ can be expressed as an explicit function of P and $\dot{P}$. If a relation is known between $\mathbf{R_t}$ and $\mathbf{R_n}$, such as for example (7.5), also $\mathbf{R_t}$ can be expressed as an explicit function of P and $\dot{P}$. This implies that the right-hand side of the second equation of (7.6) is a known explicit function of P and $\dot{P}$, and therefore the integration can be effected.

In the case of a double fixed constraint, the geometric trajectory γ is known, but not its temporal parameterization. Parameterizing it by its arc length s and introducing its intrinsic triad $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, the system (7.2) can be rewritten as

$$\begin{aligned} m\ddot{s} &= \mathbf{F} \cdot \mathbf{t} + \mathbf{R}_{\mathbf{t}}, \\ m\kappa\dot{s}^2 &= \mathbf{F} \cdot \mathbf{n} + \mathbf{R}_{\mathbf{n}}, \\ 0 &= \mathbf{F} \cdot \mathbf{b} + \mathbf{R}_{\mathbf{b}}, \end{aligned} \tag{7.7}$$

with given $s(t_o) = s_o$ and $\dot{s}(t_o) = v_o$. If the constraint is smooth, then $\mathbf{R}_{\mathbf{t}} = \mathbf{R}_{\mathbf{b}} = 0$ and the first equation of (7.7) can be integrated to resolve the motion $t \to s(t)$. The remaining two equations permit one to compute the reaction $\lambda_j \nabla f_j$ in terms of the two functions $t \to \lambda_j(t)$, $j = 1, 2$.

If the constraints are rough, the second equation of (7.7) implies that $\mathbf{R}_{\mathbf{n}}$ is an explicit function of $(s, \dot{s}; t)$, and in view of (7.5), the same is true of $\mathbf{R}_{\mathbf{t}}$. Therefore the right-hand side of the first equation of (7.7) is an explicit function of $(s, \dot{s}; t)$. Thus the integration can be effected to resolve the motion $t \to s(t)$.

7.4 Remarks on the Case of F Conservative

For smooth fixed constraints, $\delta P = dP$ and the reactions are workless. An elemental variation of kinetic energy is balanced only by the elemental work $dL_{\mathbf{F}} = \mathbf{F} \cdot dP$ done by the the forces $\mathbf{F}$ applied to $\{P; m\}$. If $\mathbf{F}$ is conservative, then by the integral of the energy,

$$T - U(P) = T_o - U(P_o), \qquad \nabla U = \mathbf{F}.$$

As a particular case consider the motion of $\{P; m\}$ subject to gravity $-mg\mathbf{u}_3$, and constrained to move on a smooth and fixed surface or curve. Irrespective of the constraint, the integral of the energy can be given the form

$$\|\dot{P}\|^2 = 2g(a - x_3), \tag{7.8}$$

where a is the level at which P has zero velocity. The constraint might keep P from reaching such a level; however, (7.8) continues to hold.

8 The Mathematical Pendulum

A point mass $\{P; m\}$ is constrained to move on a vertical, smooth, fixed circumference $\mathcal{C} = \{x^2 + z^2 = \ell^2\}$, subject to gravity $-mg\mathbf{k}$. The point P is kept on $\mathcal{C}$ by a rigid, weightless rod of length ℓ, called the length of the pendulum, with one of its extremities hinged on the center of $\mathcal{C}$. The system has one degree of freedom, and as Lagrangian coordinate one might take the

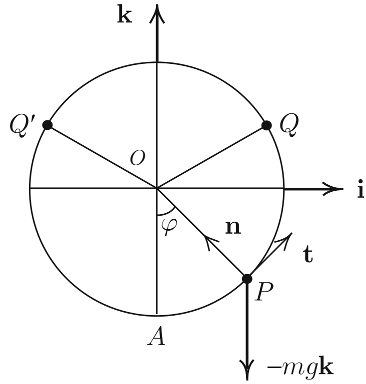

Fig. 8.1.

angle φ between $A - O$ and $P - O$ spanned counterclockwise starting from $A - O$. The trajectory is known and has intrinsic tangent and normal

$$\mathbf{t} = \cos\varphi \mathbf{i} + \sin\varphi \mathbf{k}, \qquad \mathbf{n} = -\sin\varphi \mathbf{i} + \cos\varphi \mathbf{k}.$$

Therefore

$$-mg\mathbf{k} = -mg(\cos\varphi\, \mathbf{n} + \sin\varphi\, \mathbf{t}),$$

and (7.2) written with respect to the intrinsic triad as in (7.7) becomes

$$m\ddot{s} = -mg\sin\varphi, \qquad m\frac{\dot{s}^2}{\ell} = -mg\cos\varphi + R, \tag{8.1}$$

where R is the reaction due to the constraint acting only along $\mathbf{n}$, since $\mathcal{C}$ is smooth. By the energy integral (7.8),

$$2mg\frac{(a-z)}{\ell} = -mg\cos\varphi + R.$$

From this one computes the reaction R in terms of the level z of P and the level a where $\dot{P} = 0$, e.g.,[21]

$$R = mg\frac{(2a-3z)}{\ell} \qquad (\text{since } z = -\ell\cos\varphi). \tag{8.2}$$

Putting this in the second equation of (8.1) gives $\dot{s}^2 = 2g(a-z)$. Therefore if $a > \ell$, the point P never stops and revolves indefinitely on $\mathcal{C}$. If $a < \ell$, the point P comes to a stop at some $Q \in \mathcal{C}$, at level $z_Q < \ell$, and then swings to $Q' \in \mathcal{C}$, which is symmetric to Q with respect to the coordinate vertical

[21] Such a level might or might not be attainable. See (7.8) and the related remarks.

axis. It then continues to perform periodic oscillations between Q and Q' with some period T. Assume $a < \ell$, set $s = \ell\varphi$ and $a = -\ell\cos\alpha$, and rewrite the energy integral in the form

$$\dot{\varphi}^2 = \frac{2g}{\ell}(\cos\varphi - \cos\alpha) = \frac{4g}{\ell}\left(\sin^2\tfrac{1}{2}\alpha - \sin^2\tfrac{1}{2}\varphi\right).$$

Separating the variables yields

$$2\sqrt{\frac{g}{\ell}}t = \int_0^{\varphi}\frac{d\theta}{\sqrt{\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta}}.$$

This is an implicit resolution of the motion $t \to \varphi(t)$ in terms of the Lagrangian parameter φ. Choosing the initial datum $\varphi(0) = 0$, the point P will reach Q, for $\varphi = \alpha$, after a time equal to one-fourth of the period of oscillations. Therefore

$$T = 2\sqrt{\frac{\ell}{g}}\int_0^{\alpha}\frac{d\theta}{\sqrt{\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta}}.$$

To compute this integral, introduce the change of variables

$$\eta = \frac{\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\alpha}, \qquad k = \sin\tfrac{1}{2}\alpha,$$
$$\sin^2\tfrac{1}{2}\alpha - \sin^2\tfrac{1}{2}\theta = k^2(1-\eta^2),$$
$$k\,d\eta = \tfrac{1}{2}\cos\tfrac{1}{2}\theta\, d\theta = \tfrac{1}{2}\sqrt{1-k^2\eta^2}\,d\theta.$$

Therefore the period T is computed from the *elliptic integral*

$$T = 4\sqrt{\frac{\ell}{g}}\int_0^1\frac{d\eta}{\sqrt{1-\eta^2}\sqrt{1-k^2\eta^2}}.$$

Expand the integrand in a Taylor series with respect to the parameter k and integrate term by term to get

$$T = 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \frac{1}{4}k^2 + \left(\frac{13}{24}\right)^2 k^4 + \cdots\right], \qquad k = \sin^2\tfrac{1}{2}\alpha.$$

For *small oscillations*, e.g., $\alpha \approx 0$, one recovers Galileo's approximate formula $T = 2\pi\sqrt{\ell/g}$. Such a formula can be arrived at directly by setting $\sin\varphi \approx \varphi$ in (8.1). The approximate equation of motion would then be

$$\ddot{\varphi} + \frac{g}{\ell}\varphi = 0, \qquad \text{which implies} \qquad \varphi = A\cos\left(\sqrt{\frac{g}{\ell}}t + B\right)$$

for two real constants A and B. This describes an oscillatory motion of period $T = 2\pi\sqrt{\ell/g}$.

Problems and Complements

3c General Theorems of Point-Mass Dynamics

3.1c Elastic Forces

Let $\{P; m\}$ be subject to (3.4) starting from the initial position $P_o \not\equiv O$ with initial velocity $\dot{P}_o$, which will be assumed to be parallel to $P_o - O$. The motion is central, and it takes place in the plane through O and normal $(P_o - O) \wedge \dot{P}_o$. On such a plane introduce Cartesian axes x_1, x_2, with origin at O and the x_1-axis oriented as $P_o - O$. Then (3.4) yields the system

$$\ddot{x}_j + \omega^2 x_j = 0, \quad j = 1, 2; \quad \omega^2 = \frac{k}{m}, \tag{3.1c}$$

whose general integral is

$$x_j = A_j \sin(\omega t + \alpha_j), \quad A_j,\, \alpha_j \in \mathbb{R}, \quad j = 1, 2.$$

Compute the constants A_j and α_j in terms of the initial data. The system admits the energy integral; moreover, the areolar velocity is constant. Therefore

$$\|\dot{P}\|^2 + \omega^2 \|P - O\|^2 = \text{const}_1, \quad \dot{P} \wedge (P - O) = \text{const}_2. \tag{3.2c}$$

Compute these constants in terms of the initial data and show that (3.1c) and (3.2c) are equivalent.

3.2c Point Mass Moving in a Fluid

The fluid opposes the motion of $\{P; m\}$ with a resistance $\mathbf{R} = -f(\|\dot{P}\|)\dot{P}$, where f is a smooth, nonnegative function whose form is determined from experiments. For sufficiently slow motions, $f(\|\dot{P}\|) = \text{const}$ (in the air $\|\dot{P}\| \le 2\,\text{m/s}$). In such a case the motion is said to be in *viscous regime*. If $\{P; m\}$ is assimilated to a ball of sufficiently small radius ρ, then

$$f(\|\dot{P}\|) = 6\pi\mu\rho \quad \text{for} \quad \|\dot{P}\| \ll 1 \quad \text{(viscous regime)},$$

where μ is the *kinematic viscosity* of the fluid.[22] For larger speeds, $f(\|\dot{P}\|)$ is proportional to $\|\dot{P}\|$ and the motion is said to be in *hydraulic regime* (in the air

[22] The *dynamic viscosity* is a measure of a resistance offered by a fluid when forced to change its shape. It is a sort of internal friction measured as the resistance elicited by two ideal parallel planes immersed in the fluid when forced into a mutual sliding motion. The unit of measure is the *poise*, after J.L.M. Poiseuille. It is measured in dyne/s per cm^2 and is the force distributed tangentially on a planar surface of $1\,\text{cm}^2$, needed to cause a variation of velocity of $1\,\text{cm/s}$ between two ideal parallel planes immersed in the fluid and separated by a distance of $1\,\text{cm}$. For water at $20°\text{C}$, the dynamic viscosity is 0.01002 poise. The *kinematic viscosity* is the ratio of the dynamic viscosity to the density of the fluid. The c.g.s. unit of kinematic viscosity is the *stoke*, after G.G. Stokes. Numerical values of dynamic and kinematic viscosity for several fluids are in [76, F–36–45.].

$2\,\text{m/s} < \|\dot{P}\| \leq 200\,\text{m/s}$). For a point mass assimilated to a ball of sufficiently small radius ρ,

$$f(\|\dot{P}\|) = 5\pi\mu\rho^2\|\dot{P}\| \qquad \text{(hydraulic regime)}.$$

3.3c Elastic Motions in Viscous Media

A point mass $\{P; m\}$, assimilated to a ball of radius $\rho \ll 1$, is attracted to a fixed point O by a spring of elastic constant k. Assume that $P_o - O$ is parallel to $\dot{P}_o$, so that the motion takes place along the line through O and direction $\dot{P}_o$. Assume, moreover, that $\|\dot{P}\| \ll 1$, so that the motion is in viscous regime. Denoting by x the coordinate variable along the trajectory of P, the only nontrivial equation of motion is

$$\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x = 0, \qquad \omega^2 = \frac{k}{m}, \qquad 2\varepsilon = 6\pi\frac{\mu\rho}{m}, \tag{3.3c}$$

whose general integral is

$$x = e^{-\varepsilon t}\left(C_1 e^{\beta t} + C_2 e^{-\beta t}\right), \qquad \beta = \sqrt{\varepsilon^2 - \omega^2},$$

where C_j, $j = 1, 2$, are real or complex arbitrary constants. If $\varepsilon^2 > \omega^2$, the general integral is

$$x = e^{-\varepsilon t}\left(A \sinh \beta t + B \cosh \beta t\right),$$

where A and B are real constants to be determined from the initial data. If, for example, $x(0) = 0$ and $\dot{x}(0) = 1$, one computes $B = 0$ and $A = 1/\beta$, and

$$x = \frac{1}{\beta}e^{-\varepsilon t}\sinh \beta t, \quad x_{\max} = \frac{2}{\omega}\left(\frac{\omega}{\varepsilon + \beta}\right)^{\varepsilon/\beta}, \quad \lim_{t\to\infty} x = 0.$$

If $\varepsilon^2 = \omega^2$, the general integral is

$$x = (At + B)\,e^{-\varepsilon t}.$$

If $x(0) = 0$ and $\dot{x}(0) = 1$, the solution increases from zero to its maximum value $1/(e\varepsilon)$, and goes to zero as $t \to \infty$. If $\varepsilon^2 < \omega^2$, the general integral is

$$x = Ae^{-\varepsilon t}\sin\left(\beta_* t + \alpha\right), \qquad \beta_* = \sqrt{\omega^2 - \varepsilon^2}.$$

These are damped oscillations ($\varepsilon > 0$), of period $T_* = 2\pi/\sqrt{\omega^2 - \varepsilon^2}$. Moreover,

$$x\left(t + nT_*\right) = e^{-n\varepsilon T_*}x(t), \qquad \forall\, n \in \mathbb{N},$$

so that the amplitude of the damped oscillations decays exponentially to zero. If $\varepsilon = 0$, the oscillations are free and have period $T = 2\pi/\omega$. Thus the presence of the fluid ($\varepsilon > 0$) lengthens the period of the free oscillations.

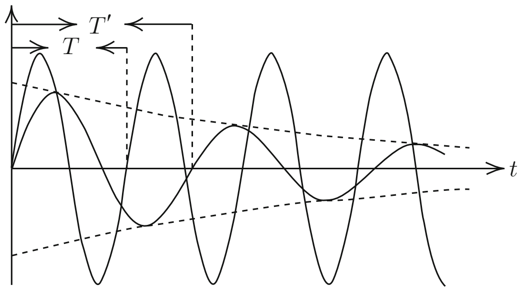

Fig. 3.1c.

3.4c Forced Oscillation

On the previous system impose an external forcing term, periodic with period $2\pi/\theta$, acting on the same line of the trajectory. Then (3.3c) is modified into

$$\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x = A\sin\theta t, \tag{3.4c}$$

where A and θ are given constants. The general integral of (3.4c) is given by the general integral $x_h(\cdot)$ of the associated homogeneous equation (3.3c), e.g., one of the previous cases, augmented by a particular integral $x_p(\cdot)$, which will be sought of the form

$$x_p = B\sin(\theta t - \gamma) = B(\sin\theta t\cos\gamma - \cos\theta t\sin\gamma).$$

Employing to this to solve (3.4c) yields

$$\tan\gamma = \frac{2\varepsilon\theta}{\omega^2 - \theta^2}, \qquad B(\omega^2 - \theta^2)\cos\gamma + 2B\varepsilon\theta\sin\gamma = A.$$

From these we compute

$$1 + \tan^2\gamma = \frac{\phi^2}{(\omega^2 - \theta^2)^2}, \qquad \phi = \sqrt{(\omega^2 - \theta^2)^2 + 4\varepsilon^2\theta^2},$$

$$\sin\gamma = \frac{2\varepsilon\theta}{\phi}, \quad \cos\gamma = \frac{\omega^2 - \theta^2}{\phi}, \quad B = \frac{A}{\phi}. \tag{3.5c}$$

The general integral of (3.4c) is then $x = x_h + x_p$, which is interpreted as the superposition of the damped oscillations x_h and the forced harmonic oscillation x_p.

3.5c Phase Delay

The phase of the forced oscillation is $\theta t - \gamma$, whereas the phase of the external forcing term is θt. Therefore x_p exhibits a phase delay of γ with respect to the external forcing term, given by

$$\gamma(\theta) = \tan^{-1}\left(\frac{2\varepsilon\theta}{\omega^2 - \theta^2}\right).$$

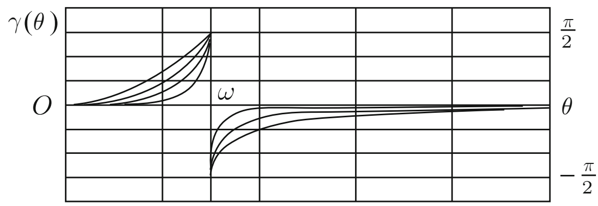

Fig. 3.2c.

If θ increases from zero to ω, the phase delay γ increases from zero to $\pi/2$. If θ increases from ω to $+\infty$, then γ increases from $\pi/2$ to π. If the forcing term $A \sin \theta t$ has the same frequency $\omega/2\pi$ of the free oscillations, then the phase delay is always $\pi/2$, e.g., the forced vibrations are in *quadrature of phase* with respect to the external forcing term. If $\theta = 0$, the oscillations are in *concurrence of phase* with the forcing term.

3.6c Amplitude of Forced Oscillations

The amplitude B of the forced oscillations is given by the third equation of (3.5c). Its behavior with respect to θ hinges on the behavior of the function $\theta \to \phi(\theta)$. By taking the derivative of ϕ with respect to θ, we get

$$2\phi\phi' = 4\theta \left[\theta^2 - \left(\omega^2 - 2\varepsilon^2\right)\right].$$

Assume first $\omega^2 - 2\varepsilon^2 > 0$. If θ increases from zero to $\sqrt{\omega^2 - 2\varepsilon^2}$, then ϕ decreases, and the amplitude of the forced oscillations increases from A/ω^2 to its maximum

$$B_{\max} = \frac{A}{2\varepsilon\sqrt{\omega^2 - \varepsilon^2}}.$$

When such a maximum is reached, the external forcing term and the resulting forced oscillations are in *resonance*. When θ increases from $\sqrt{\omega^2 - 2\varepsilon^2}$ to ∞, then ϕ increases and the amplitude $B(\theta)$ decreases to zero. If $\omega^2 - 2\varepsilon^2 \le 0$, then $B(\theta)$ decreases for all values of θ, and tends to zero as $\theta \to \infty$.

Thus in an oscillating mechanical system, forced by an external vibration of frequency $\theta/2\pi$, for small forcing frequencies the forced vibrations reach a maximum for $\theta = \sqrt{\omega^2 - 2\varepsilon^2}$ and tend to disappear as $\theta \to \infty$.

4c The Two-Body Problem

4.1c Resolving Kepler's Motions

Write the energy integral (4.5) in polar coordinates ρ and φ:

$$\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right) - U = E, \qquad U = \gamma\frac{m(m + m_o)}{\rho}.$$

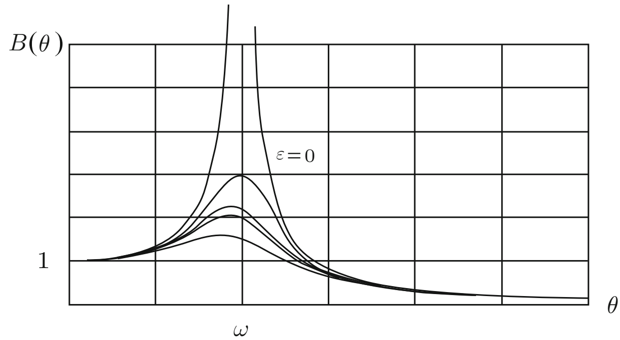

Fig. 3.3c.

Taking into account that $\rho^2\dot\varphi = a_o$, these can be rewritten as

$$\frac{1}{2}m\dot\rho^2 - U_{\rm eff} = E, \qquad U_{\rm eff} = U - \frac{1}{2}m\frac{a_o^2}{\rho^2}. \tag{4.1c}$$

The function $U_{\rm eff}$ is called *effective potential.* This relation provides a time resolution of Kepler's motion in the implicit form

$$t = \pm\sqrt{\frac{m}{2}}\int_{\rho_o}^{\rho}\frac{dr}{\sqrt{E+U_{\rm eff}(r)}}. \tag{4.2c}$$

4.2c Stable Circular Orbits

These relations may be interpreted as the energy integral of a mechanical system with one degree of freedom and subject to the potential $U_{\rm eff}$. The only equation of motion of such a system is

$$m\ddot\rho = \frac{d}{d\rho}U_{\rm eff} \tag{4.3c}$$

with some given initial data. Formula (4.2c) is the implicit time resolution of the motion generated by (4.3c). The energy E in (4.1c) depends on the initial data, e.g., $E_o = E(\rho_o, \dot\rho_o)$. In what follows, rather than prescribing ρ_o and $\dot\rho_o$, we will prescribe equivalently ρ_o and E_o. By (4.1c)–(4.3c) the orbit of P is circular if and only if ρ_o and E_o are solutions of

$$\frac{d}{d\rho}U_{\rm eff}\Big|_{\rho=\rho_o} = \left[-\frac{\gamma m(m+m_o)}{\rho^2} + \frac{ma_o^2}{\rho^3}\right]\Big|_{\rho=\rho_o} = 0,$$

$$E_o = -\left[\frac{\gamma m(m+m_o)}{\rho} - \frac{1}{2}m\frac{a_o^2}{\rho^2}\right]\Big|_{\rho=\rho_o}$$

e.g.,

$$\rho_o = \frac{a_o^2}{\gamma(m+m_o)}, \qquad E_o = \frac{1}{2}\, m\, \frac{a_o^2}{\rho_o^2}. \tag{4.4c}$$

In the trivial case that the area constant a_o is zero, the circular orbit degenerates to a point. If $a_o > 0$, then

$$\frac{d^2}{d\rho^2} U_{\text{eff}} \Big|_{\rho=\rho_o} = -\frac{\gamma}{\rho^3}(m+m_o) < 0,$$

so that U_{eff} has a maximum at ρ_o and the corresponding potential energy V_{eff} has a minimum at ρ_o. Physically, a point mass $\{P; m\}$ tends to take a configuration that minimizes its potential energy, e.g., small variations from that position tend, roughly speaking, to be damped so that P can resume the position of minimum potential energy. In this sense $\rho = \rho_o$ is a configuration of *stable equilibrium*.[23]

In the context of the systems Earth–Sun, if Earth were in the position ρ_o with energy E_o given by (4.4c), it would move along a stable, circular trajectory about the Sun. If $a_o = 0$, equation (4.3c), irrespective of the initial data, forces P to be attracted by O. Thus if Earth did not rotate with respect to the Sun (i.e., $a_o = 0$), then irrespective of its initial position ρ_o and its initial speed $\dot{\rho}_o$, it would ultimately fall into the Sun.

4.3c Radial Potentials

The analysis of the two-body problem is independent of the particular form of the gravitational potential. An energy integral similar to (4.5) would continue to hold starting from a potential that would ensure that the motion is central and thus planar. Potentials that generate central motions are radial (Proposition 3.1). Therefore (4.1c)–(4.2c) continue to hold for any smooth radial function $\rho \to U(\rho)$ defined in $(0, \infty)$. As an example consider the case

$$U = \gamma\rho^\alpha \quad \text{and} \quad U_{\text{eff}} = \gamma\rho^\alpha - \frac{1}{2} m \frac{a_o^2}{\rho^2},$$

where γ and α are given real constants. For $\gamma > 0$ and $\alpha = -1$ this is the gravitational potential; for $\gamma > 0$ and $\alpha = 2$ this is the elastic potential. Other values of γ and α occur in atomic potentials [14, Chap. IX, §67].

Denote by $\{P_o, m_o\}$ a fixed point and by $\{P; m\}$ a point mass in relative motion with respect to $\{P_o; m_o\}$. One might then ask whether the motion generated by (4.3c), with these potentials, admits circular orbits and whether such orbits, if any, are stable. One might also investigate whether $\{P; m\}$ will

[23] We are referring here to an intuitive notion of stability. A mathematical notion is in §1.1 of Chapter 8. By this notion, maxima for the potential correspond to configurations of stable equilibrium (Dirichlet stability criterion, §4 of Chapter 8).

ultimately fall onto $\{P_o; m_o\}$, or whether the two points move apart indefinitely. Circular orbits $\rho = \rho_o$ are possible if and only if the initial data ρ_o and E_o satisfy

$$\frac{d}{d\rho}U_{\text{eff}}\Big|_{\rho=\rho_o} = \frac{1}{\rho^3}\left(\gamma\alpha\rho^{\alpha+2} + ma_o^2\right)\Big|_{\rho=\rho_o} = 0,$$

$$E_o = -\left(\gamma\rho^{\alpha} - \frac{1}{2}m\frac{a_o^2}{\rho^2}\right)\Big|_{\rho=\rho_o}.$$

These orbits, if they exist, are stable if

$$\frac{d^2}{d\rho^2}U_{\text{eff}}\Big|_{\rho=\rho_o} = \frac{1}{\rho^4}\left[\gamma\alpha(\alpha-1)\rho^{\alpha+2} - 3ma_o^2\right]\Big|_{\rho=\rho_o} < 0.$$

These orbits are *unstable* if this condition is violated. That is, the sole absence of maximality for the potential suffices for one to conclude that the configuration is unstable (Corollary 6.1 of Chapter 8).

4.3.1c Circular Orbits

Prove that circular orbits are possible only if $\gamma\alpha \le 0$. Moreover, if $\gamma\alpha > 0$, then regardless of the initial data, $\|P - P_o\| \to \infty$ as $t \to \infty$. Finally, if $\gamma\alpha < 0$ and the area constant a_o is zero, then P falls onto P_o.

Assume next that $\gamma\alpha < 0$ and that $a_o > 0$. Prove that if $\alpha \neq -2$ then circular orbits are admissible and

$$\rho_o^{\alpha+2} = \frac{-ma_o^2}{\gamma\alpha}, \qquad E_o = \frac{ma_o^2}{\rho_o^2}\,\frac{\alpha+2}{2\alpha}. \tag{4.5c}$$

Moreover, for these values,

$$\frac{d^2}{d\rho^2}U_{\text{eff}}(\rho)\Big|_{\rho=\rho_o} = -\frac{ma_o^2}{\rho_o^4}(\alpha+2). \tag{4.6c}$$

Therefore, these possible circular trajectories are stable for $\alpha > -2$ and unstable for $\alpha < -2$.

4.3.2c The Case $-2 < \alpha < 0$

This includes the gravitational potentials for $\gamma > 0$ and $\alpha = -1$. Prove that regardless of initial data (ρ_*, E_*), P never falls onto P_o, e.g., there exists some $R_o > 0$, determined by (ρ_*, E_*), such that $\rho(t) \ge R_o$ for all $t \ge 0$. Moreover, if $E_* > 0$, then $\|P - P_o\| \to \infty$ as $t \to \infty$, whatever the initial radius ρ_*.

4.3.3c The Case $\alpha > 0$

This includes the elastic potentials for $\gamma < 0$ and $\alpha = 2$. Prove that regardless of initial data (ρ_*, E_*) however fixed, P never falls onto P_o, nor will it be arbitrarily far from it, e.g., there exist two positive numbers $R_o < R_1$, determined in terms of (ρ_*, E_*), such that $R_o \le \rho(t) \le R_1$, for all times.

4.3.4c The Case $\alpha < -2$

Let (ρ_o, E_o) be the initial data given by (4.5c)–(4.6c) characterizing possible circular orbits. Prove that if $E_* > E_o$, then regardless of the initial datum ρ_*, the point P will ultimately fall onto P_o. Moreover,

$$\begin{aligned} E_* < E_o \quad \text{and} \quad \rho_* > \rho_o \quad &\Longrightarrow \quad \lim_{t\to\infty} \|P(t) - P_o\| = \infty, \\ E_* < E_o \quad \text{and} \quad \rho_* < \rho_o \quad &\Longrightarrow \quad \lim_{t\to\infty} \|P(t) - P_o\| = 0. \end{aligned}$$

4.3.5c The Case $\alpha = -2$

Prove that if $2\gamma \neq m a_o^2$, circular orbits are not admissible. Moreover, if $2\gamma > m a_o^2$, then P ultimately falls onto P_o, and if $2\gamma < m a_o^2$, then $\|P - P_o\| \to \infty$ as $t \to \infty$. Finally, if $2\gamma = m a_o^2$, the only orbits that are solutions of (4.1c)–(4.2c) are circular.

4.4c Closed Orbits

Return now to (4.1c), where $U(\cdot)$ is any any smooth radial function defined in $(0, \infty)$. Regarding ρ as a function of φ and using the formal differentiation formula (2.4) of Chapter 1, (4.1c) can be rewritten as

$$\frac{1}{2} m \frac{a_o^2}{\rho^4} \left(\frac{d\rho}{d\varphi} \right)^2 = E + U_{\text{eff}}(\rho).$$

Integrating this by separation of variables gives the polar equation $\rho = \rho(\varphi)$ of the trajectory, in the implicit form

$$\varphi - \varphi_* = \pm a_o \sqrt{\frac{m}{2}} \int_{\rho_*}^{\rho} \frac{dr}{r^2 \sqrt{E + U_{\text{eff}}(r)}}.$$

Assume now that the trajectory is confined between two limiting circles centered at P_o with radii $0 < R_o < R_1$, for example as in the case $\alpha > 0$. Choosing $\varphi_* = 0$ as the angle for which $\rho(\varphi_*) = R_o$, the previous formula takes the form

$$\varphi = c_o \int_{R_o}^{\rho} \frac{dr}{r^2 \sqrt{E + U_{\text{eff}}(r)}}, \qquad c_o = a_o \sqrt{\frac{m}{2}}.$$

An orbit generated by (4.1c) for such a choice of initial data is *closed* if after φ having spanned the unit circle an integer number of times, the point P returns to its initial position. One then asks whether (4.1c) generates orbits that although not necessarily circular or elliptic, are closed.

Prove that closed orbits are possible if and only if there exist positive integers m and n such that

$$2\pi\, m \;=\; n\, c_o \int_{R_o}^{R_1} \frac{dr}{r^2\sqrt{E+U_{\text{eff}}(r)}}.$$

Such an occurrence is rather special and, in general, will depend on the nature of the potential $U(\rho)$ or the initial data (ρ_*, E_*), or both. The following theorem underscores the importance of the gravitational and elastic potentials.

Theorem 4.1c (Bertrand [9]). *The gravitational and elastic potentials are the only ones for which (4.1c) generates closed orbits for any choice of initial data (ρ_*, E_*).*

4.4.1c More on the Polar Equation of the Trajectory

The polar equation $\rho = \rho(\varphi)$ of the trajectory could be derived from (4.3c) by the formal differentiation formula (2.4) of Chapter 1. Thus

$$m\frac{a_o}{\rho^2}\frac{d}{d\varphi}\left(\frac{a_o}{\rho^2}\frac{d}{d\varphi}\rho\right) = \frac{d}{d\rho}U_{\text{eff}}(\rho).$$

For gravitational potentials this formula was derived by Clairaut in [30].

4.5c The n-Body Problem

Given n material points $\{P_i; m_i\}$, $i = 0, 1, \dots, n-1$, subject to their mutual gravitational attraction, one would like to describe their motion with respect to one of them, say for example $\{P_o; m_o\}$. Let Σ be an inertial system and let S be a triad centered at P_o whose axes, along the motion of $\{P_o; m_o\}$, are parallel to those of Σ. Thus S is in rigid motion with respect to Σ, with characteristics $\mathbf{v}_\Sigma(P_o)$ and $\boldsymbol{\omega} = 0$. It follows from Coriolis's theorem that

$$\mathbf{a}_S(P_i) = \mathbf{a}_\Sigma(P_i) - \mathbf{a}_\Sigma(P_o), \qquad i = 0, 1, \dots, n-1. \tag{4.7c}$$

Moreover, by (1.1), for all $i = 0, 1, \dots, n-1$,

$$m_i \mathbf{a}_\Sigma(P_i) = \sum_{\substack{j=0\\ j\neq i}}^{n-1} \gamma \frac{m_i m_j}{\|P_j - P_i\|^2}\,\frac{P_j - P_i}{\|P_j - P_i\|}.$$

For $i = 0$ this gives

$$m_o \mathbf{a}_\Sigma(P_o) = \sum_{j=1}^{n-1} \gamma \frac{m_o m_j}{\|P_j - P_o\|^2} \frac{P_j - P_o}{\|P_j - P_o\|}.$$

Multiplying the ith equation (4.7c) by m_i gives

$$\begin{aligned} m_i \mathbf{a}_S(P_i) &= \sum_{\substack{j=0 \\ j\neq i}}^{n-1} \gamma \frac{m_i m_j}{\|P_j - P_i\|^2} \frac{P_j - P_i}{\|P_j - P_i\|} - \sum_{j=1}^{n-1} \gamma \frac{m_i m_j}{\|P_j - P_o\|^2} \frac{P_j - P_o}{\|P_j - P_o\|} \\ &= -\gamma \frac{m_i(m_i + m_o)}{\|P_i - P_o\|^2} \frac{P_i - P_o}{\|P_i - P_o\|} \\ &\quad + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} \gamma m_i m_j \left(\frac{P_j - P_i}{\|P_j - P_i\|^3} - \frac{P_j - P_o}{\|P_j - P_o\|^3} \right). \end{aligned}$$

For $n = 1$ this reduces to the two-body problem. While the two-body problem is solvable, as indicated in §4, the n-body problem is, in general, still open (see [121]). A further discussion on the n-body problem is in §§8–12 of Chapter 6. The three-body problem, with the further assumption that the motion is *planar*, was solved by Lagrange and Euler (§§11–12 of Chapter 6).

6c Dynamics of a Point Mass Subject to Gravity [133]

Expand (6.6)–(6.8) in a Maclaurin series in t and discard the terms of order higher than four to obtain

$$\begin{aligned} x &= \tfrac{1}{12} g\omega^2 \sin 2\lambda\, t^4 + \cdots, \\ y &= \tfrac{1}{3} g\omega \cos\lambda\, t^3 + \cdots, \\ z &= -\tfrac{1}{2} g t^2 + \tfrac{1}{6} g\omega^2 \cos^2\lambda\, t^4 + \cdots. \end{aligned}$$

For $t \ll 1$, discarding the terms of order higher than three, this exhibits only an eastward deflection along $\mathbf{j}$, and the trajectory is

$$9gy^2 + \left(8\omega^2 \cos^2\lambda\right) z^3 = 0.$$

In the second series discard the terms of order higher than three and take $\lambda \approx 0.287\,\pi$ (astronomical latitude of Rome, Italy). Then $y(t) = 0.0172 t^3\,\text{cm/s}^3$. After a fall of 4 s, one computes an eastward deflection of ≈10 mm. Assume next that in (6.4), $\mathbf{v}_o \neq 0$ and prove that the vertical and meridian deflections are of the order of ω, and therefore not negligible with respect to the eastward deflection.

6.1c Motion of $\{P; m\}$ in the Air: Viscous Regime

To extract the effect of air resistance, discard the Coriolis force and assume that initial velocity is vertical, so that the motion takes place along the vertical axis, of unit ascending vector **k**. The point mass is assimilated to a ball of radius $\rho \ll 1$; it is also assumed that its speed is so low that the motion occurs in viscous regime, as defined in §3c of the Complements. This yields the single equation of motion and initial conditions

$$\ddot{z} + \beta\dot{z} + g = 0, \quad \beta = 6\pi\frac{\mu\rho}{m}; \qquad z(0) = 0, \quad \dot{z}(0) = v_o,$$

whose integral is

$$z = \frac{1}{\beta}\Big(v_o + \frac{g}{\beta}\Big)\left(1 - e^{-\beta t}\right) - \frac{g}{\beta}t.$$

Prove that as $\beta \to 0$ one recovers the classical Torricelli's laws for the free gravitational fall of a point mass. If $v_o = -g/\beta$, then $\{P; m\}$ falls in uniform straight-line motion. Moreover, regardless of v_o,

$$\lim_{t\to\infty} \dot{z} = \dot{z}_{\text{lim}} = -\frac{g}{\beta} \qquad \text{(limiting velocity)}.$$

This limiting velocity is reached with good approximation in finite time. From the values of g and $\dot{z}_{\text{lim}}$ one computes the kinematic viscosity μ.

6.2c Motion of $\{P; m\}$ in the Air: Hydraulic Regime

With the same framework as the viscous regime, the motion is driven by the initial value problem

$$m\ddot{z} + \lambda|\dot{z}|\dot{z} + mg = 0, \quad \lambda = 5\pi\mu\rho^2, \quad z(0) = 0, \quad \dot{z}(0) = v_o.$$

This mathematical problem is well posed, at least for small times. However, physically it is meaningful only for speeds that are sufficiently large to justify the hydraulic regime. Thus we will assume that $|v_o|$ is large enough so that at least for small times, the hydraulic regime is in force.

6.3c Upward Initial Velocity ($v_o > 0$)

For sufficiently small times that guarantee $\dot{z}(t) \geq 0$,

$$\ddot{z} = -g\Big(1 + \frac{\dot{z}^2}{a^2}\Big), \qquad a^2 = \frac{mg}{\lambda}.$$

From this we obtain

$$\dot{z} = a\Big(\frac{v_o - a\tan(gt/a)}{a - v_o\tan(gt/a)}\Big), \quad t \in \left(0, \frac{a}{g}\arctan\Big(\frac{v_o}{a}\Big)\right) = (0, \bar{t}\,).$$

Since $\ddot{z} < 0$, there exists a time $t' \in (0, \bar{t})$ such that $\dot{z}(t')$ is so small as to evoke the viscous regime. From such a time forward such a regime will be in force, and for $t \to \infty$ the point mass will have the same limiting velocity $\dot{z}_{\text{lim}}$ as before.

6.4c Downward Initial Velocity ($v_o < 0$)

The equation of motion becomes

$$\ddot{z} = -g\left(1 - \frac{\dot{z}^2}{a^2}\right), \qquad \dot{z}(0) = v_o < 0, \tag{6.1c}$$

so long as $\dot{z} \le 0$. If $v_o = -a$, the only solution is $\dot{z} = v_o$ and $\{P; m\}$ falls in uniform straight-line motion along the vertical. If $v_o \neq a$, then (6.1c) admits the implicit integral

$$t = -\frac{a}{2g} \ln\left(\frac{a + \dot{z}}{a + v_o}\,\frac{a - v_o}{a - \dot{z}}\right), \quad \text{provided} \quad \frac{a + \dot{z}}{a + v_o}\,\frac{a - v_o}{a - \dot{z}} > 0. \tag{6.2c}$$

Lemma 6.1c $\dot{z} \le 0$ *at all times.*

Proof. If $v_o < -a$, then $\ddot{z} > 0$ in some time interval $t \in (0, t_*)$ and $\dot{z}$ increases from $v_o < 0$ to $-a$. If $t_* < \infty$, then starting from t_*,

$$\ddot{z} = -g\left(1 - \frac{\dot{z}^2}{a^2}\right), \qquad \dot{z}(t_*) = -a, \qquad t \ge t_*.$$

This has the only solution $\dot{z} = -a$. If $v_o > -a$, then $\ddot{z} < 0$ in some time interval $t \in (0, t^*)$ and $\dot{z}$ decreases from $v_o < 0$ to $-a$. Starting from t^*,

$$\ddot{z} = -g\left(1 - \frac{\dot{z}^2}{a^2}\right), \qquad \dot{z}(t^*) = -a, \qquad t \ge t^*. \qquad \blacksquare$$

It follows from the lemma that $(a - v_o)/(a - \dot{z}) > 0$ at all times. Therefore the inequality in (6.2c) is verified if $(a + \dot{z})/(a + v_o) > 0$. If $v_o < -a$, then $(\dot{z}(t) + a) < 0$ for $t \in (0, t_*)$. In such a time interval the inequality of (6.2c) is verified, and the implicit integral can be rewritten in the explicit form

$$\frac{a + \dot{z}}{a - \dot{z}} = \frac{a + v_o}{a - v_o}\, e^{-2gt/a}, \qquad v_o < -a, \qquad t \in (0, t_*).$$

This shows that the value $\dot{z} = -a$ is approached asymptotically, and therefore $t_* = \infty$. In a similar manner one proves that if $v_o > -a$, then $t^* = \infty$.

7c The Mathematical Pendulum

7.1c The String Pendulum

The point mass $\{P; m\}$ is fixed at one of the extremities of an ideal, inextensible, massless string of length ℓ, whose other extremum is fixed at O. This is an example of unilateral constraint, since P is allowed to abandon the circumference $\mathcal{C}$. The string exerts a traction $R_{\mathbf{n}}$ directed toward O. Therefore for P not to abandon $\mathcal{C}$ we must have

$$R_{\mathbf{n}} \geq 0, \quad \text{i.e., from (8.2)}, \quad 2a \geq 3z.$$

If this holds, the pendulum never abandons $\mathcal{C}$, and it behaves as if it were held by a rigid rod. For this to hold, the level z of P, the level a, real or ideal, and the length ℓ must satisfy $a > -\ell$; otherwise, the pendulum would not move; $z < a$ by the energy integral; $-\ell \leq z \leq \ell$ by the equation of the constraint. Given such restrictions, the condition on the traction is verified in the following two cases:

$$-\ell < a \leq 0 \quad \text{oscillatory case}, \qquad 2a \geq 3\ell \quad \text{revolving case}.$$

Except for these two cases, there exists on $\mathcal{C}$ a point at level $0 < 3z \leq 2a < 3\ell$ where $R_{\mathbf{n}} = 0$. Starting from this point, $\{P; m\}$ abandons $\mathcal{C}$ and it describes a parabola. The equation of this parabola is found by observing at such an instant, taken as initial, one has

$$z(0) = \tfrac{2}{3}a, \quad v(0) = \sqrt{\tfrac{2}{3}ag} \neq 0 \quad \text{along the tangent to } \mathcal{C}.$$

Write down the equation of such a parabola and show that it is tangent to $\mathcal{C}$ at the point of level $3z(0) = 2a$. Show that the reaction $R(\varphi)$ at such a point. Compute the instant t_1 when the parabola intercepts $\mathcal{C}$ and show that the function $t \to P(t)$ is continuous in a neighborhood of t_1.

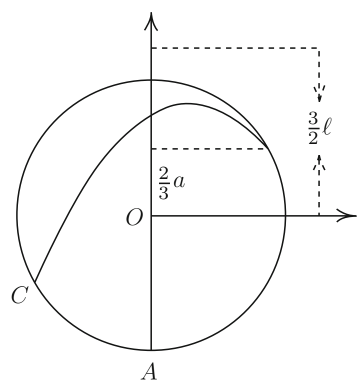

Fig. 7.1c.

7.2c The Cycloidal Pendulum (Huygens [81])

Let $\{P; m\}$ be constrained on a smooth, fixed cycloid

$$x = R(\varphi + \sin\varphi), \quad z = R(1 - \cos\varphi), \quad \varphi \in (-\pi, \pi).$$

The arc length is $s = 4R\sin\frac{1}{2}\varphi$, which implies $s^2 = 8Rz$. Therefore the z-component of the unit tangent to the cycloid is

$$\mathbf{k}\cdot\mathbf{t} = \frac{dz}{ds} = \frac{s}{4R}.$$

By the energy integral, $\|\dot{P}\|^2 = 2g(a-z)$. To resolve the motion in terms of the parameter t write (1.1) in the triad intrinsic to the cycloid as in (2.2). The equation along $\mathbf{t}$ is

$$m\ddot{s} = \mathbf{F}\cdot\mathbf{t} = -mg\mathbf{k}\cdot\mathbf{t} = -\frac{mg}{4R}s.$$

This describes an oscillatory motion of period $T = 2\pi\sqrt{4R/g}$. Thus the approximate law of Galileo is exact for oscillations of a point mass along an arc of cycloid [81], [80, Vol. XVIII].

7.3c The Spherical Pendulum (Lagrange [101])

A point mass $\{P;m\}$ moves in the cavity of a smooth, fixed sphere of center O and radius ℓ, subject to gravity. Introduce a triad $S = \{O;\mathbf{i},\mathbf{j},\mathbf{k}\}$ with $\mathbf{k}$ along the ascending vertical and set

$$x = \ell\sin\theta\cos\varphi, \qquad y = \ell\sin\theta\sin\varphi, \qquad z = \ell\cos\theta.$$

By the energy integral,

$$\tfrac{1}{2}m\ell^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) - mg\ell\cos\theta = E_o.$$

Since the constraint is smooth, the reaction is directed as $P-O$, and its projection on the plane $z=0$ is on a line through O. It follows that the motion of the projection P_* of P in the plane $z=0$ is central, and its areolar velocity is constant, i.e.,

$$\|(P_*-O)\wedge\dot{P}_*\| = \ell^2\dot{\varphi}\sin^2\theta = a_o.$$

If $a_o = 0$, the motion takes place in a plane and $\{P;m\}$ behaves like a mathematical pendulum. Assuming $a_o > 0$, the energy integral and the area integral imply

$$\begin{gathered}\tfrac{1}{2}m\ell^2\dot{\theta}^2\sin^2\theta = A_o + A_1\cos\theta + A_2\cos^2\theta + A_3\cos^3\theta,\\ A_o = E - \frac{m}{2\ell^2}a_o^2, \quad A_1 = mg\ell, \quad A_2 = -E, \quad A_3 = -mg\ell.\end{gathered}$$

The right-hand side is nonnegative for θ in some interval $[\theta_1,\theta_2]$. Therefore the motion takes place in the annular, spherical sector $\theta\in(\theta_1,\theta_2)$.

Choosing the initial time so that $\theta(0) = \theta_1$, the implicit time resolution of the motion is

$$t = \pm\frac{1}{2}m\ell^2 \int_{\cos\theta_1}^{\cos\theta} \frac{d\eta}{\sqrt{A_o + A_1\eta + A_2\eta^2 + A_3\eta^3}}.$$

To compute the reaction, observe that along the motion, $\|P - O\|^2 = \ell^2$. From this by double differentiation,

$$\dot{P}^2 + (P - O)\cdot\ddot{P} = 0, \qquad \text{and moreover,} \qquad m\ddot{P} = -mg\mathbf{k} + \mathbf{R}.$$

Therefore

$$m\dot{P}^2 - mg\mathbf{k}\cdot(P - O) + \mathbf{R}\cdot(P - O) = 0,$$

from which, by the energy integral,

$$\tfrac{1}{2}m\dot{P}^2 + mgz - \tfrac{3}{2}mgz = \tfrac{1}{2}R_{\mathbf{n}}\ell \quad \Longrightarrow \quad R_{\mathbf{n}} = \frac{2E_o - 3mgz}{\ell}.$$

7.4c Small Oscillations of a Spherical Pendulum

Assume $\theta \approx \pi$, $\dot{z}, \ddot{z} \approx 0$ and $z \approx -\ell$. Since the sphere is smooth, $\mathbf{R} = \lambda(P - O)/\ell$. If $z \approx -\ell$, one also has $\mathbf{R} \approx mg\mathbf{k}$, so that $\lambda \approx -mg$, and the approximate equations of motion for $\theta \approx 0$ take the form

$$\ddot{x} + (g/\ell)x = 0, \qquad \ddot{y} + (g/\ell)y = 0. \tag{7.1c}$$

Each of these describes a harmonic oscillation of period $2\pi\sqrt{\ell/g}$. The motion takes place in the plane $z = -\ell$ and the trajectory is an ellipse, possibly degenerate.

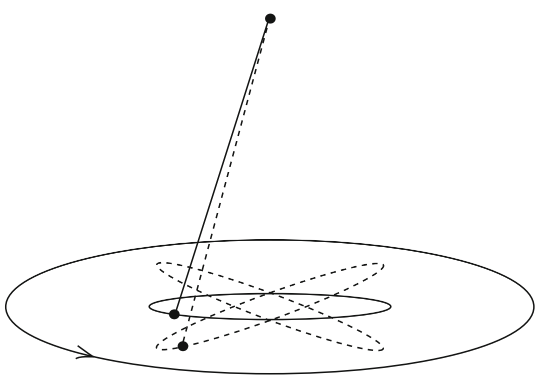

Fig. 7.2c.

7.5c The Foucault Pendulum [11, 57]

This is a spherical pendulum, suspended at one of the extremities of a long, ideal, inextensible, massless string of length ℓ, whose second extreme is fixed at O. Its relevance is that it elucidates the influence of the rotation of Earth on the motion of a spherical pendulum. Let Σ be an inertial system and fix a triad $S = \{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with origin at the center of the spherical pendulum, and with unit vectors chosen as in §6.2. By Coriolis's theorem,

$$\begin{aligned}\mathbf{a}_S(P) &= \mathbf{a}_\Sigma(P) - 2\boldsymbol{\omega} \wedge \mathbf{v}_S(P) \\ &= \mathbf{a}_\Sigma(P) + 2\omega\,(\cos\lambda\mathbf{i} - \sin\lambda\mathbf{k}) \wedge (\dot{x}, \dot{y}, \dot{z}),\end{aligned}$$

where ω is the angular speed of the rotation of Earth about its axis, and λ is the astronomical latitude of the point of coordinates $(0, 0, -\ell)$. Assuming the oscillations are small, the equations of motion is S are

$$\ddot{x} + \frac{g}{\ell}x - 2\omega_o\dot{y} = 0, \qquad \ddot{y} + \frac{g}{\ell}y + 2\omega_o\dot{x} = 0, \qquad \omega_o = \omega\sin\lambda.$$

Multiply the second by the imaginary unit i and add it to the first to get

$$\ddot{z} + 2i\omega_o\dot{z} + (g/\ell)z = 0, \qquad \text{where} \qquad z = x + iy.$$

This has the general integral

$$z = e^{-i\omega_o t}\left(Ae^{i\sqrt{(g/\ell)+\omega_o^2}\,t} + Be^{-i\sqrt{(g/\ell)+\omega_o^2}\,t}\right),$$

where A and B are constants to be determined in terms of the initial conditions. To interpret the motion of z, assume that ω_o^2 is negligible with respect to (g/ℓ). In such a case,

$$\begin{aligned}\left(Ae^{i\sqrt{(g/\ell)+\omega_o^2}\,t} + Be^{-i\sqrt{(g/\ell)+\omega_o^2}\,t}\right) &\approx \left(Ae^{i\sqrt{g/\ell}\,t} + Be^{-i\sqrt{g/\ell}\,t}\right) \\ &= (x_* + iy_*) = z_*,\end{aligned}$$

where the components x_* and y_* are solutions of (7.1c). Therefore $t \to z_*(t)$ describes an ellipse in the complex plane, and it represents the motion of small oscillations of a spherical pendulum when the effects of the rotation of Earth are neglected. With these approximations, the previous general integral can be rewritten in the approximate form

$$z = e^{-i\omega_o t}z_*,$$

and it is interpreted as a point that describes ellipses on planes that are ideally superposed to the plane $z = 0$ and that are themselves rotating about $\mathbf{k}$ with angular velocity $\omega_o\mathbf{k}$. This was indeed the outcome of Foucault's experiments [57, 1851], mathematically formalized by Binet [11].

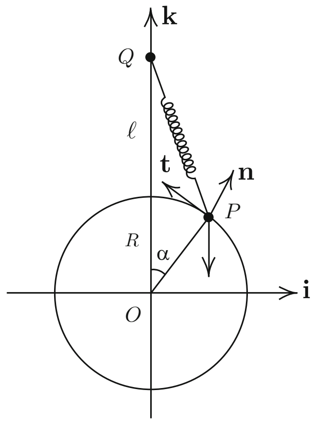

Fig. 7.3c.

7.6c Point Mass Sliding on a Circle

A point mass $\{P; m\}$ is constrained on a smooth, fixed circle of center O and radius R, and set in a vertical plane. The point P is subject to gravity and is acted upon by an elastic force $\mathbf{F} = k(Q - P)$ as in **Figure 7.3c**. Compute the reaction due to the constraint in terms of the position of P. Let α be the angle between $P - O$ and $Q - O$. Prove that if initially $\alpha \in [0, \frac{1}{2}\pi)$, then α remains in such an interval for all times, provided $k(R + \ell) > mg$. In such a case prove that the motion is oscillatory, and compute the period of small oscillations ($\sin \alpha \approx \alpha$).

Since the constraint is smooth, $\mathbf{R} = \lambda \mathbf{n}$ for $\lambda \in \mathbb{R}$, whereas the external forces are given by

$$\begin{aligned} \mathbf{F}^{(e)} &= k(Q - P) - mg \cos \alpha \mathbf{n} - mg \sin \alpha \mathbf{t} \\ &= [k(R + \ell) \cos \alpha - kR - mg \cos \alpha]\, \mathbf{n} \\ &\quad + [k(R + \ell) \sin \alpha - mg \sin \alpha]\, \mathbf{t}. \end{aligned}$$

Therefore, in terms of the intrinsic triad to the trajectory,

$$\begin{aligned} mR\ddot{\alpha} - [k(R + \ell) - mg] \sin \alpha = 0, \\ mR\dot{\alpha}^2 - [k(R + \ell) \cos \alpha - kR - mg \cos \alpha] - \lambda = 0. \end{aligned}$$

The first of these determines the motion, in terms of the Lagrangian coordinate α, starting from some initial conditions. Therefore $t \to \alpha(t)$ is known, and the second determines λ, and thus the reaction due to the constraints. Starting from $\alpha(0) \in (0, \frac{1}{2}\pi)$, the first equation keeps $\alpha(t)$ in the same interval only if

$k(R+\ell) > mg$. In such a case the motion is oscillatory and the period of the small oscillations is

$$T = 2\pi\sqrt{\frac{mR}{k(R+\ell) - mg}}.$$

7.7c Point Mass Sliding on a Sphere

A point mass $\{P; m\}$, subject to gravity, starts moving from its rest position at the top of a sphere having been activated by an elemental disturbance. Find the point and the time when it leaves the sphere.

4

GEOMETRY OF MASSES

1 Material Systems and Measures

A distribution of masses within a bounded set $E \subset \mathbb{R}^3$ is described by a measure μ. The symbol $d\mu(P)$ is the elemental mass about P as measured by μ. The measure μ is required to be finite, that is,

$$\{\text{total mass of the system}\} = \int_E d\mu < \infty,$$

and supported in E, that is,

$$\int_K d\mu = 0 \quad \text{for every compact} \quad K \subset \mathbb{R}^3 - \bar{E}.$$

The mechanical quantities of a material system are described by scalar- or vector-valued functions

$$Q \longrightarrow \Phi(Q) = \int f(P - Q) d\mu(P),$$

where $f(\cdot - Q)$ is μ-integrable for all Q. The pair $\{\mathcal{M}; d\mu\}$ will mean a material system $\mathcal{M}$ whose masses are measured by a given such a measure μ. In what follows we will refer to either continuous or discrete systems.

1.1 Continuous Distribution of Masses

Let dx denote the Lebesgue measure in E. A nonnegative, Lebesgue measurable function $\rho(\cdot)$ defined in E may be regarded as the density function of a continuous distribution of masses in Ω. In such a case $d\mu = \rho\, dx$. The measure $d\mu$ is absolutely continuous with respect to the Lebesgue measure and ρ is its Radon–Nikodym derivative. The total mass contained in E is

$$m = \int_E \rho\, dx = \int d\mu, \qquad \text{provided} \;\; \rho \in L^1(E).$$

E. DiBenedetto, *Classical Mechanics: Theory and Mathematical Modeling*, Cornerstones, DOI 10.1007/978-0-8176-4648-6_4,

In the last integral the domain of integration has been omitted, since μ is supported in $\bar{E}$. If $\{\mathcal{M}; d\mu\}$ is distributed over a line or a plane, then μ is the Lebesgue measure in $\mathbb{R}^1$ or $\mathbb{R}^2$ respectively, and ρ is the linear or planar density of the system.

1.2 Discrete Distribution of Masses

The measure describing a system of n material points $\{P_j; m_j\}$ is

$$d\mu = \sum_{j=1}^{n} m_j \delta(P_j),$$

where $\delta(P_j)$ is the Dirac mass concentrated in P_j. The total mass is

$$m = \int d\mu = \sum_{j=1}^{n} m_j \int \delta(P_j) = \sum_{j=1}^{n} m_j.$$

2 Center of Mass and First-Order Moments

Given a system $\{\mathcal{M}; d\mu\}$ and $O \in \mathbb{R}^3$, the equation

$$m(P_o - O) = \int (P - O) d\mu(P), \qquad m = \int d\mu, \tag{2.1}$$

defines a point P_o, independent of O, called *center of mass* of $\{\mathcal{M}; d\mu\}$. For continuous or discrete systems, (2.1) takes the form

$$m(P_o - O) = \int (P - O)\rho\, dx, \qquad m(P_o - O) = \sum_{j=1}^{n} (P_j - O) m_j.$$

Let $\pi\{Q; \mathbf{e}\}$ denote the plane through Q and normal to the unit vector $\mathbf{e}$. The moment of $\{\mathcal{M}; d\mu\}$ with respect to $\pi\{Q; \mathbf{e}\}$ is

$$\int (P - Q) \cdot \mathbf{e} d\mu(P) = m(P_o - Q) \cdot \mathbf{e}. \tag{2.2}$$

Thus the moment of $\{\mathcal{M}; d\mu\}$ with respect to $\pi\{Q; \mathbf{e}\}$ equals the moment of the point mass $\{P_o; m\}$ with respect to the same plane. The integrals in (2.1) and (2.2) are moments of the *first order*, or of *degree one*.

3 Second-Order Moments and Huygens's Theorem

For a material system $\{\mathcal{M}; d\mu\}$ define [81]:

(i) Polar Moment of Inertia of $\{\mathcal{M}; d\mu\}$ with respect to a point O,

$$I_o = \int \|P - O\|^2 d\mu(P).$$

(ii) Planar Moment of Inertia of $\{\mathcal{M}; d\mu\}$ with respect to a plane $\pi\{Q;\mathbf{e}\}$,

$$I_{\pi\{Q;\mathbf{e}\}} = \int |(P-Q)\cdot\mathbf{e}|^2\, d\mu(P).$$

(iii) Axial Moment of Inertia of $\{\mathcal{M}; d\mu\}$ with respect to an axis $\ell\{Q;\mathbf{e}\}$ through Q and directed as the unit vector $\mathbf{e}$,

$$I_{\ell\{Q;\mathbf{e}\}} = \int \|(P-Q)\wedge\mathbf{e}\|^2 d\mu(P).$$

(iv) Deflection Moment of Inertia[1] of $\{\mathcal{M}; d\mu\}$ with respect to a pair of nonparallel planes $\pi\{Q;\mathbf{e}\}$ and $\pi\{Q;\mathbf{e}'\}$,

$$I_{\pi\{Q;\mathbf{e}\};\pi\{Q;\mathbf{e}'\}} = \int \big((P-Q)\cdot\mathbf{e}\big)\big((P-Q)\cdot\mathbf{e}'\big) d\mu(P).$$

The moments of inertia are nonnegative, whereas the deflection moments of inertia are of variable sign. Introduce a triad S with origin in O and set

$$I_o = \int \sum_{i=1}^{3} x_i^2 d\mu, \qquad I_i = I_{\text{plane}\{x_i=0\}} = \int x_i^2 d\mu,$$
$$I_{ii} = I_{\text{axis } x_i} = \int \sum_{j\neq i} x_j^2 d\mu.$$

Then

$$I_{ii} = \sum_{j\neq i} I_j, \qquad \sum_{i=1}^{3} I_{ii} = 2\sum_{j=1}^{3} I_j = 2I_o. \tag{3.1}$$

The distances $\|P-O\|$ are intrinsic to $\{\mathcal{M}; d\mu\}$ and are invariant under a rotation of S. Thus I_o is also an intrinsic quantity of $\{\mathcal{M}; d\mu\}$. The deflection moments of inertia with respect to pairs of coordinate planes are denoted by

$$-I_{ij} = I_{\{x_i=0\};\{x_j=0\}} = \int x_i x_j d\mu, \qquad i,j = 1,2,3, \quad i\neq j.$$

These inertia quantities, with the indicated notation, are then organized into the symmetric matrix

$$\sigma = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix},$$

called an *inertia tensor*. From the previous remarks, the trace of σ is $2I_o$, which is invariant under rotations of S.

[1]The terminology *deflection moments* will be justified in §5 of Chapter 7 and in particular, Remark 5.1.

Theorem 3.1 (Huygens [82]). *For every* $O \in \mathbb{R}^3$ *and every plane* $\pi\{Q;\mathbf{e}\}$,

$$\begin{aligned} I_o &= I_{P_o} + m\|P_o - O\|^2, && (3.2)\\ I_{\pi\{Q;\mathbf{e}\}} &= I_{\pi\{P_o;\mathbf{e}\}} + m|(P_o - Q)\cdot\mathbf{e}|^2, && (3.3)\\ I_{\ell\{Q;\mathbf{e}\}} &= I_{\ell\{P_o;\mathbf{e}\}} + m\|(P_o - Q)\wedge\mathbf{e}\|^2, && (3.4)\\ I_{\pi\{Q;\mathbf{e}\};\pi\{Q;\mathbf{e}'\}} &= I_{\pi\{P_o;\mathbf{e}\};\pi\{P_o;\mathbf{e}'\}}, && (3.5)\\ &\quad + m\big((P_o - Q)\cdot\mathbf{e}\big)\big((P_o - Q)\cdot\mathbf{e}'\big). \end{aligned}$$

Proof. To prove (3.2) compute

$$\begin{aligned} I_o &= \int \|P-O\|^2 d\mu(P) = \int \|(P-P_o)+(P_o-O)\|^2 d\mu(P)\\ &= \int \|P-P_o\|^2 d\mu(P) + \|P_o-O\|^2 \int d\mu(P) + 2(P_o-O)\cdot\int (P-P_o)d\mu(P)\\ &= I_{P_o} + m\|P_o-O\|^2, \end{aligned}$$

since the last term vanishes by definition of center of mass. As for (3.4), compute

$$\begin{aligned} I_{\ell\{Q;\mathbf{e}\}} &= \int \|(P-Q)\wedge\mathbf{e}\|^2 d\mu(P)\\ &= \int \|(P-P_o)\wedge\mathbf{e} + (P_o-Q)\wedge\mathbf{e}\|^2 d\mu(P)\\ &= I_{\ell\{P_o;\mathbf{e}\}} + m\|(P_o-Q)\wedge\mathbf{e}\|^2\\ &\quad + 2\big((P_o-Q)\wedge\mathbf{e}\big)\cdot\Big(\int (P-P_o)d\mu(P)\wedge\mathbf{e}\Big). \end{aligned}$$

From this (3.4) follows, since the last term vanishes. The remaining formulas are proved analogously. ■

Remark 3.1 The center of mass P_o of a system $\{\mathcal{M}; d\mu\}$ is the point for which the polar moment is minimal.

Remark 3.2 Given a bundle of parallel lines (planes), the one through P_o is the axis (plane) whose axial (planar) moment of inertia is minimal.

4 Ellipsoid of Inertia and Principal Axes

Proposition 4.1 *For any two unit vectors* $\mathbf{a}$ *and* $\mathbf{b}$,

$$\mathbf{a}^t\sigma\mathbf{b} = I_o\mathbf{a}\cdot\mathbf{b} - I_{\pi\{O;\mathbf{a}\};\pi\{O;\mathbf{b}\}}, \qquad \mathbf{a}^t\sigma\mathbf{a} = I_{\ell\{O;\mathbf{a}\}}.$$

Proof. From the definition of deflection moment of inertia with respect to a pair of planes,

$$\begin{aligned} I_{\pi\{O;\mathbf{a}\};\pi\{O;\mathbf{b}\}} &= \int \big((P-O)\cdot\mathbf{a}\big)\big((P-O)\cdot\mathbf{b}\big)d\mu(P) \\ &= \int x_i a_i x_j b_j d\mu = \int x_i^2 a_i b_i d\mu + \int \sum_{i\neq j} x_i x_j a_i b_j d\mu \\ &= (I_o - I_{ii})a_i b_i - \sum_{i\neq j} I_{ij} a_i b_j \\ &= I_o \mathbf{a}\cdot\mathbf{b} - \mathbf{a}^t\sigma\mathbf{b}. \end{aligned}$$

This establishes the first statement of the proposition. Putting now $\mathbf{a}=\mathbf{b}$ and using the definition of planar moment of inertia yields

$$\begin{aligned} \mathbf{a}^t\sigma\mathbf{a} &= I_o\|\mathbf{a}\|^2 - I_{\pi\{O;\mathbf{a}\}} \\ &= \int \|P-O\|^2 d\mu(P) - \int |(P-O)\cdot\mathbf{a}|^2 d\mu(P) \\ &= \int \left(\|P-O\|^2 - |(P-O)\cdot\mathbf{a}|^2\right) d\mu(P) \\ &= \int \|(P-O)\wedge\mathbf{a}\|^2\, d\mu(P) = I_{\ell\{O;\mathbf{a}\}}. \end{aligned}$$

■

Corollary 4.1 *The matrix* σ *is positive semidefinite. Moreover,*

$$x^t\sigma x = \|x\|^2 I_x \qquad \textit{for all } x\in\mathbb{R}^3\backslash\{0\},$$

where I_x *is the axial moment of inertia of* $\{\mathcal{M};d\mu\}$ *with respect to the axis through the origin and direction* x.

Proof. By Proposition 4.1, $x^t\sigma x\geq 0$ for all $x\in\mathbb{R}^3$. The corollary follows by putting $\mathbf{a}=x/\|x\|$ into the second equation of Proposition 4.1.[2] ■

This formula gives a practical way of computing the axial moments of inertia of a system $\{\mathcal{M};d\mu\}$ with respect to axes through the origin whenever the inertia tensor σ is known. Assume now that σ is not degenerate. Having fixed $x_o\in\mathbb{R}^3\backslash\{O\}$, compute I_{x_o} and set

$$\lambda^2 = \|x_o\|^2 I_{x_o}.$$

The points $x\in\mathbb{R}^3$ satisfying the equation $x^t\sigma x=\lambda^2$ define an ellipsoid, denoted by $\mathcal{E}_\lambda$, and called an *inertia ellipsoid* or Cauchy ellipsoid [23]. It follows from the definition that

$$I_x = \frac{\lambda^2}{\|x\|^2}, \qquad \text{for every } x\in\mathcal{E}_\lambda.$$

[2]In general σ is not positive definite. For an axial distribution of masses σ is degenerate.

Thus, knowing the axial moment of inertia with respect to some point $x_o \in \mathcal{E}_\lambda$ permits one to compute the axial moment with respect to any other point of the inertia ellipsoid $\mathcal{E}_\lambda$. These remarks suggest putting σ into diagonal form, or equivalently, writing $\mathcal{E}_\lambda$ in its canonical form. Since σ is symmetric and positive definite, its eigenvalues $\mathcal{I}_i$ are nonnegative, and there exists a unitary matrix $\mathcal{U}$ such that

$$\mathcal{U}\sigma\mathcal{U}^t = \begin{pmatrix} \mathcal{I}_1 & 0 & 0 \\ 0 & \mathcal{I}_2 & 0 \\ 0 & 0 & \mathcal{I}_3 \end{pmatrix} = (\mathcal{I}_i\delta_{ij})\,.$$

The matrix $\mathcal{U}$ rotates S into a new system S', whose coordinates we denote by $X = (X_1, X_2, X_3)$. If $x \in \mathcal{E}_\lambda$, then

$$\lambda^2 = x^t\sigma x = (\mathcal{U}x)^t\mathcal{U}\sigma\mathcal{U}^t(\mathcal{U}x) = X^t(\mathcal{U}\sigma\mathcal{U}^t)X = X^t(\mathcal{I}_i\delta_{ij})X.$$

Therefore, in the rotated system S', the ellipsoid $\mathcal{E}_\lambda$ takes its canonical form

$$\mathcal{I}_i X_i^2 = \lambda^2.$$

The semiaxes $a_i = \lambda/\sqrt{\mathcal{I}_i}$ of $\mathcal{E}_\lambda$ are inversely proportional to the eigenvalues of σ. In particular, the largest semiaxis corresponds to the least eigenvalue.

The next proposition asserts that the transformed matrix $\mathcal{U}\sigma\mathcal{U}^t$ keeps the same material-geometric significance as σ. Thus σ identifies a tensor.

Proposition 4.2 *Let σ' be the inertia tensor of $\{\mathcal{M}; d\mu\}$ with respect to S'. Then $\sigma' = \mathcal{U}\sigma\mathcal{U}^t$.*

Proof. The entries I_{ij} of σ can be written concisely as

$$I_{ij} = \delta_{ij}\int \|P - O\|^2 d\mu(P) - \int x_i x_j d\mu.$$

Since $\mathcal{U}$ is unitary, the entries $\mathcal{I}_{ij}$ of the transformed matrix $\mathcal{U}\sigma\mathcal{U}^t$ are

$$\begin{aligned} \mathcal{I}_{ij} &= \alpha_{i\ell}I_{\ell k}\alpha_{jk} = \alpha_{i\ell}I_o\alpha_{j\ell} - \int \alpha_{i\ell}x_\ell x_k\alpha_{jk}d\mu \\ &= I_o\delta_{ij} - \int X_i X_j d\mu. \end{aligned}$$

Thus $\mathcal{I}_{ij}$ are the entries of the inertia tensor of $\{\mathcal{M}; d\mu\}$ in S'. ■

Corollary 4.2 *The eigenvalues $\mathcal{I}_i$ of σ, for $i = 1, 2, 3$, are the axial moments of inertia of $\{\mathcal{M}; d\mu\}$ with respect to the coordinate axes of S'.*

The new coordinate axes are called *principal axes of inertia* and are directed as the eigenvectors of σ. In S', the inertia tensor σ' is diagonal, and the elements of the diagonal are the axial moments of inertia of $\{\mathcal{M}; d\mu\}$ with respect to the principal axes of inertia.

5 Miscellaneous Remarks

Assume that the entries of the ith row and ith column of σ are all zero except I_{ii}. For example, assume that σ is of the form

$$\sigma = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}. \tag{5.1}$$

Since I_{33} is an eigenvalue of σ, the x_3-axis is a principal axis of inertia. The ellipsoids $\mathcal{E}_\lambda$ are of the form

$$I_{11}x_1^2 + 2I_{12}x_1x_2 + I_{22}x_2^2 + I_{33}x_3^2 = \lambda^2.$$

From the definitions and the form of σ it follows that the x_3-axis is a principal axis of inertia if and only if I_{13} and I_{23} are zero, equivalently, if and only if $\mathcal{E}_\lambda$ is symmetric with respect to the plane $\{x_3 = 0\}$.

5.1 Center of Mass and Principal Axes of Inertia

The inertia tensor σ and the principal axes of inertia depend on the choice of O and are called principal axes of inertia with respect to O. If the principal triad of inertia has origin at the center of mass P_o of the material system $\{\mathcal{M}; d\mu\}$, it is called a *central principal* triad of inertia, and the coordinate axes are *central principal* axes of inertia.

Proposition 5.1 *If an axis is principal with respect to O and contains P_o, then it is central principal.*

Proof. If $P_o = O$, the conclusion is trivial. If $P_o \neq O$, let ℓ be the axis through O and P_o and let S_O be the principal triad of inertia with origin at O. By assumption, ℓ is a principal axis of inertia, and without loss of generality we label it as the x_1-axis of S_O, so that

$$-I_{12} = \int x_1x_2 d\mu = 0, \qquad -I_{13} = \int x_1x_3 d\mu = 0.$$

Moreover, since $P_o = (a, 0, 0)$ for some $a \neq 0$,

$$\int x_3 d\mu = \int x_2 d\mu = 0.$$

From these one computes the deflection moment of inertia:

$$I_{\{x_1=a\};\{x_i=0\}} = \int (x_1 - a)x_i \, d\mu = 0, \qquad i = 2, 3.$$

Thus ℓ is a central principal axis of inertia. ∎

Corollary 5.1 *The principal axes of inertia with respect to a point of a central principal axis are parallel to the central principal axes of inertia.*

Corollary 5.2 *If a principal axis of inertia with respect to a point O contains P_o, then the principal axes with respect to O are parallel to the central principal axes of inertia.*

5.2 Planar Systems

Let $\{\mathcal{M}; d\mu\}$ be a system of masses distributed in a plane π. Since π can be regarded as a plane of material symmetry for $\{\mathcal{M}; d\mu\}$, any axis normal to π is a principal axis of inertia. Let S be a triad with origin $O \in \pi$ and with $\mathbf{u}_3$ normal to π. With respect to S, the tensor of inertia σ has the form (5.1). Moreover,

$$I_{11} = \int x_2^2 d\mu, \qquad I_{22} = \int x_1^2 d\mu \qquad I_{33} = I_{11} + I_{22}. \tag{5.2}$$

6 Computing Some Moments of Inertia

6.1 Homogeneous Material Segment

A segment of length $a > 0$ is identified with the interval $(-\frac{1}{2}a, \frac{1}{2}a)$ of the x_1-axis. By material symmetry the coordinate axes are principal axes of inertia. We compute $I_{11} = 0$ and $I_{ij} = 0$, $i \neq j$. Moreover,

$$I_O = I_{22} = I_{33} = \rho \int_{-a/2}^{a/2} x_1^2 dx_1 = \tfrac{1}{12} m a^2, \quad m = \rho a. \tag{6.1}$$

The axial moment with respect to any axis ℓ normal to the segment through one of the extremities is computed by Huygens's theorem as

$$I_\ell = \tfrac{1}{12} m a^2 + \tfrac{1}{4} m a^2 = \tfrac{1}{3} m a^2.$$

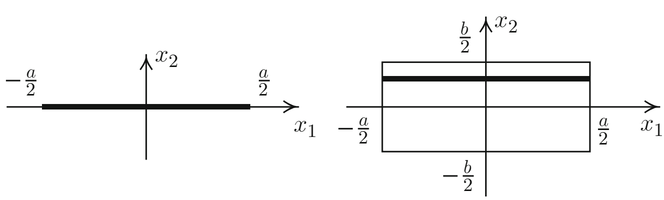

Fig. 6.1.

6.2 Homogeneous Material Rectangle

The triad S as in the left-hand picture **Figure 6.1** is a central principal axis of inertia. One might regard the rectangle as a material segment of constant density $b\rho$, obtained by condensing on the segment MN those points at equal distance from the x_2-axis. Then by (6.1),

$$I_{22} = \tfrac{1}{12}ma^2, \qquad \text{and by symmetry} \qquad I_{11} = \tfrac{1}{12}mb^2.$$

Moreover, by (5.2),

$$I_{33} = I_{11} + I_{22} = \tfrac{1}{12}m(a^2 + b^2).$$

By Huygens's theorem,

$$I_{\substack{\text{with resp. to axis}\\ \{x_3=0\}\cap\{x_1=-a/2\}}} = \tfrac{1}{12}ma^2 + \tfrac{1}{4}ma^2 = \tfrac{1}{3}ma^2,$$
$$I_{\substack{\text{with resp. to axis}\\ \{x_3=0\}\cap\{x_2=b/2\}}} = \tfrac{1}{12}mb^2 + \tfrac{1}{4}mb^2 = \tfrac{1}{3}mb^2.$$

For a material homogeneous square of edge a,

$$I_{11} = I_{22} = \tfrac{1}{12}ma^2, \qquad I_{33} = \tfrac{1}{6}ma^2.$$

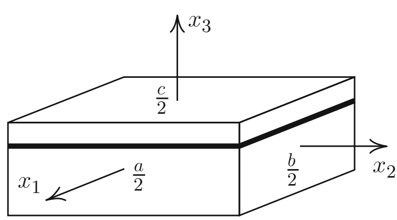

Fig. 6.2.

6.3 Homogeneous Material Parallelepiped

Referring to **Figure 6.2**, regard the parallelepiped as a material rectangle of sides a and b and of constant surface density ρc. The mass of such a rectangle is $m = \rho abc$. By the previous calculations,

$$I_{11} = \tfrac{1}{12}m(b^2 + c^2), \quad I_{22} = \tfrac{1}{12}m(a^2 + c^2), \quad I_{33} = \tfrac{1}{12}m(a^2 + b^2).$$

Let ℓ be the axis parallel to $\mathbf{u}_3$ and through the point $\frac{1}{2}(a, b)$, whose distance from the origin is $\frac{1}{2}\sqrt{a^2 + b^2}$. By Huygens's theorem,

$$I_\ell = I_{33} + \tfrac{1}{4}m(a^2 + b^2) = \tfrac{1}{3}m(a^2 + b^2).$$

Consider now the axis ℓ' through the origin and $\frac{1}{2}(b,a,c)$ and set

$$\sigma = \frac{m}{12}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}, \qquad \mathbf{v} = \frac{(a,b,c)^t}{\sqrt{a^2+b^2+c^2}}.$$

From Proposition 4.1,

$$I_{\ell'} = \mathbf{v}^t\sigma\mathbf{v} = \frac{1}{12}m\frac{a^2(a^2+c^2)+b^2(b^2+c^2)+c^2(a^2+b^2)}{a^2+b^2+c^2}.$$

6.4 Homogeneous Material Circle and Disk

The x_3-axis through P_o and normal to the plane of the circle or disk is a central principal axis of inertia. By symmetry, any pair of axes orthogonal in P_o and normal to the x_3-axis is central principal and

$$I_{11} = I_{22}, \qquad I_{33} = 2I_{11} = 2I_{22}.$$

Therefore it will suffice to compute I_{33}. For the circle,

$$I_{33} = mR^2, \qquad I_{11} = I_{22} = \tfrac{1}{2}mR^2.$$

For the disk, $I_{11} = I_{22} = \frac{1}{4}mR^2$, and

$$I_{33} = \rho\int_{x_1^2+x_2^2<R^2}(x_1^2+x_2^2)dx_1dx_2 = \tfrac{1}{2}mR^2.$$

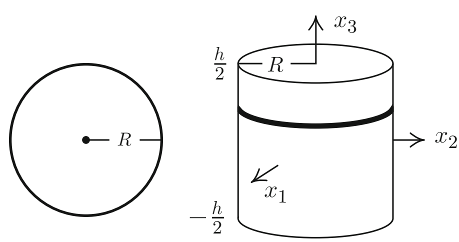

Fig. 6.3.

6.5 Homogeneous Material Right Circular Cylinder

The cylinder has height h and section a disk of radius R. The axis of the cylinder is central principal. The remaining central principal axes are any two orthogonal lines through P_o and normal to the axis of the cylinder. The density is constant and the mass of the cylinder is $m = \rho\pi R^2 h$. To compute I_{33} one regards the mass of the cylinder concentrated into a homogeneous material disk of center P_o. Therefore $I_{33} = \frac{1}{2}mR^2$. To compute $I_{11} = I_{22}$, regard the cylinder as a stack of homogeneous material disks of radius R and infinitesimal thickness dx_3, and each of mass $\pi\rho R^2 dx_3$. By Huygens's theorem, the moment of the generic of these disks whose center has distance x_3 from P_o is given by

$$\pi\rho R^2 \left(\tfrac{1}{4}R^2 + x_3^2\right) dx_3.$$

By integration in dx_3 over $(-\frac{1}{2}h, \frac{1}{2}h)$,

$$I_{11} = I_{22} = \tfrac{1}{4}m\left(R^2 + \tfrac{1}{3}h^2\right).$$

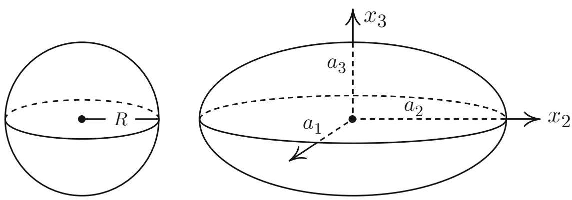

Fig. 6.4.

6.6 Homogeneous Material Sphere and Ball

By symmetry, any triad with origin in P_o is central, principal of inertia, and $I_{11} = I_{22} = I_{33}$. Moreover, from the definition of polar moment with respect to P_o,

$$I_{11} + I_{22} + I_{33} = 3I_{11} = 2I_o.$$

For the sphere of radius R,

$$I_o = mR^2, \qquad I_{ii} = \tfrac{2}{3}mR^2, \quad i = 1, 2, 3.$$

For the ball of radius R,

$$I_o = \tfrac{3}{5}mR^2, \qquad I_{ii} = \tfrac{2}{5}mR^2.$$

Problems and Complements

2c Center of Mass and First-Order Moments

2.1. Prove that P_o is independent of the choice of O. Moreover, P_o is in the convex envelope of the support of $d\mu(P)$.

2.2. In a material homogeneous triangle the center of mass is at the intersection of the medians, one-third the distance from the intersection of any one of them with the side of the triangle they bisect.

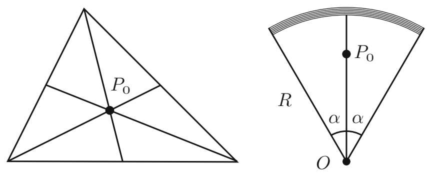

Fig. 2.1c.

2.3. The center of mass of a material homogeneous arc of circumference of center O, radius R, and aperture $2\alpha \in (0, \pi)$ is $P_o = (0, R \sin \alpha / \alpha)$.

2.4. For a material homogeneous circular sector of center O, radius R, and aperture 2α, compute $P_o = (0, 2R \sin \alpha / 3\alpha)$.

2.5. For the material homogeneous parabolic segment $x_2^2 \leq 2px_1$, $x_1 \in (0, a)$, $a, p > 0$, compute $P_o = \left(\frac{3}{5}a, 0\right)$.

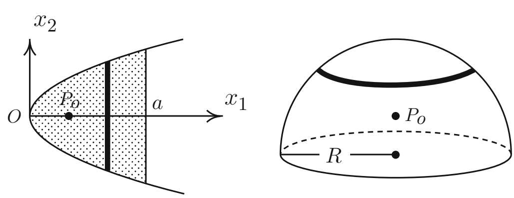

Fig. 2.2c.

2.6. The center of mass of a material homogeneous hemisphere of radius R is at distance $\frac{3}{8}R$ from the equatorial plane.

3c Second-Order Moments and Huygens's Theorem

Let σ be the tensor of inertia of a material system $\{\mathcal{M}; d\mu\}$ with respect to a triad S. Prove that $\mathrm{tr}(\sigma)$ and $\det(\sigma)$ are invariant by orthogonal transformations of S. Give a geometrical interpretation of $\det(\sigma)$.

3.1c Inertia Tensor for a Homogeneous Ellipsoid

Let $\mathcal{E}$ be a material homogeneous ellipsoid of density ρ, mass m, and semiaxes a_1, a_2, a_3, referred to a triad S as in **Figure 6.4**. By symmetry $\sigma = I_{ii}\delta_{ij}$.

The change of variables $X_i = x_i/a_i$, $i = 1, 2, 3$, whose Jacobian is $(a_1 a_2 a_3)$, maps $\mathcal{E}$ into the unit ball B_1 centered at the origin. Therefore

$$\begin{aligned}\frac{4\pi}{3m} I_{33} &= \frac{1}{a_1 a_2 a_3} \int_{\mathcal{E}} \left(x_1^2 + x_2^2\right) dx \\ &= a_1^2 \int_{B_1} X_1^2 dX + a_2^2 \int_{B_1} X_2^2 dX = F_1 + F_2.\end{aligned}$$

In polar coordinates

$$F_1 = a_1^2 \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_0^1 r^4 \cos^3\theta \cos^2\varphi \, d\rho \, d\theta \, d\varphi = \tfrac{4}{15}\pi a_1^2.$$

Analogously $F_2 = \frac{4}{15} a_2^2 \pi$. Therefore, $I_{33} = \frac{1}{5} m(a_1^2 + a_2^2)$.

5c Miscellaneous Remarks

5.1c Finding the Principal Axes of a Planar System

Let $\{\mathcal{M}; d\mu\}$ be a planar system and let S be a triad with the plane $\{x_3 = 0\}$ on the plane of the system. Let now S_φ be a triad obtained by rotating S through an angle φ, counterclockwise about $\mathbf{u}_3$. It is required to find the angle φ for which S_φ is a principal triad of inertia. The corresponding rotation matrix is

$$R = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore the tensor of inertia σ_φ with respect to S_φ is

$$\sigma_\varphi = R \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} R^t = \begin{pmatrix} \mathcal{I}_{11}(\varphi) & \mathcal{I}_{12}(\varphi) & 0 \\ \mathcal{I}_{21}(\varphi) & \mathcal{I}_{22}(\varphi) & 0 \\ 0 & 0 & I_{33} \end{pmatrix},$$

where

$$\begin{aligned}\mathcal{I}_{11}(\varphi) &= I_{11}\cos^2\varphi + 2I_{12}\sin\varphi\cos\varphi + I_{22}\sin^2\varphi, \\ 2\mathcal{I}_{12}(\varphi) &= -I_{11}\sin 2\varphi + 2I_{12}\cos 2\varphi + I_{22}\sin 2\varphi, \\ \mathcal{I}_{22}(\varphi) &= I_{11}\sin^2\varphi - 2I_{12}\sin\varphi\cos\varphi + I_{22}\cos^2\varphi.\end{aligned} \tag{5.1c}$$

Therefore S_φ is principal of inertia if the angle φ is chosen so that $\mathcal{I}_{12}(\varphi) = 0$, e.g.,

$$(I_{11} - I_{22})\sin 2\varphi = 2I_{12}\cos 2\varphi.$$

If $I_{11} \neq I_{22}$ one chooses

$$2\varphi = \tan^{-1}\left(\frac{2I_{12}}{I_{11} - I_{22}}\right) \cap \left(0, \frac{\pi}{2}\right).$$

If $I_{11} = I_{22}$ then $I_{12}\cos 2\varphi = 0$. If $I_{12} \neq 0$ then $\varphi = \pi/4$, whereas if $I_{12} = 0$, the original triad S was already principal of inertia.

Remark 5.1c Let $\ell(\varphi)$ be the axis through the origin and of unit vector $\mathbf{v} = (\cos\varphi, -\sin\varphi, 0)^t$. By Proposition 4.1 and (5.1c),

$$I_{\ell(\varphi)} = \mathbf{v}^t\sigma\mathbf{v} = \mathcal{I}_{11}(\varphi) \qquad \text{and} \qquad 2\mathcal{I}_{12} = \frac{d}{d\varphi}I_{\ell(\varphi)}.$$

Therefore the angle φ that identifies the principal axes of inertia is a stationary point for the function $\varphi \to I_{\ell(\varphi)}$. Prove that such a point is indeed an extremum.

5.2c Planar Systems and Mohr's Circle

Assume that the triad S is principal of inertia, so that $I_{12} + 0$. To examine the functions $\varphi \to \mathcal{I}(\varphi)$ starting from S, rewrite (5.1c) in the form

$$\mathcal{I}_{11} = \frac{I_{11} + I_{22}}{2} + \frac{I_{11} - I_{22}}{2}\cos 2\varphi,$$
$$\mathcal{I}_{12} = \frac{I_{22} - I_{11}}{2}\sin 2\varphi,$$
$$\mathcal{I}_{22} = \frac{I_{11} + I_{22}}{2} + \frac{I_{22} - I_{11}}{2}\cos 2\varphi.$$

From the first two equations,

$$(\mathcal{I}_{11}(\varphi) - a)^2 + \mathcal{I}_{12}^2 = b^2,$$

where

$$a = \frac{I_{11} + I_{22}}{2}, \qquad b = \frac{I_{11} - I_{22}}{2}.$$

This implies that the pair $(\mathcal{I}_{11}(\varphi), \mathcal{I}_{12}(\varphi))$ represents a point in the circle of center $(a, 0)$ and radius $|b|$, called *Mohr's circle*. It permits a graphical determination of the axial and deflection moments of inertia in a planar system as functions of φ. Having determined I_{11} and I_{22}, one constructs such a circle as in the right-hand picture **Figure 5.1c**. Then, for a given φ, one traces through $(a, 0)$ and slope $\tan 2\varphi$. The coordinates of M are $\mathcal{I}_{11}$ and $\mathcal{I}_{12}$. Prove that $\mathcal{I}_{22}(\varphi)$ is the abscissa of M'.

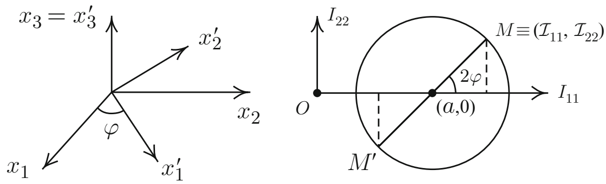

Fig. 5.1c.

6c Computing Some Moments of Inertia

6.1c Right Circular Cone

Compute the inertia tensor σ of a material homogeneous right circular cone of height h and radius R with respect to a triad centered at its vertex and with $\mathbf{u}_3$ directed as the axis of the cone as in the left-hand picture **Figure 6.1c**.

The cross section normal to the axis of the cone and at level z is a disk of radius $r = Rz/h$ whose moments of inertia with respect to its central principal axes are

$$I_{11}(z) = I_{22}(z) = \tfrac{1}{4} dm(z) r^2 = \tfrac{1}{4}\pi\rho r^4 dz,$$
$$I_{33} = \tfrac{1}{2} dm(z) r^2 = \tfrac{1}{2}\pi\rho r^4 dz,$$

where ρ is the density of the cone. Therefore by Huygens's theorem,

$$I_{11} = I_{22} = \frac{\pi\rho}{4}\left(\frac{R}{h}\right)^2 \left[\left(\frac{R}{h}\right)^2 + 4\right] \int_0^h z^4 dz = \tfrac{3}{20} m \left(R^2 + 4h^2\right).$$

Moreover,

$$I_{33} = \tfrac{1}{2}\pi\rho \int_0^h r^4 dz = \tfrac{3}{10} m R^2.$$

By symmetry, $I_{ij} = 0$ for $i \neq j$. Therefore the tensor of inertia is

$$\sigma_o = \frac{3}{20} m \begin{pmatrix} R^2 + 4h^2 & 0 & 0 \\ 0 & R^2 + 4h^2 & 0 \\ 0 & 0 & 2R^2 \end{pmatrix}.$$

6.1.1c Right Circular Cone—I

Verify that the center of mass is on the axis of the cone at distance $\frac{3}{4}h$ from its vertex. Use this and Huygens's theorem to compute the tensor of inertia with respect to the central principal triad of the cone. Such a tensor is

$$\sigma_{P_o} = \frac{3}{20} m \begin{pmatrix} R^2 + \frac{1}{4}h^2 & 0 & 0 \\ 0 & R^2 + \frac{1}{4}h^2 & 0 \\ 0 & 0 & 2R^2 \end{pmatrix}.$$

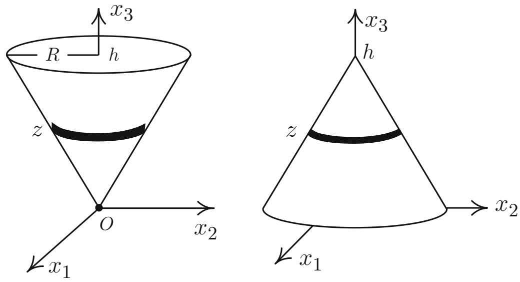

Fig. 6.1c.

6.1.2c Right Circular Cone—II

Compute the tensor of inertia of the previous cone with respect to a triad with origin at the center of its base and $\mathbf{u}_3$ on the axis of the cone as in the right-hand picture **Figure 6.1c**. The cross section normal to the axis of the cone and at distance z from the plane $\{x_3 = 0\}$ is a disk of radius $r = R(h - z)/h$ whose moments of inertia with respect to its central principal axes are computed as before. Then by Huygens's theorem,

$$I_{11} = I_{22} = \pi\rho \int_0^h \left(\tfrac{1}{4}r^4 + r^2 z^2\right) dz = \tfrac{3}{10} m \left(\tfrac{1}{2}R^2 + \tfrac{1}{3}h^2\right).$$

Moreover,

$$I_{33} = \tfrac{1}{2}\pi\rho \int_0^h r^4 dz = \tfrac{3}{10} m R^2.$$

By symmetry, $I_{ij} = 0$ for $i \neq j$. Therefore

$$\sigma_o = \tfrac{3}{10} m \begin{pmatrix} \tfrac{1}{2}R^2 + \tfrac{1}{3}h^2 & 0 & 0 \\ 0 & \tfrac{1}{2}R^2 + \tfrac{1}{3}h^2 & 0 \\ 0 & 0 & R^2 \end{pmatrix}.$$

Deduce σ_o from σ_{P_o} and Huygens's theorem.

6.2c Homogeneous Material Triangle

Let $\Delta(ABC)$ be a homogeneous material triangle of density ρ, mass m, vertices A, B, C, and sides $a = \overline{BC}$, $b = \overline{AC}$, $c = \overline{AB}$. Assume that $\overline{BC}$ is the largest side, denote by O the projection of A on BC, and set $h = \overline{OA}$. Introduce a fixed triad S centered at O, with $\mathbf{u}_3$ directed as $(C - O)$ and $\mathbf{u}_1$ directed as $(A - O)$. Compute the tensor of inertia σ_S of the triangle with respect to S.

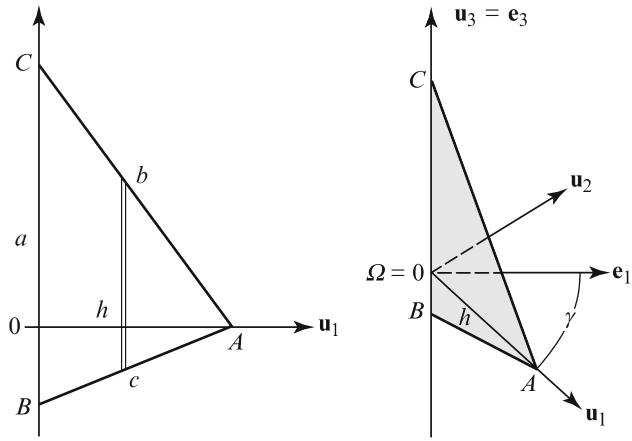

Fig. 6.2c.

In computing I_{33}, the triangle is regarded as the infinite union of contiguous rectangles of infinitesimal width dx_1 whose base is parallel to BC. Such a rectangle is at distance x_1 from BC and has infinitesimal mass $\rho a(h - x_1)dx_1/h$. Therefore

$$I_{33} = \frac{\rho a}{h} \int_0^h x_1^2 (h - x_1) dx_1 = \tfrac{1}{6} m h^2.$$

In computing I_{11}, the triangle is regarded as the union of the triangles $\Delta_1(OAB)$ and $\Delta_2(AOC)$. Therefore using the previous calculation,

$$\begin{aligned} I_{11} &= \rho \int_{\Delta_1} x_3^2 dx_1 dx_3 + \rho \int_{\Delta_2} x_3^2 dx_1 dx_3 \\ &= \tfrac{1}{6} m_1 \big(a - \sqrt{b^2 - h^2}\big)^2 + \tfrac{1}{6} m_2 (b^2 - h^2) \\ &= \tfrac{1}{6} m \big[a^2 - 3\sqrt{b^2 - h^2}\big(a - \sqrt{b^2 - h^2}\big)\big]. \end{aligned}$$

As for I_{13}, we compute

$$\begin{aligned} I_{13} &= -\rho \int_{\Delta_1} x_1 x_3 dx_1 dx_3 - \rho \int_{\Delta_2} x_1 x_3 dx_1 dx_3 \\ &= \tfrac{1}{24} \rho h^2 \big[\big(a - \sqrt{b^2 - h^2}\big)^2 - (b^2 - h^2)\big] \\ &= \tfrac{1}{6} m h \big(\tfrac{1}{2} a - \sqrt{b^2 - h^2}\big). \end{aligned}$$

Finally, since the system is planar, $I_{12} = 0$ and $I_{22} = I_{11} + I_{33}$.

6.3c Homogeneous Material Right Triangle

If $\Delta(ABC)$ is a right triangle with hypotenuse b and legs h and $a = \sqrt{b^2 - h^2}$, then $B = O$ and

$$\sigma_S = \frac{1}{6}m \begin{pmatrix} a^2 & 0 & -\frac{1}{2}ah \\ 0 & a^2 + h^2 & 0 \\ -\frac{1}{2}ah & 0 & h^2 \end{pmatrix}.$$

6.4c Spiral Ramp

Compute the moment of inertia with respect to the x_3-axis of the homogeneous material spiral ramp given in cylindrical coordinates by $x_3 = \theta$, $\rho \in [0,1]$, $\theta \in [0,\pi]$.

5

SYSTEMS DYNAMICS

1 Flow Map and Derivatives of Integrals

Let $G(t)$ be the configuration at time t of a material system $\{\mathcal{M}; d\mu\}$ in motion from its initial configuration G_o. Every point $P_o \in G_o$ follows its trajectory to arrive at the position $P(t) \in G(t)$ at time t; vice versa, a point $P \in G(t)$ may be regarded as originating from the motion of some $P_o \in G_o$. The *flow map* is the transformation

$$P = \Phi(P_o; t) : G_o \longrightarrow G(t); \qquad \Phi \in C^\infty(G_o \times I),$$

where I is the interval of time on which where the motion is defined. It is assumed that for each fixed $t \in I$, the flow map is a smooth bijection between G_o and $G(t)$, with smooth inverse. From the definition,

$$\dot{P} = \Phi_t(P_o; t), \quad \ddot{P} = \Phi_{tt}(P_o; t), \quad \dots \quad \text{for all } t \in I.$$

Kinematic and dynamic information on the motion are expressed by scalar- or vector-valued smooth functions $(P, \dot{P}; t) \to \mathcal{F}(P, \dot{P}; t)$. These can be regarded as defined in $G_o \times I$ by composing them with the flow map

$$\mathcal{F}(P, \dot{P}; t) = \mathcal{F}\left(\Phi(P_o; t), \Phi_t(P_o; t); t\right).$$

Consider material systems for which the elemental mass $d\mu(P)$ about P is the same as the elemental mass $d\mu(P_o)$ about the initial position P_o, e.g.,

$$d\mu(P) = d\mu\big(\Phi(P_o; t)\big) = d\mu(P_o).$$

This occurs, for example, if $\{\mathcal{M}; d\mu\}$ is discrete or a nondeformable continuum.[1] In such a case

[1] If it is a continuum, then $d\mu(P) = \rho(P)dV(P)$, where $\rho(\cdot)$ is the density and $dV(P)$ is the Lebesgue measure of an elemental volume about P. If it is nondeformable, the configuration $G(t)$ is obtained from G_o by a rigid motion, so that the Jacobian of the transformation is one.

E. DiBenedetto, *Classical Mechanics: Theory and Mathematical Modeling*, Cornerstones, DOI 10.1007/978-0-8176-4648-6_5,

$$\int_{G(t)} \mathcal{F}(P,\dot{P};t)d\mu(P) = \int_{G_o} \mathcal{F}\left(\Phi,\Phi_t;t\right)d\mu(P_o).$$

Proposition 1.1 *Let $\{\mathcal{M};d\mu\}$ be nondeformable or discrete. Then*

$$\frac{d}{dt}\int_{G(t)} \mathcal{F}(P,\dot{P};t)d\mu(P) = \int_{G(t)} \frac{d}{dt}\mathcal{F}(P,\dot{P};t)d\mu(P).$$

The proof is a direct consequence of the definition and the assumed smoothness of the flow map and $\mathcal{F}$. In what follows we consider only discrete systems or nondeformable continua, so that the proposition is in force.

2 General Theorems of System Dynamics

2.1 D'Alembert's Principle

A material system $\{\mathcal{M};d\mu\}$ is regarded as the union of point masses $\{P;d\mu(P)\}$ of elemental mass $d\mu(P)$. If $\{\mathcal{M};d\mu\}$ is in motion, each of its points is acted upon by forces

$$\mathbf{f}(P,\dot{P};t)d\mu(P) = \left[\mathbf{f}^{(\mathrm{i})}(P,\dot{P};t) + \mathbf{f}^{(\mathrm{e})}(P,\dot{P};t)\right]d\mu(P).$$

The function $\mathbf{f}$ is smooth and it has the physical dimensions of a force per unit of mass. The force $\mathbf{f}d\mu(P)$ is the resultant of forces $\mathbf{f}^{(\mathrm{i})}d\mu(P)$ internal to the system and forces $\mathbf{f}^{(\mathrm{e})}d\mu(P)$ external to it. The former may be regarded as solicitations exerted on $\{P;d\mu(P)\}$ by the remaining points of the system, whereas the latter are generated by external causes. If $\{\mathcal{M};d\mu\}$ is subject to constraints, these will exert on each $\{P;d\mu(P)\}$ a reaction $\mathbf{r}(P,\dot{P};t)d\mu(P)$, which in general is unknown. Since these are due to causes external to the system, they are compounded in $\mathbf{f}^{(\mathrm{e})}d\mu(P)$. The total force acting on $\{\mathcal{M};d\mu\}$ is $\mathbf{F} = \mathbf{F}^{(\mathrm{i})} + \mathbf{F}^{(\mathrm{e})}$, where

$$\mathbf{F}^{(\mathrm{e,i})} = \int \mathbf{f}^{(\mathrm{e,i})}(P,\dot{P};t)d\mu(P).$$

The moment of $\mathbf{f}d\mu(P)$ with respect to a pole O is

$$(P-O)\wedge\left(\mathbf{f}^{(\mathrm{i})} + \mathbf{f}^{(\mathrm{e})}\right)d\mu(P).$$

The total resulting moment is $\mathbf{M} = \mathbf{M}^{(\mathrm{i})} + \mathbf{M}^{(\mathrm{e})}$, where

$$\mathbf{M}^{(\mathrm{e,i})} = \int (P-O)\wedge\mathbf{f}^{(\mathrm{e,i})}d\mu(P).$$

By Newton's third law, the system of the internal forces consists of couples of zero wrench. Therefore $\mathbf{F}^{(\mathrm{i})} = \mathbf{M}^{(\mathrm{i})} = 0$.

The d'Alembert principle states that every $\{P; d\mu(P)\}$ making up $\{\mathcal{M}; d\mu\}$ moves according to Newton's first law, as if it were removed from $\{\mathcal{M}; d\mu\}$, provided the internal forces are included in the compound of forces acting on, it, e.g., [36],

$$\ddot{P} d\mu(P) = \big(\mathbf{f}^{(\mathrm{i})} + \mathbf{f}^{(\mathrm{e})}\big) d\mu(P). \tag{2.1}$$

Its relevance is in that every dynamic problem can be regarded as a problem of pointwise equilibrium of all forces acting on every point of the system, provided one includes the internal and external forces, including possible reactions due to constraints, and inertial forces. From this, integrating in $d\mu(P)$,

$$\mathbf{F}^{(\mathrm{e})} = \int \ddot{P}\, d\mu(P) \qquad\qquad (\text{since } \mathbf{F}^{(\mathrm{i})} = 0). \tag{2.2}$$

2.2 Momentum

The momentum of the elemental point mass is $\dot{P} d\mu(P)$. Integrating gives the momentum of the system

$$\mathbf{Q} = \int \dot{P} d\mu(P).$$

Theorem 2.1 (of the Momentum). *The derivative of the momentum of the system $\{\mathcal{M}; d\mu\}$ equals the resultant $\mathbf{F}^{(e)}$ of all external forces acting on it.*

Proof. Take the derivative in the expression of $\mathbf{Q}$ and use Proposition 1.1 and (2.2) to get

$$\dot{\mathbf{Q}} = \int \ddot{P}\, d\mu(P) = \mathbf{F}^{(\mathrm{e})}.$$ ■

Theorem 2.2 (Center of Mass).

(i) *The momentum of the system equals the momentum of the center of mass P_o where ideally the entire mass m of $\{\mathcal{M}; d\mu\}$ is concentrated, e.g., $\mathbf{Q} = m\dot{P}_o$.*

(ii) *The center of mass P_o moves as a point mass $\{P_o; m\}$ solicited by the resultant $\mathbf{F}^{(e)}$ of the external forces, e.g., $m\ddot{P}_o = \mathbf{F}^{(e)}$.*

Proof. From the equation of the center of mass, by taking the time derivative,

$$m\dot{P}_o = \int \dot{P} d\mu(P).$$

This implies **(i)**. Differentiating a second time and taking into account the theorem of the momentum proves **(ii)**. ■

2.3 Angular Momentum

The angular momentum of the elemental point mass $\{P; d\mu(P)\}$ with respect to a pole O is

$$d\mathbf{K} = (P - O) \wedge \dot{P} d\mu(P).$$

Integrating in $d\mu(P)$ gives the resultant angular momentum

$$\mathbf{K} = \int (P - O) \wedge \dot{P} d\mu(P).$$

The point O might be fixed or mobile and it might or might not belong to the system. By taking the derivative of $\mathbf{K}$, we have

$$\begin{aligned} \dot{\mathbf{K}} &= \int (P - O) \wedge \left(\mathbf{f}^{(\mathrm{i})} + \mathbf{f}^{(\mathrm{e})}\right) d\mu(P) - m\dot{O} \wedge \dot{P}_o \\ &= \mathbf{M}^{(\mathrm{e})} + m\dot{P}_o \wedge \dot{O}. \end{aligned}$$

The last term vanishes if $O = P_o$ or if O is fixed.

Theorem 2.3 (of the Angular Momentum). *The time derivative of the resultant angular momentum of a system $\{\mathcal{M}; d\mu\}$, taken with respect to the center of mass P_o of a fixed point O, equals the resultant total moment $\mathbf{M}^{(e)}$ of the external forces, taken with respect to the same point.*

Remark 2.1 If $\{\mathcal{M}; d\mu\}$ is isolated then $\mathbf{M}^{(e)} = 0$, and if moments are taken with respect to the center of mass P_o, then $\mathbf{K} = \text{const}$. Therefore if $\mathbf{K} \neq 0$, the plane through P_o and normal $\mathbf{K}$ is constant. The solar system is approximately isolated and its center of mass is approximately in the Sun. The plane through the Sun and normal $\mathbf{K}$ was called by Laplace *the invariant plane of the solar system* ([109]; see also §§4–5 of Chapter 3).

2.4 Energy

The kinetic energy of an elemental point mass is $\frac{1}{2}\dot{P}^2 d\mu(P)$. Integrating in $d\mu(P)$ gives the energy of the system

$$T = \frac{1}{2} \int \dot{P}^2 d\mu(P).$$

Taking the differential in dt yields

$$\begin{aligned} dT &= \int \dot{P} \cdot \ddot{P} d\mu(P) = \int \left[\mathbf{f}^{(\mathrm{i})}(P, \dot{P}; t) + \mathbf{f}^{(\mathrm{e})}(P, \dot{P}; t)\right] \cdot \dot{P} d\mu(P) \\ &= dL^{(\mathrm{i})} + dL^{(\mathrm{e})}, \end{aligned}$$

where the differential

$$dL^{(\mathrm{e,i})} = \int \mathbf{f}^{(\mathrm{e,i})} \cdot dP d\mu(P)$$

is the elemental work done by all the internal and external forces respectively. Integrating over the time interval (t_o, t) gives $T(t) - T(t_o) = L$, where L is the work done by the internal and external forces in the same time interval. Notice that the internal forces being in equilibrium ($\mathbf{F}^{(\mathrm{i})} = \mathbf{M}^{(\mathrm{i})} = 0$) does not imply that the system of internal forces is workless.

Theorem 2.4 (Kinetic Energy). *The variation in kinetic energy in a time interval* (t_o, t) *equals the total work done in the same time interval by the internal and external forces acting on* $\{\mathcal{M}; d\mu\}$.

If the force densities $\mathbf{f}^{(\mathrm{i,e})}$ are conservative, then $\mathbf{f}^{(\mathrm{e,i})\mathrm{i}} = \nabla U^{(\mathrm{e,i})}$ for two smooth functions $P \to U^{(\mathrm{e,i})}(P)$. Setting $U = U^{(\mathrm{i})} + U^{(\mathrm{e})}$, the differential form of the kinetic energy implies

$$d(T - U) = 0, \qquad \text{e.g.,} \qquad t \to [T(t) - U(t)] = \text{const.}$$

The function $t \to -U(t)$ is the *potential energy* of the system. Thus the sum of the kinetic and potential energies of the system is conserved along the motion.

3 Cardinal Equations

A consequence of the previous remarks is that the motion of a material system $\{\mathcal{M}; d\mu\}$ must satisfy the equations

$$m\ddot{P}_o = \mathbf{F}^{(\mathrm{e})} = \dot{\mathbf{Q}}, \tag{3.1}$$

$$\dot{\mathbf{K}} = \mathbf{M}^{(\mathrm{e})} + m\dot{P}_o \wedge \dot{O} = \mathbf{M}^{(\mathrm{e})} + \mathbf{Q} \wedge \dot{O}. \tag{3.2}$$

If the pole O is either fixed or chosen as the center of mass P_o, then $\dot{\mathbf{K}} = \mathbf{M}^{(\mathrm{e})}$.

The resultant $\mathbf{F}^{(\mathrm{e})}$ is inclusive of possible reactions due to constraints; similarly, $\mathbf{M}^{(\mathrm{e})}$ includes the moments of such possible reactions.

The general problem of system dynamics is to determine, starting from some initial mechanical configuration of a system $\{\mathcal{M}; d\mu\}$, its evolution and the reactions that constrain its evolving configurations.

Equations (3.1)–(3.2) are *necessary* for any system $\{\mathcal{M}; d\mu\}$ and for this reason are called the *cardinal* equations of dynamics. However, they are not sufficient for the actual determination of the motion. Indeed, they are global in nature (e.g., integral) and provide no information on the motion of the single points of the system. They consist of six scalar equations and thus sufficient only for a mechanical system with at most six degrees of freedom. For example, they are necessary and sufficient to determine the motion of an *unconstrained* rigid system. The cardinal equations involve only free vectors in space, and as such they are independent of a reference system. However, having been derived from Newton's laws, they have mechanical meaning only when referred to an inertial system $\Sigma = \{\Omega; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, which we assume has been selected and fixed.

3.1 Cardinal Equations in Noninertial Systems

Let $S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a triad in rigid motion with respect to Σ with given characteristics $\mathbf{v}(O)$ and $\boldsymbol{\omega}$. Given a vector-valued smooth function $t \to \mathbf{v}(t)$, denote by $x_j(t)$ its components in S, so that $\mathbf{v} = x_j \mathbf{u}_j$. By differentiation,

$$\dot{\mathbf{v}} = \dot{x}_j \mathbf{u}_j + \boldsymbol{\omega} \wedge \mathbf{v} = \left(\frac{d\mathbf{v}}{dt}\right)_S + \boldsymbol{\omega} \wedge \mathbf{v},$$

where $(d\mathbf{v}/dt)_S$ denotes the derivative of $\mathbf{v}$ relative to S. Applying such a differentiation rule to the vectors $\mathbf{Q}$ and $\mathbf{K}$ in (3.1)–(3.2) gives

$$\left(\frac{d\mathbf{Q}}{dt}\right)_S + \boldsymbol{\omega} \wedge \mathbf{Q} = \mathbf{F}^{(e)}, \qquad \left(\frac{d\mathbf{K}}{dt}\right)_S + \boldsymbol{\omega} \wedge \mathbf{K} = \mathbf{M}^{(e)}. \tag{3.3}$$

3.2 Motion Relative to the Center of Mass

Let S be a triad with origin at the center of mass P_o and translating with respect to Σ with velocity $\dot{P}_o$. The motion of $\{\mathcal{M}; d\mu\}$ described with respect to S is called *relative to the center of mass* [51,65]. By the formula for relative velocity,

$$\dot{P} = (\dot{P})_S + \dot{P}_o, \qquad \text{e.g.,} \qquad \mathbf{v}_\Sigma(P) = \mathbf{v}_S(P) + \mathbf{v}_\Sigma(P_o).$$

Therefore the momentum of the system is

$$\mathbf{Q} = \int (\dot{P})_S d\mu(P) + m\dot{P}_o.$$

By the theorem of the center of mass, $m\dot{P}_o = \mathbf{Q}$. Therefore

$$\mathbf{Q}_S = \int (\dot{P})_S d\mu(P) = 0. \tag{3.4}$$

The elemental angular momentum, taken with respect to a fixed point $O \in \Sigma$, is given by

$$(P - O) \wedge \big[(\dot{P})_S + \dot{P}_o\big] d\mu(P).$$

By integration in $d\mu(P)$,

$$\begin{aligned}\mathbf{K} &= \int (P - O) \wedge (\dot{P})_S d\mu(P) + \int (P - O) \wedge \dot{P}_o d\mu(P) \\ &= \mathbf{K}_S + (P_o - O) \wedge m\dot{P}_o \\ &= \mathbf{K}_S + (P_o - O) \wedge \mathbf{Q}.\end{aligned}$$

3.3 König's Theorem [24, 96]

The kinetic energy of $\{\mathcal{M}; d\mu\}$, expressed in terms of the motion relative to the center of mass is

$$\begin{aligned} 2T &= \int \left[(\dot{P})_S + \dot{P}_o\right]^2 d\mu(P) \\ &= \int (\dot{P})_S^2 d\mu(P) + \int \dot{P}_o^2 d\mu(P) + 2\int (\dot{P})_S \dot{P}_o d\mu(P). \end{aligned}$$

From (3.4) it follows that the last term is zero. Therefore

$$T = \frac{1}{2} m \dot{P}_o^2 + T_S, \qquad \text{where} \qquad T_S = \frac{1}{2}\int (\dot{P})_S^2 d\mu(P). \tag{3.5}$$

The first term on the right-hand side is the kinetic energy of the center of mass, where one ideally concentrates the entire mass of the system and T_S is the kinetic energy relative to S. Formula (3.5) is König's theorem.

4 Mechanical Quantities of Rigid Systems

If $\{\mathcal{M}; d\mu\}$ is rigid, choose an inertial triad Σ and a triad $S = \{\Omega; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ fixed with the system, in rigid motion with respect to Σ with characteristics $\mathbf{v}(O)$ and $\boldsymbol{\omega}$. The momentum of the elemental point mass $\{P; d\mu(P)\}$ is

$$[\mathbf{v}(O) + \boldsymbol{\omega} \wedge (P - O)]\, d\mu(P).$$

From this by integration,

$$\mathbf{Q} = m\,[\mathbf{v}(O) + \boldsymbol{\omega} \wedge (P_o - O)] = m\dot{P}_o.$$

The angular momentum of $\{P; d\mu(P)\}$ with respect to a pole O is

$$(P - O) \wedge [\mathbf{v}(O) + \boldsymbol{\omega} \wedge (P - O)]\, d\mu(P),$$

and by integration,

$$\mathbf{K} = m(P_o - O) \wedge \mathbf{v}(O) + \int (P - O) \wedge \left[\boldsymbol{\omega} \wedge (P - O)\right] d\mu(P).$$

Write $(P - O) = x_i \mathbf{u}_i$ and $\boldsymbol{\omega} = \omega_i \mathbf{u}_i$ in the coordinates of S, and compute

$$\begin{aligned} (P - O) \wedge \big(\boldsymbol{\omega} \wedge (P - O)\big) = &\left[\left(x_2^2 + x_3^2\right)\omega_1 - x_1 x_2 \omega_2 - x_1 x_3 \omega_3\right] \mathbf{u}_1 \\ &+ \left[-x_1 x_2 \omega_1 + \left(x_1^2 + x_3^2\right)\omega_2 - x_2 x_3 \omega_3\right] \mathbf{u}_2 \\ &+ \left[-x_1 x_3 \omega_1 - x_2 x_3 \omega_2 + \left(x_1^2 + x_2^2\right)\omega_3\right] \mathbf{u}_3. \end{aligned}$$

Therefore

$$\int (P-O)\wedge\big[\boldsymbol{\omega}\wedge(P-O)\big]\,d\mu(P) = (I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3)\,\mathbf{u}_1 + (I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3)\,\mathbf{u}_2 + (I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3)\,\mathbf{u}_3,$$

where $(-1)^{1+\delta_{ij}} I_{ij}$ are the moments of inertia of $\{\mathcal{M}; d\mu\}$, axial $(i = j)$, or deflection $(i \neq j)$, computed in S. Thus

$$\mathbf{K} = m(P_o - O)\wedge\mathbf{v}(O) + \sigma\boldsymbol{\omega}, \tag{4.1}$$

where $\sigma = (I_{ij})$ is the inertia tensor of $\{\mathcal{M}; d\mu\}$ with respect to the triad S fixed with $\{\mathcal{M}; d\mu\}$. If the system has a fixed point, choosing the origin of S to coincide with such a fixed point, the first term in the right-hand side of (4.1) is zero. Such a term also vanishes it the origin of S is at the center of mass of $\{\mathcal{M}; d\mu\}$. In either of these cases (4.1) becomes

$$\mathbf{K} = \sigma_o\boldsymbol{\omega}, \tag{4.1$'$}$$

where σ_o is the inertia tensor of $\{\mathcal{M}; d\mu\}$ with respect to the fixed triad S with origin either in a fixed point O or in the center of mass P_o. If S is principal of inertia relative to O, then

$$\mathbf{K} = \mathcal{I}_1\omega_1\mathbf{u}_1 + \mathcal{I}_2\omega_2\mathbf{u}_2 + \mathcal{I}_3\omega_3\mathbf{u}_3, \tag{4.1$''$}$$

where $\mathcal{I}_j$ are the axial moments of inertia with respect to the coordinate principal axes through O. These formulas continue to hold instantaneously, e.g., for elemental motions for which $\mathbf{v}(O) = 0$. This occurs, for example, if O is a point of the instantaneous axis of rotation and in addition $\mathbf{v}(O)\cdot\boldsymbol{\omega} = 0$.

4.1 Kinetic Energy

The kinetic energy of the elemental point mass $\{P; d\mu(P)\}$ is

$$dT = \tfrac{1}{2}[\mathbf{v}(O) + \boldsymbol{\omega}\wedge(P-O)]^2 d\mu(P),$$

and by integration,

$$2T = m\mathbf{v}^2(O) + 2m\mathbf{v}(O)\cdot\boldsymbol{\omega}\wedge(P_o - O) + \int \|(P-O)\wedge\boldsymbol{\omega}\|^2\,d\mu(P). \tag{4.2}$$

Let $\ell(O;\boldsymbol{\omega})$ be the axis through O with direction $\boldsymbol{\omega}$ and let $I_{\ell(O;\boldsymbol{\omega})}$ denote the moment of inertia of $\{\mathcal{M}; d\mu\}$ with respect to such an axis. From the definition of axial moment of inertia,

$$\int \|(P-O)\wedge\boldsymbol{\omega}\|^2\,d\mu(P) = I_{\ell(O;\boldsymbol{\omega})}\|\boldsymbol{\omega}\|^2.$$

With this notation,

$$2T = m\mathbf{v}^2(O) + 2m\mathbf{v}(O) \cdot \boldsymbol{\omega} \wedge (P_o - O) + I_{\ell(O;\boldsymbol{\omega})}\|\boldsymbol{\omega}\|^2. \qquad (4.2)'$$

Corollary 4.1 *If $\{\mathcal{M}; d\mu\}$ has a fixed point O, then*

$$2T = I_{\ell(O;\boldsymbol{\omega})}\|\boldsymbol{\omega}\|^2 = \omega^t \sigma_o \omega = \mathbf{K} \cdot \boldsymbol{\omega}, \qquad (4.3)$$

where σ_o is the inertia tensor with respect to a triad S fixed with $\{\mathcal{M}; d\mu\}$ and with origin in O.

Another special case occurs if the origin of S coincides with the center of mass of the system.

Corollary 4.2 *Let S_{P_o} be a triad fixed with $\{\mathcal{M}; d\mu\}$ and with origin at the center of mass P_o of the system. Then*

$$\begin{aligned} 2T &= m\dot{P}_o^2 + I_{\ell(P_o;\boldsymbol{\omega})}\|\boldsymbol{\omega}\|^2 \\ &= m\dot{P}_o^2 + \boldsymbol{\omega}^t \sigma_{P_o} \boldsymbol{\omega} = m\dot{P}_o^2 + \mathbf{K} \cdot \boldsymbol{\omega}, \end{aligned} \qquad (4.4)$$

where σ_{P_o} is the tensor of inertia of $\{\mathcal{M}; d\mu\}$ with respect to S_{P_o} and $I_{\ell(P_o;\boldsymbol{\omega})}$ is the axial moment of inertia with respect to the axis $\ell(P_o; \boldsymbol{\omega})$, through P_o and directed as $\boldsymbol{\omega}$.

Remark 4.1 If O is a point of the instantaneous axis of rotation, then $\mathbf{v}(O)$ is parallel to $\boldsymbol{\omega}$ and (4.2)′ takes the form

$$2T = m\mathbf{v}^2(O) + I_{\ell(O;\boldsymbol{\omega})}\|\boldsymbol{\omega}\|^2, \qquad O \in \{\text{axis of motion}\}. \qquad (4.2)''$$

Remark 4.2 By Huygens's theorem,

$$I_{\ell(O;\boldsymbol{\omega})} = I_{\ell(P_o;\boldsymbol{\omega})} + \frac{m\|(P_o - O) \wedge \boldsymbol{\omega}\|^2}{\|\boldsymbol{\omega}\|^2} \qquad (\boldsymbol{\omega} \neq 0).$$

Therefore (4.2) can be given the form

$$\begin{aligned} 2T =& m\mathbf{v}^2(O) + 2m\mathbf{v}(O) \cdot \boldsymbol{\omega} \wedge (P_o - O) \\ &+ I_{\ell(P_o;\boldsymbol{\omega})}\|\boldsymbol{\omega}\|^2 + m\|(P_o - O) \wedge \boldsymbol{\omega}\|^2. \end{aligned} \qquad (4.2)'''$$

If O is a point of the instantaneous axis of rotation, then

$$2T = m\mathbf{v}^2(O) + I_{\ell(P_o;\boldsymbol{\omega})}\|\boldsymbol{\omega}\|^2 + m\|(P_o - O) \wedge \boldsymbol{\omega}\|^2. \qquad (4.2)^{\text{iv}}$$

5 Workless Constraints: Discrete Systems

A discrete material system $\{\mathcal{M}; d\mu\}$ consisting of n point masses $\{P_\ell; m_\ell\}$ is subject to m independent constraints $[f_j = 0]$. These exert reactions $\mathbf{R}_\ell$ on the points $\{P_\ell; m_\ell\}$, which are in general unknown. Assuming $m < 3n$, the equations of motion are

$$\begin{aligned} m_\ell \ddot{P}_\ell &= \mathbf{F}_\ell + \mathbf{R}_\ell, & \ell &= 1, 2, \dots, n, \\ f_j(P_1, \dots, P_n; t) &= 0, & j &= 1, 2, \dots, m < 3n, \end{aligned} \tag{5.1}$$

where $\mathbf{F}_\ell$ includes the internal and external forces acting on P_ℓ, but not the reactions due to the constraints. This system consists of $(3n+m)$ scalar equations with the $6n$ scalar unknowns $P_\ell(t)$ and $\mathbf{R}_\ell(t)$ and therefore is not sufficient to describe the motion of $\{\mathcal{M}; d\mu\}$, and further information is needed on the nature of the constraints. The elemental virtual work done by $\mathbf{R}_\ell$ during an elemental virtual displacement δP_ℓ compatible with the constraints is $\mathbf{R}_\ell \cdot \delta P_\ell$. The elemental virtual work done by the system of the reactions is

$$\delta \Lambda = \sum_{\ell=1}^{n} \mathbf{R}_\ell \cdot \delta P_\ell .$$

The constraints $[f_j = 0]$ are *smooth* or *workless* if[2]

$$\delta \Lambda = 0 \qquad \text{for all virtual displacements compatible with the constraints.} \tag{5.2}$$

Theorem 5.1 (Lagrange [101] Chapter XV). *Let the constraints $[f_j = 0]$ be workless. Then there exist m constants λ_j such that*

$$\mathbf{R}_\ell = \sum_{j=1}^{m} \lambda_j \nabla_{P_\ell} f_j, \qquad \ell = 1, 2, \dots, n. \tag{5.3}$$

Corollary 5.1 *The equations of motion of a discrete system subject to workless constraints are*

$$\begin{aligned} m_\ell \ddot{P}_\ell &= \mathbf{F}_\ell + \sum_{j=1}^{n} \lambda_j \nabla_{P_\ell} f_j, & \ell &= 1, 2, \dots, n, \\ f_j(P_1, P_2, \dots, P_n; t) &= 0, & j &= 1, 2, \dots, m < 3n. \end{aligned} \tag{5.4}$$

These are $(3n+m)$ scalar equations in the $(3n+m)$ unknowns $t \to P_\ell(t)$ for $\ell = 1, \dots, n$ and $t \to \lambda_j(t)$ for $j = 1, \dots, m$. Therefore the motion of the system and its reactions are determined from its initial conditions.

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[2]A more general notion of *workless* that would include unilateral constraints would be that $\delta \Lambda \geq 0$ for every elemental virtual displacement δP compatible with the constraints.

5.1 Proof of Lagrange's Theorem

Relabel the coordinates of the points P_ℓ by setting

$$\begin{array}{lll} y_1 = x_{1,1}, & y_2 = x_{1,2}, & y_3 = x_{1,3}, \\ y_4 = x_{2,1}, & y_5 = x_{2,2}, & y_6 = x_{2,3}, \\ \dots & \dots & \dots \\ y_{3n-2} = x_{n,1}, & y_{3n-1} = x_{n,2}, & y_{3n} = x_{n,3}. \end{array}$$

Likewise relabel the components of reactions $\mathbf{R}_\ell$ as

$$\begin{array}{lll} R_1 = R_{1,1}, & R_2 = R_{1,2}, & R_3 = R_{1,3}, \\ R_4 = R_{2,1}, & R_5 = R_{2,2}, & R_6 = R_{2,3}, \\ \dots & \dots & \dots \\ R_{3n-2} = R_{n,1}, & R_{3n-1} = R_{n,2}, & R_{3n} = R_{n,3}. \end{array}$$

With this notation, an elemental virtual displacement is denoted by

$$\delta P = (\delta P_1, \dots, \delta P_n) = (\delta y_1, \dots, \delta y_{3n}) = \delta y.$$

Compatibility of such a displacement with the constraints and the assumption that the constraints are workless are written in the form

$$\sum_{i=1}^{m} \frac{\partial f_j}{\partial y_i} \delta y_i = - \sum_{s=m+1}^{3n} \frac{\partial f_j}{\partial y_s} \delta y_s, \quad j = 1, \dots, m, \tag{5.5}$$

$$\sum_{i=1}^{m} R_i \delta y_i = - \sum_{s=m+1}^{3n} R_s \delta y_s. \tag{5.6}$$

Since the constraints are independent, the matrix $(\partial f_j / \partial y_i)$ has rank m. Without loss of generality assume that the minor of the first m rows and m columns has nonzero determinant. The first m equations in (5.5) can be regarded as a linear algebraic system in the m unknowns δy_i, $i = 1, \dots, m$. These unknowns are uniquely determined by an arbitrary choice of the $(3n - m)$-tuple δy_s for $s = m+1, \dots, 3n$. The assumption states that every solution δy of (5.5) is also a solution of (5.6). This implies that the last equation of the system (5.5)–(5.6) must be a linear combination of the preceeding ones. Therefore there exist m real parameters λ_j such that

$$R_i = \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial y_i}, \qquad i = 1, \dots, m,$$

and

$$\sum_{s=(m+1)}^{3n} R_s \delta y_s = \sum_{j=1}^{m} \lambda_j \left(\sum_{s=m+1}^{3n} \frac{\partial f_j}{\partial y_s} \delta y_s \right).$$

Rewrite the last relation as

$$\sum_{s=m+1}^{3n} \left(R_s - \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial y_s} \right) \delta y_s = 0$$

and observe that the $(3n-m)$-tuple δy_s is arbitrary to deduce

$$R_s = \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial y_s}, \qquad s = m+1, \dots, 3n.$$

This implies the theorem by recalling the relabeling of the coordinates of P_ℓ and of the components of $\mathbf{R}_\ell$.

6 The Principle of Virtual Work

A constraint $[f=0]$ imposed on a point mass $\{P;m\}$ is *smooth* if it generates reactions normal to the surface $[f=0]$. This in turn implies that the work done by the reaction is zero for every elemental virtual displacement δP compatible with the constraint. Conversely, if the virtual work of the reaction is zero for every elemental virtual displacement δP compatible with the constraint, then the reaction is normal to the surface $[f=0]$. The theorem of Lagrange asserts that for a discrete system of n points subject to m constraints, the constraints are *workless* if and only if the reactions $\mathbf{R}_\ell$ due to the constraints are normal to the intersection of all surfaces $[f_j=0]$. Equivalently, the constraints are *smooth* if and only if the total work done by the reactions $\mathbf{R}_\ell$ is zero for every virtual displacement δP_ℓ of the points P_ℓ compatible with the constraints. This notion of *smooth* constraints will be adopted for all mechanical systems, discrete or not. We will say that a set of constraints imposed on the motion of a material system $\{\mathcal{M};d\mu\}$ is *smooth* or *workless* if the total work done by the reactions is zero for every virtual displacement of the system compatible with the constraints. Such an assumption placed on the constraints is the *principle of virtual work.*

6.1 The Principle of Virtual Work for Rigid Systems

The principle of virtual work as expressed by (5.2) is satisfied by the rigidity constraints of a rigid system $\{\mathcal{M};d\mu\}$. Let S be a triad fixed with $\{\mathcal{M};d\mu\}$ and in rigid motion with respect to an inertial triad Σ. The rigidity constraints require that the mutual distance of any two points $P,Q\in S$ be constant in time. The constraints are time independent, so that virtual and actual displacements coincide. These constraints generate on P a reaction

$$\mathbf{R}(P,Q) = \gamma \frac{P-Q}{\|P-Q\|} \qquad \text{for some } \gamma \in \mathbb{R}.$$

By Newton's third law the constraints generate on Q a reaction $-\mathbf{R}$. Therefore the work done by the reactions acting on the pair of points $P,Q\in S$ is

$$\begin{aligned}\delta\Lambda(P,Q) &= \gamma\frac{P-Q}{\|P-Q\|}\delta P - \gamma\frac{P-Q}{\|P-Q\|}\delta Q \\ &= \gamma\frac{(P-Q)\cdot\delta(P-Q)}{\|P-Q\|} = \gamma\delta\|P-Q\| = 0.\end{aligned}$$

Problems and Complements

3c Cardinal Equations

3.1c Disk Rolling on a Slanted Guide as in Figure 3.1c

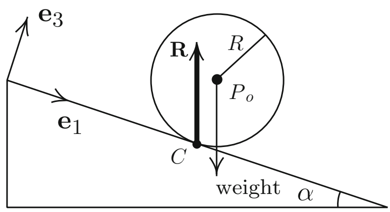

Fig. 3.1c.

It is a rigid motion of characteristics $\mathbf{v}_{P_o}$ and $\boldsymbol{\omega}$. The center of mass P_o moves on a line parallel to the guide, with acceleration $g \sin\alpha \mathbf{e}_1$. Therefore $\mathbf{v}(P_o) = gt \sin\alpha \mathbf{e}_1$. The angular momentum with respect to P_o is $\mathbf{K} = \frac{1}{2}MR^2\boldsymbol{\omega}$. If the constraint is smooth, the reaction $\mathbf{R}$ is on the normal to the guide through P_o and the weight is applied in P_o. Therefore $\dot{\mathbf{K}} = 0$ and $\boldsymbol{\omega} = \text{const}$. The velocity of C is computed from Poisson's formula, as $\mathbf{v}(C) = (g \sin\alpha t - R\omega)\mathbf{e}_1$, where $\omega = \|\boldsymbol{\omega}\|$. Thus if the guide is smooth, the disk never rolls without slipping. Assume now that the guide is rough and exerts a reaction $\mathbf{R}$ so that the disk rolls without slipping. Decomposing $\mathbf{R} = \mathbf{R}^{\|} + \mathbf{R}^{\perp}$ into its components, parallel and normal to the guide, impose that $\|\mathbf{R}^{\|}\| \le \gamma\|\mathbf{R}^{\perp}\|$, where γ is the friction coefficient. Let x be the abscissa of P_o on the guide. By the theorems of the momentum and of the center of mass,

$$M\ddot{x} = Mg \sin\alpha - \|\mathbf{R}^{\|}\|, \qquad \tfrac{1}{2}MR^2\dot{\omega} = R\|\mathbf{R}^{\|}\|.$$

Rolling without slipping implies $\|\dot{P}_o\| = \dot{x} = \omega R$. Therefore

$$\|\mathbf{R}^{\|}\| = \tfrac{1}{3}Mg \sin\alpha \le \gamma\mathbf{R}^{\perp} = \gamma Mg\cos\alpha \quad \Longrightarrow \quad \tan\alpha \le 3\gamma.$$

Therefore the constraints resulting from the guide and the "rolling without slipping" impose a limitation on α and the friction coefficient γ. Assuming the latter being satisfied, $\mathbf{v}(C) = 0$ and $\mathbf{R} \cdot \delta C = 0$. Therefore the resulting constraints are workless. The potential U and the kinetic energy T of the disk are

$$U = -Mgx \sin\alpha + \text{const}, \qquad T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{4}MR^2\omega^2 = \tfrac{3}{4}M\dot{x}^2.$$

In computing T we have used König's theorem. From the above, taking into account the expression of $\|\mathbf{R}^{\|}\|$,

$$\dot{E}_{\text{disk}} = M(\tfrac{3}{2}\ddot{x} - g\sin\alpha)\dot{x} = 0.$$

Thus the energy is conserved.

3.2c Rod Moving in a Plane with a Fixed Extreme

A rigid material rod of length ℓ is constrained to move, subject to its weight, in a vertical plane π with one of its extremities O kept fixed by a workless cylindrical hinge. The second extreme Q is connected to a spring, of elasticity constant k, whose fixed center C is on the horizontal through O and at distance ℓ from O. Assume that the density function on the rod is $\rho(P) = \|P - O\|$, and that initially $Q(0) = C$ and $\dot{Q}(0) = 0$.

Compute the momentum of the system and the angular momentum with respect to O, and write down the cardinal equations. Compute the period of the small oscillations and the reactions due to the constraints.

Let φ be the angle between $(C-O)$ and $(Q-O)$ and, for the generic point on the rod, set $x = \|P - O\|$. Every point P on the rod moves over a circle centered at O of radius x and with of unit tangent $\mathbf{t}(\varphi)$ and unit normal $\mathbf{n}(\varphi)$. Mass and position of the center of mass are computed from

$$m = \int_0^\ell x dx = \tfrac{1}{2}\ell^2, \qquad \tfrac{1}{2}\ell^2 x_o = \int_0^\ell x^2 dx = \tfrac{1}{3}\ell^3, \qquad P_o = \tfrac{2}{3}\ell\mathbf{n}.$$

The expression of the elastic force in terms of $\mathbf{t}$ and $\mathbf{n}$ is

$$\mathbf{F} = -k(Q - C) = -k\ell\left[\sin\varphi\mathbf{t} + (1 - \cos\varphi)\mathbf{n}\right].$$

Let $d\mathbf{w}$, $d\mathbf{Q}$, and $d\mathbf{K}$, denote the elemental weight, momentum, and angular momentum with respect to O of the element of rod contiguous to P. In terms of $\mathbf{t}$ and $\mathbf{n}$,

$$d\mathbf{w} = xg(\cos\varphi\mathbf{t} + \sin\varphi\mathbf{n})dx, \quad d\mathbf{Q} = \dot{\varphi}x^2\mathbf{t}dx, \quad d\mathbf{K} = \dot{\varphi}x^3\mathbf{n}\wedge\mathbf{t}dx.$$

By integration over the rod,

$$\mathbf{w} = \tfrac{1}{2}g\ell^2(\cos\varphi\mathbf{t} + \sin\varphi\mathbf{n}), \qquad \mathbf{Q} = \tfrac{1}{3}\ell^3\dot{\varphi}\mathbf{t}, \qquad \mathbf{K} = \tfrac{1}{4}\ell^4\dot{\varphi}\mathbf{e},$$

where $\mathbf{e} = \mathbf{n}\wedge\mathbf{t}$. The moment of $d\mathbf{w}$ with respect to O is $x^2 g\cos\varphi\mathbf{e}dx$, and by integration the moment of the weight is $\frac{1}{3}\ell^3 g\cos\varphi\mathbf{e}$. The moment of the elastic force with respect to O is $-k\ell^2\sin\varphi\mathbf{e}$. Next compute $\dot{P}_o = -\frac{2}{3}\ell\dot{\varphi}\mathbf{t}$ and $\ddot{P}_o = \frac{2}{3}\ell\left(\ddot{\varphi}\mathbf{t} - \dot{\varphi}^2\mathbf{n}\right)$. Therefore the first cardinal equation is

$$\begin{aligned}\tfrac{1}{3}\ell^3(\ddot{\varphi}\mathbf{t} - \dot{\varphi}^2\mathbf{n}) = &(-k\ell\sin\varphi + \tfrac{1}{2}g\ell^2\cos\varphi + R_{\mathbf{t}})\mathbf{t}\\ &+ (-k\ell(1 - \cos\varphi) + \tfrac{1}{2}g\ell^2\sin\varphi + R_{\mathbf{n}})\mathbf{n},\end{aligned}$$

where $R_{\mathbf{t}}$ and $R_{\mathbf{n}}$ are the components of the reactions due to the constraints along $\mathbf{t}$ and $\mathbf{n}$ respectively. In components,

$$\begin{aligned}\ddot{\varphi} &= \frac{3g}{2\ell}\cos\varphi - \frac{3k}{\ell^2}\sin\varphi + \frac{3}{\ell^3}R_{\mathbf{t}},\\ \dot{\varphi}^2 &= -\frac{3g}{2\ell}\sin\varphi + \frac{3k}{\ell^2}(1-\cos\varphi) - \frac{3}{\ell^3}R_{\mathbf{n}}.\end{aligned} \qquad (*)$$

By the theorem of angular momentum,

$$\frac{1}{4}\ell^4\ddot{\varphi} = \frac{1}{3}\ell^3 g\cos\varphi - k\ell^2\sin\varphi \implies \ddot{\varphi} = \frac{4g}{3\ell}\cos\varphi - \frac{4k}{\ell^2}\sin\varphi. \qquad (**)$$

Therefore

$$\frac{3}{\ell^3}R_{\mathbf{t}}(\varphi) = \frac{k}{\ell^2}\sin\varphi - \frac{g}{6\ell}\cos\varphi.$$

Multiply the second equation of $(**)$ by $\dot{\varphi}$ and integrate in dt. Using the initial conditions $\dot{\varphi}(0) = 0$ gives

$$\dot{\varphi}^2 = \frac{8g}{3\ell}\sin\varphi - \frac{8k}{\ell^2}(1-\cos\varphi).$$

From this and the second equation of $(*)$ one computes

$$\frac{3}{\ell^3}R_{\mathbf{n}} = -\frac{25g}{6\ell}\sin\varphi + \frac{11k}{\ell^2}(1-\cos\varphi).$$

3.3c Ring Sliding on a Material Spinning Segment

A homogeneous rod of length ℓ and mass M spins on a horizontal plane about its midpoint P_o with angular speed ω, starting from some angular speed ω_o. A point mass $\{P; m\}$ slides with no friction on the rod starting from P_o. Determine the velocity of P with respect to the rod in terms of its distance x from P_o.

The weight on P and the reactions are workless, so that the kinetic energy is conserved and equals

$$T = \tfrac{1}{2}I\omega_o^2, \qquad \text{where} \qquad I = \tfrac{1}{12}ML^2.$$

Therefore

$$(I + mx^2)\omega^2 + m\dot{x}^2 = I\omega^2, \quad \text{i.e.,} \quad \dot{x}^2 = I\omega_o^2 - (I + mx^2)\omega^2.$$

Since the angular momentum is conserved, $(I + mx^2)\omega = I\omega_o$. Therefore

$$|\dot{x}| = \omega_o\sqrt{Im}\frac{x}{\sqrt{I + mx^2}}.$$

3.4c A Material Segment with Extremities on Rectilinear Guides

A material homogeneous rigid rod of mass m and length ℓ moves with its extremities A and B constrained on two smooth rectilinear guides intersecting at O and forming an angle $\alpha = \widehat{AOB} \in [\frac{1}{2}\pi, \pi)$. The points A and B are attracted to O by two elastic forces of elasticity constants k_A and k_B.

(i) Find the instantaneous center of motion C of the corresponding plane rigid motion and determine the fixed and moving centrodes.
(ii) As the sole Lagrangian coordinate take the angle θ between $(B-A)$ and $(O-A)$ and write down the cardinal equations taking C as the pole. Justify such a choice of pole.
(iii) Assuming $\alpha = \frac{1}{2}\pi$ and $k_A = 2k_B$, determine the configurations of equilibrium and the corresponding reactions due to the constraints at the points A and B. (See also §4.4c of the Complements of Chapter 8.)
(iv) Assuming further $k_A = k_B = 0$ and $\theta \ll 1$, prove that the system effects small oscillations about the equilibrium configuration $\theta = 0$, with period $T = 2\pi\sqrt{2g/3\ell}$.

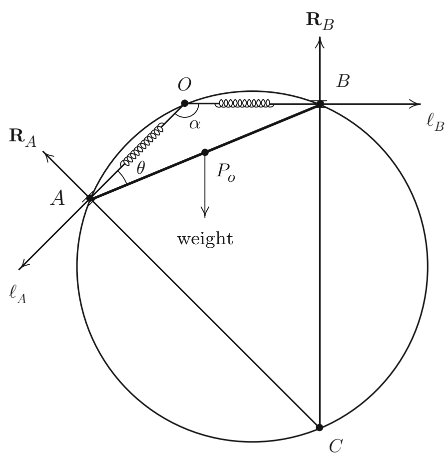

Fig. 3.2c.

3.4.1c Geometry of the Rigid Motion

The trajectories of A and B are the lines ℓ_A and ℓ_B respectively. By Chasles's theorem the center of instantaneous rotation C is at the intersection of the normals to ℓ_A and ℓ_B respectively. Since $\widehat{OBC} = \widehat{OAC} = \frac{1}{2}\pi$, the quadrilateral of vertices O, A, C, B can be inscribed in the circle through O, A, B and diameter $\overline{OC}$. The segment $\overline{AB}$ is a chord of such a circle, seen by O under a constant angle α. Therefore the moving centrode is such a circle. Since $\overline{OC} = \ell/\sin\alpha$, the distance from C to O remains constant along the motion. Therefore the fixed centrode is the circle of center O and radius $\ell/\sin\alpha$ (§§11.3c–11.4c of Chapter 1.)

3.4.2c Second Cardinal Equation

First compute

$$\begin{aligned}
\overline{OB} &= \ell\cos\big(\pi-(\alpha+\theta)\big) - \overline{OA}\cos(\pi-\alpha),\\
\overline{OA} &= \ell\cos\theta - \overline{OB}\cos(\pi-\alpha),
\end{aligned}$$

$$\overline{OB} = \frac{\ell}{\sin\alpha}\sin\theta;\ \overline{OA} = \frac{\ell}{\sin\alpha}\sin(\alpha+\theta),$$

$$\overline{BC} = \frac{\ell}{\sin\alpha}\cos\theta;\ \overline{AC} = -\frac{\ell}{\sin\alpha}\cos(\alpha+\theta).$$

Introduce a fixed triad with origin in O, with $\mathbf{e}_1$ the unit vector of $(B-O)$, with $\mathbf{e}_2$ ascending vertically and $\mathbf{e}_3$ such that the triad $\Sigma = \{O;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is positive. With respect to Σ,

$$\begin{aligned}
C-O =& \frac{\ell}{\sin\alpha}\left(\sin\theta\mathbf{e}_1 - \cos\theta\,\mathbf{e}_2\right),\\
P_o - O =& \Big[\frac{\ell}{\sin\alpha}\sin\theta - \frac{\ell}{2}\cos\big(\pi-(\alpha+\theta)\big)\Big]\mathbf{e}_1 - \frac{\ell}{2}\sin\big(\pi-(\alpha+\theta)\big)\mathbf{e}_2,\\
P_o - C =& \frac{\ell}{2}\cos(\alpha+\theta)\mathbf{e}_1 + \big((P_o-C)\cdot\mathbf{e}_2\big)\mathbf{e}_2\\
=& \frac{\ell}{2}\cos(\alpha+\theta)\mathbf{e}_1 + \Big[\frac{\ell}{\sin\alpha}\cos\theta - \frac{\ell}{2}\sin(\alpha+\theta)\Big]\mathbf{e}_2,
\end{aligned}$$

where P_o is the center of mass of the rod. The moments of the elastic forces $\mathbf{F}_A = -k_A(A-O)$ and $\mathbf{F}_B = -k_B(B-O)$, with respect to C, are

$$\begin{aligned}
(A-C)\wedge\mathbf{F}_A &= k_A\Big(\frac{\ell}{\sin\alpha}\Big)^2\sin(\alpha+\theta)\cos(\alpha+\theta)\mathbf{e}_3,\\
(B-C)\wedge\mathbf{F}_B &= k_B\Big(\frac{\ell}{\sin\alpha}\Big)^2\sin\theta\cos\theta\mathbf{e}_3.
\end{aligned}$$

The moment of the weight, still with respect to C, is

$$-mg(P_o-C)\wedge\mathbf{e}_2 = -\tfrac{1}{2}\ell mg\cos(\alpha+\theta)\mathbf{e}_3.$$

The moment of the reactions $\mathbf{R}_A$ and $\mathbf{R}_B$ with respect to C are zero. Therefore the resultant moment of the external forces is

$$\begin{aligned}\mathbf{M}^{(e)} = &\Big(\frac{\ell}{\sin\alpha}\Big)^2\big[k_A\sin(\alpha+\theta)\cos(\alpha+\theta)+k_B\sin\theta\cos\theta\big]\mathbf{e}_3\\ &-\tfrac{1}{2}\ell mg\cos(\alpha+\theta)\mathbf{e}_3.\end{aligned}$$

The angular momentum with respect to C is computed from

$$\begin{aligned}\mathbf{K} &= \int_{\mathrm{rod}}(P-C)\wedge\big[\dot{P}_o+\boldsymbol{\omega}\wedge(P-P_o)\big]\,d\mu(P)\\ &= m(P_o-C)\wedge\dot{P}_o+\int_{\mathrm{rod}}(P-C)\wedge[\boldsymbol{\omega}\wedge(P-C)]\,d\mu(P)\\ &\quad+\int_{\mathrm{rod}}(P-C)\wedge[\boldsymbol{\omega}\wedge(C-P_o)]\,d\mu(P)\\ &= m(P_o-C)\wedge\big[\dot{P}_o+\boldsymbol{\omega}\wedge(C-P_o)\big]\\ &\quad+\int_{\mathrm{rod}}(P-C)\wedge[\boldsymbol{\omega}\wedge(P-C)]\,d\mu(P).\end{aligned}$$

The velocity of C as part of the rigid motion of the rigid material rod is zero, and it is given by Poisson's formula

$$\mathbf{v}(C)=\dot{P}_o+\boldsymbol{\omega}\wedge(C-P_o)=0.$$

This is the reason for having chosen C as the pole with respect to which moments are computed. From these remarks, since also $\boldsymbol{\omega}=-\dot{\theta}\mathbf{e}_3$,

$$\mathbf{K}=\int_{\mathrm{rod}}(P-C)\wedge[\boldsymbol{\omega}\wedge(P-C)]\,d\mu(P)=-\dot{\theta}I_{\ell(C;\mathbf{e}_3)}\mathbf{e}_3.$$

By Huygens's theorem,

$$\begin{aligned}I_{\ell(C;\mathbf{e}_3)} &= I_{\ell(P_o;\mathbf{e}_3)}+m\|C-P_o\|^2\\ &= \frac{1}{12}m\ell^2+m\Big[\frac{\ell^2}{4}+\Big(\frac{\ell}{\sin\alpha}\Big)^2\cos^2\theta-\frac{\ell^2}{\sin\alpha}\cos\theta\sin(\alpha+\theta)\Big]\\ &= \frac{1}{3}m\ell^2+m\Big(\frac{\ell}{\sin\alpha}\Big)^2\cos\alpha\cos\theta\cos(\alpha+\theta).\end{aligned}$$

Therefore the second cardinal equation for this system is

$$\begin{aligned}\frac{1}{3}m\ell^2\ddot{\theta}&+m\Big(\frac{\ell}{\sin\alpha}\Big)^2\cos\alpha\frac{d}{dt}\big[\dot{\theta}\cos\theta\cos(\alpha+\theta)\big]\\ &= -\Big(\frac{\ell}{\sin\alpha}\Big)^2\big[k_A\sin(\alpha+\theta)\cos(\alpha+\theta)\\ &\qquad+k_B\sin\theta\cos\theta\big]+\frac{1}{2}\ell mg\cos(\alpha+\theta).\end{aligned}$$

3.4.3c First Cardinal Equation

The resultant of the external forces is

$$\begin{aligned}\mathbf{F}^{(e)} = &-\Big[\frac{\ell}{\sin\alpha}\big(k_A\sin(\alpha+\theta)\cos\alpha + k_B\sin\theta\big) - \mathbf{R}_A\cdot\mathbf{e}_1\Big]\mathbf{e}_1\\ &-\Big[\frac{\ell}{\sin\alpha}k_A\sin(\alpha+\theta)\sin\alpha + mg - \mathbf{R}_A\cdot\mathbf{e}_2 - \mathbf{R}_B\cdot\mathbf{e}_2\Big]\mathbf{e}_2.\end{aligned}$$

Moreover,

$$P_o - O = \Big[\frac{\ell}{\sin\alpha}\sin\theta + \frac{\ell}{2}\cos(\alpha+\theta)\Big]\mathbf{e}_1 - \frac{\ell}{2}\sin(\alpha+\theta)\mathbf{e}_2,$$

from which

$$\ddot{P}_o = \ddot{\theta}\frac{d}{d\theta}(P_o - O) - \dot{\theta}^2(P_o - O).$$

Therefore the first cardinal equation takes the form

$$m\Big[\ddot{\theta}\frac{d}{d\theta}(P_o - O) - \dot{\theta}^2(P_o - O)\Big] = \mathbf{F}^{(e)}.$$

3.4.4c The Case $\alpha = \frac{1}{2}\pi$ and $k_A = 2k_B$

From the second cardinal equation,

$$\tfrac{1}{3}m\ell^2\ddot{\theta} = \tfrac{1}{2}\ell^2 k_B\sin 2\theta - \tfrac{1}{2}\ell mg\sin\theta.$$

From this one finds the equilibrium configurations $\theta = 0$ and $\cos\theta = mg/2\ell k_B$. For the configuration $\theta = 0$, the reactions are computed from the first cardinal equation $\mathbf{R}_A = 0$ and $\|\mathbf{R}_B\| = mg + \ell k_A$. The second equilibrium configuration holds only if $k_B \geq mg/2\ell$. In such a case,

$$\|\mathbf{R}_A\| = \ell k_B\sqrt{1 - \Big(\frac{mg}{2k_B\ell}\Big)^2}, \qquad \|\mathbf{R}_B\| = 2mg.$$

3.4.5c The Case $\alpha = \frac{1}{2}\pi$ and $k_A = k_B = 0$

This case yields $\ddot{\theta} = -(g/L)\sin\theta$, where $L = 2\ell/3$. For $\theta \ll 1$ the rod effects oscillations about its equilibrium configuration, of period $T = 2\pi\sqrt{g/L}$.

3.5c Vibrations of a Flywheel

A homogeneous flywheel of mass M is mounted on springs of elasticity constant k and is equipped with a damping device as in **Figure 3.3c**. The flywheel spins about its axis, so that its center O can move only on the vertical y-axis starting from its rest position Ω. The flywheel has a material impurity modeled by a small mass m placed at the point P_*, at distance ρ from the center O. The system has two degrees of freedom, and as Lagrangian coordinates we take the ordinate y of the center of the flywheel with respect to its rest position $y = 0$, and the angle θ as in **Figure 3.3c**. The generic point P of the flywheel has coordinates

$$P = \|P - O\| \cos\theta \mathbf{u}_1 + (y + \|P - O\| \sin\theta)\, \mathbf{u}_2,$$

and it has velocity

$$\dot{P} = -\|P - O\|\dot{\theta} \sin\theta \mathbf{e}_1 + \big(\dot{y} + \|P - O\|\dot{\theta}\cos\theta\big)\mathbf{e}_2.$$

The kinetic energy and the momentum of the system are

$$2T = (\mathcal{I} + m\rho^2)\dot{\theta}^2 + (M + m)\dot{y}^2 + 2m\rho\dot{\theta}\dot{y}\cos\theta,$$
$$\mathbf{Q} = m\rho\dot{\theta}\sin\theta\mathbf{e}_1 + \Big((M + m)\dot{y} + m\rho\dot{\theta}\cos\theta\Big)\,\mathbf{e}_2.$$

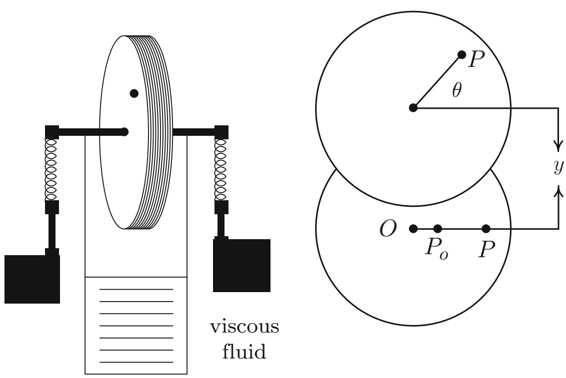

Fig. 3.3c.

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The damping device, realized by a grid in a viscous fluid, opposes the motion with a force $-c\dot{y}\mathbf{e}_2$ for a given positive constant c depending on the fluid. The spring opposes the motion with an elastic force $-ky\mathbf{e}_2$, and the force of gravity of the flywheel is $-(M+m)g\mathbf{e}_2$. An external motor keeps the flywheel spinning at constant angular speed $\dot{\theta} = \omega$. At such constant angular speed, the first cardinal equation, along the $\mathbf{e}_2$-axis, takes the form

$$\ddot{y} + \frac{c}{M + m}\dot{y} + \frac{k}{M + m}y = -\frac{m}{M + m}\rho\omega^2 \cos\omega t - g.$$

Prove that as $\omega \to \infty$, the y-component of the velocity of the center of mass tends to zero. Analyze the reactions due to the constraints, as functions of ω.

4c Mechanical Quantities of Rigid Systems

4.1c Disk Rolling without Slipping on a Line

A homogeneous disk of radius R and mass m rolls without slipping on a horizontal rectilinear guide, as in §2.2c of the Complements of Chapter 2. Compute the angular momentum $\mathbf{K}_\Omega$ and the kinetic energy T.

The constraint of *rolling without slipping* implies $\boldsymbol{\omega} = -(\dot{y}_1/R)\mathbf{e}_3$. Moreover, by the theorem of the momentum, $\mathbf{Q} = m\dot{y}_1\mathbf{e}_1$. Let σ_{P_o} be the inertia tensor with respect to a central principal triad with the axis $\ell(P_o, \mathbf{e}_2)$ normal to the disk. From §6.4 of Chapter 4,

$$\sigma_{P_o} = \frac{1}{4}mR^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Next,

$$\begin{aligned} \mathbf{K}_\Omega &= \int (P - P_o) \wedge \dot{P} d\mu(P) + \int (P - \Omega) \wedge \dot{P} d\mu(P) \\ &= \mathbf{K}_{P_o} + (P_o - \Omega) \wedge \mathbf{Q} \\ &= \sigma_{P_o} (0, 0, -\dot{y}_1/R)^t + (y_1\mathbf{e}_1 + R\mathbf{e}_2) \wedge m\dot{y}_1\mathbf{e}_1 \\ &= -\tfrac{3}{2}mR\dot{y}_1\mathbf{e}_3. \end{aligned}$$

The kinetic energy is

$$T = \tfrac{1}{2}m\dot{P}_o^2 + \tfrac{1}{2}I_{33}\|\boldsymbol{\omega}\|^2 = \tfrac{3}{4}m\dot{y}_1^2.$$

4.2c Disk Rolling in a Plane

Assume now that the disk is constrained to move only in the vertical plane $y_3 = 0$. The system has three degrees of freedom, and as Lagrangian coordinates one may choose the coordinates (y_1, y_2) of P_o and the angle φ formed by $\mathbf{e}_1$ with a fixed radius of the disk. With these choices,

$$\mathbf{Q} = m\left(\dot{y}_1\mathbf{e}_1 + \dot{y}_2\mathbf{e}_2\right), \quad \mathbf{K}_{P_o} = \sigma_{P_o}\left(0, 0, \dot{\varphi}\right)^t = \tfrac{1}{2}mR^2\dot{\varphi}\mathbf{e}_3.$$

For a fixed pole Ω,

$$\mathbf{K}_\Omega = \mathbf{K}_{P_o} + (P_o - \Omega) \wedge \mathbf{Q}, \qquad 2T = m\dot{P}_o^2 + I_{33}\|\boldsymbol{\omega}\|^2. \tag{4.1c}$$

Therefore

$$\mathbf{K}_\Omega = m\left(\tfrac{1}{2}R^2\dot{\varphi} + \dot{y}_1 y_2 - y_1\dot{y}_2\right)\mathbf{e}_3, \qquad T = \tfrac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2) + \tfrac{1}{4}mR^2\dot{\varphi}^2.$$

4.3c Rigid Rod with Constrained Extremities

A material homogeneous rigid rod of length ℓ and mass m moves with its extremities A and B constrained on two rectilinear orthogonal guides intersecting at Ω, as in the Cardano device (§11.3c of Chapter 1). Compute $\mathbf{K}_\Omega$ and T. Choose Σ with origin in Ω and $\mathbf{e}_1$ and $\mathbf{e}_2$ along the guides. Choose also $S = \{P_o; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ centered at P_o, with $\mathbf{u}_2$ along $(B - A)$ and $\mathbf{u}_3 = \mathbf{e}_3$. The system has one degree of freedom, and as Lagrangian coordinate choose the angle $\varphi = \widehat{\mathbf{e}_1\mathbf{u}_1}$. For these choices,

$$2P_o = \ell(\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2), \qquad 2\mathbf{Q} = m\ell\dot{\varphi}\,(-\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2)\,.$$

Using the calculations of §6.1 of Chapter 4, compute the inertia tensor σ_{P_o} of the rod with respect to S. Then, using (4.1c) one computes $3\mathbf{K}_\Omega = m\ell^2\dot{\varphi}\mathbf{e}_3$ and $6T = m\ell^2\dot{\varphi}^2$.

4.4c Rigid Rod Moving in a Plane

If the rod is constrained to the plane $y_3 = 0$, the system has three degrees of freedom, which are chosen as the coordinates (y_1, y_2) of P_o and the angle $\varphi = \widehat{\mathbf{e}_1\mathbf{u}_1}$. For such choices,

$$\begin{aligned}\mathbf{K}_\Omega &= \tfrac{1}{12}m\ell^2\dot{\varphi}\mathbf{e}_3 + m(y_1\mathbf{e}_1 + y_2\mathbf{e}_2) \wedge (\dot{y}_1\mathbf{e}_1 + \dot{y}_2\mathbf{e}_2),\\ T &= \tfrac{1}{24}m\ell^2\dot{\varphi}^2 + \tfrac{1}{2}m\left(\dot{y}_1^2 + \dot{y}_2^2\right).\end{aligned}$$

Compute $\mathbf{K}_\Omega$ and T for a rod with one of the extremes constrained on a guide and the other free in the plane $y_3 = 0$.

4.5c The Double Pendulum

A homogeneous material rod of length ℓ and mass m is hinged at one of its extremities to a fixed point Ω. The second extremity, denoted by P_*, is hinged to one extremity of a second homogeneous material rod, of length L and mass M. The system is constrained to move in the plane $y_3 = 0$. Compute $\mathbf{K}_\Omega$ and T.

The system has two degrees of freedom for the choice of the angles φ and θ as in **Figure 4.1c**. For these choices,

$$\begin{gathered}2P_o = \ell\,(\sin\varphi\mathbf{e}_1 - \cos\varphi\mathbf{e}_2)\,, \qquad P_* = \ell\,(\sin\varphi\,\mathbf{e}_1 - \cos\varphi\,\mathbf{e}_2)\,,\\ 2Q_o = (2\ell\sin\varphi + L\sin\theta)\,\mathbf{e}_1 - (2\ell\cos\varphi + L\cos\theta)\,\mathbf{e}_2.\end{gathered}$$

Therefore

$$\begin{aligned}2\mathbf{Q} =&\big(\ell\dot{\varphi}(m + 2M)\cos\varphi + LM\dot{\theta}\cos\theta\big)\mathbf{e}_1\\ &+ \big(\ell\dot{\varphi}(m + 2M)\sin\varphi + LM\dot{\theta}\sin\theta\big)\mathbf{e}_2.\end{aligned}$$

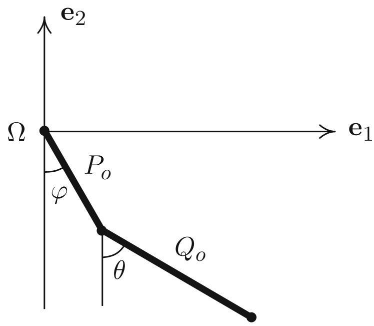

Fig. 4.1c.

The first rod is in precession with $\boldsymbol{\omega} = \dot{\varphi}\mathbf{e}_3$. Choose S fixed with the rod with origin at Ω, and $\mathbf{u}_1$ as $(P_o - \Omega)$. The second rod is in rigid motion with characteristics $\dot{P}_*$ and $\boldsymbol{\omega}' = \dot{\theta}\mathbf{e}_3$. Choose a moving triad S', fixed with the second rod, with origin in P_* and $\mathbf{u}'_1$ as $(Q_o - P_*)$.

The inertia tensors σ_Ω and σ_{P_*} with respect to S and S' are

$$\sigma_\Omega = \frac{1}{3}m\ell^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \sigma'_{P_*} = \frac{1}{3}ML^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The angular momentum is computed from

$$\begin{aligned} \mathbf{K}_\Omega &= \int_{\text{first rod}} (P - \Omega) \wedge \dot{P} d\mu(P) + \int_{\text{second rod}} (P - \Omega) \wedge \dot{P} d\mu(P) \\ &= \sigma_\Omega (0, 0, \dot{\varphi})^t + \int_{\text{second rod}} (P - P_*) \wedge \dot{P} d\mu(P) + (P_* - \Omega) \wedge M\dot{Q}_o \\ &= \sigma_\Omega (0, 0, \dot{\varphi})^t + \sigma_{P_*}(0, 0, \dot{\theta})^t + M(Q_o - P_*) \wedge \dot{P}_* + (P_* - \Omega) \wedge M\dot{Q}_o. \end{aligned}$$

From the expression of Q_o compute $\dot{Q}_o$ and substitute it above to conclude that

$$\mathbf{K}_\Omega = \tfrac{1}{3}\big(m\ell^2\dot{\varphi} + ML^2\dot{\theta}\big)\mathbf{e}_3 - \tfrac{1}{2}M\ell\big(2\ell\dot{\varphi} + L(\dot{\varphi} - \dot{\theta})\cos(\theta - \varphi)\big)\mathbf{e}_3.$$

To compute T, regard the second rod as in rigid motion, with characteristics $\dot{Q}_o$ and $\dot{\theta}\mathbf{e}_3$. Then

$$\begin{aligned} 2T &= \int_{\text{first rod}} \dot{P}^2 d\mu(P) + \int_{\text{second rod}} \dot{P}^2 d\mu(P) \\ &= \tfrac{1}{3}m\ell^2\dot{\varphi}^2 + \int_{\text{second rod}} \big(\dot{Q}_o - \dot{\theta}\mathbf{e}_3 \wedge (P - Q_o)\big)^2 d\mu(P) \\ &= \tfrac{1}{3}m\ell^2\dot{\varphi}^2 + \tfrac{1}{12}ML^2\dot{\theta}^2 + M\dot{Q}_o^2. \end{aligned}$$

4.6c Cone of §10.1c of Chapter 1

Assume that the cone is homogeneous with mass m and that the plane π is smooth. Assume that $\theta \in (0, \frac{\pi}{2})$ and compute the kinetic energy and the angular momentum with respect to O. Moreover, assuming $\|\boldsymbol{\omega}\| = \text{const}$, compute the reactions offered by the plane π and the vertical axis through O.

4.6.1c Mechanical Quantities of the Cone

In the elemental time dt the contact point C covers a circular arc of elemental length $\dot{\varphi} dt\, R \sin\theta / \sin\alpha$ in the fixed plane $y_3 = 0$. In the same time interval, the extreme C of the vector $(C - Q)$ covers a circular arc of elemental length $\dot{\psi} dt\, R$ in the moving plane $x_3 = h$ containing the base of the cone. Since the cone rolls without slipping,

$$\sin\alpha \dot{\psi} = \dot{\varphi} \sin\theta \qquad \text{(which defines } \psi\text{)}.$$

The vector $\boldsymbol{\omega}$ in the coordinates of the moving triad S is

$$\boldsymbol{\omega} = -\|\boldsymbol{\omega}\| \text{sign}\{\dot{\psi}\} \left[(\cos\psi \mathbf{u}_1 - \sin\psi \mathbf{u}_2) \sin\alpha + \cos\alpha \mathbf{u}_3 \right].$$

The inertia tensor σ_o with respect to S has been computed in §6.1c of the Complements of Chapter 4 (see also the left **Figure 6.1c**). Therefore the angular momentum with respect to O is given, in the coordinates of S, by

$$\mathbf{K} = \sigma_o \boldsymbol{\omega} = -\frac{3}{20} \text{sign}\{\dot{\psi}\} m \|\boldsymbol{\omega}\| \begin{pmatrix} (R^2 + 4h^2) \cos\psi \sin\alpha \\ -(R^2 + 4h^2) \sin\psi \sin\alpha \\ 2R^2 \cos\alpha \end{pmatrix}.$$

For the kinetic energy,

$$2T = \boldsymbol{\omega}^t \sigma_o \boldsymbol{\omega} = \tfrac{3}{20} m \|\boldsymbol{\omega}\|^2 \left[2R^2 + \left(4h^2 - R^2\right) \sin^2\alpha \right].$$

The trajectory of the center of mass P_o is the circle of radius $\frac{3}{4} h \sin(\alpha + \theta)$ in the fixed plane $y_3 = d - \frac{3}{4} h \cos(\alpha + \theta)$. Since its angular velocity is $\dot{\varphi} \mathbf{e}_3$, one computes

$$\dot{P}_o = \tfrac{3}{4} \dot{\varphi} h \sin(\alpha + \theta) (-\sin\varphi \mathbf{e}_1 + \cos\varphi \mathbf{e}_2).$$

Verify that the same expression for T could be derived using Corollary 4.2 and the previous expression for $\dot{P}_o$. Use also the inertia tensor σ_{P_o} computed in §6.1.1c of the Complements of Chapter 4.

4.6.2c Cardinal Equations and Reactions

The component of the reaction in O normal to the vertical axis is

$$\mathbf{R}_{\perp \mathbf{e}_3} = -\tfrac{3}{4} m \dot{\varphi}^2 h \sin(\alpha + \theta) (\cos\varphi\, \mathbf{e}_1 + \sin\varphi\, \mathbf{e}_2).$$

The moment of the reaction in C with respect to O has modulus $\|\mathbf{R}_C\| h \sin\theta$, lies on the moving plane $x_3 = 0$, and is normal to the projection of $\boldsymbol{\omega}$ on such a plane. Moreover, it forms a positive triad with such a projection and the moving axis of $\mathbf{u}_3$.

Denote by ψ the angle formed by $\mathbf{u}_1$ with the projection of $\boldsymbol{\omega}$ on the moving plane $x_3 = 0$. Then the expression of the moment of $\mathbf{R}_C$ with respect to O, written in the coordinates of S, takes the form

$$-\|\mathbf{R}_C\| h \sin\theta \mathbf{u}, \qquad \text{where } \mathbf{u} = (\sin\psi \mathbf{u}_1 + \cos\psi \mathbf{u}_2 + 0\mathbf{u}_3)\,.$$

By analogous considerations, the moment of the weight with respect to O, in the coordinates of S, is $\frac{3}{4}hmg\sin(\alpha+\theta)\mathbf{u}$. Therefore the resultant moment with respect to O of the external forces in the coordinates of S is

$$\mathbf{M}^{(\mathrm{e})} = \big(\tfrac{3}{4}hmg\sin(\alpha+\theta) - \|\mathbf{R}_C\| h \sin\theta\big)\mathbf{u}.$$

From the expression of $\mathbf{K}$ in S,

$$\begin{aligned}\dot{\mathbf{K}} &= \left(\frac{d\mathbf{K}}{dt}\right)_S + \boldsymbol{\omega}\wedge\mathbf{K} = \tfrac{3}{20}m\|\boldsymbol{\omega}\||\dot{\psi}|(R^2+4h^2)\sin\alpha\mathbf{u} \\ &\quad + \tfrac{3}{20}m\|\boldsymbol{\omega}\|^2 \begin{pmatrix}(R^2+4h^2)\cos\psi\sin\alpha \\ -(R^2+4h^2)\sin\psi\sin\alpha \\ 2R^2\cos\alpha\end{pmatrix} \wedge \begin{pmatrix}\cos\psi\sin\alpha \\ -\sin\psi\sin\alpha \\ \cos\alpha\end{pmatrix}.\end{aligned}$$

The exterior product of the last two vectors equals $(R^2-4h^2)\sin\alpha\cos\alpha\mathbf{u}$. Therefore the second cardinal equation, in the coordinates of S, takes the form

$$\begin{aligned}\dot{\mathbf{K}} &= \tfrac{3}{20}m\|\boldsymbol{\omega}\|\big[(R^2+4h^2)|\dot{\psi}|\sin\alpha + \|\boldsymbol{\omega}\|(R^2-4h^2)\sin\alpha\cos\alpha\big]\mathbf{u} \\ &= \big(\tfrac{3}{4}hmg\sin(\alpha+\theta) - \|\mathbf{R}_C\| h\sin\theta\big)\mathbf{u} = \mathbf{M}^{(\mathrm{e})}.\end{aligned}$$

In the particular case $d = R$, e.g., $(\alpha+\theta) = \frac{1}{2}\pi$,

$$\|\boldsymbol{\omega}\| = \frac{|\dot{\varphi}|}{\sin\alpha} = \frac{|\dot{\psi}|}{\cos\alpha}$$

and

$$\tfrac{3}{20}m\|\boldsymbol{\omega}\|^2R^2\sin 2\alpha = \big(\tfrac{3}{4}hmg - \|\mathbf{R}_C\| h\cos\alpha\big).$$

4.7c Square Plate with a Vertex on a Rectilinear Guide

A homogeneous material square plate of edge ℓ and mass m moves on a horizontal plane $x_3 = 0$, with one of the extremities A constrained on a guide as in **Figure 4.2c**. The point A is attracted to a point O, fixed on the horizontal guide, by a spring of elastic constant k. The plane $x_3 = 0$ rotates about the $\mathbf{e}_3$-axis with constant angular velocity $\omega\mathbf{e}_3$. Resolve the motion.

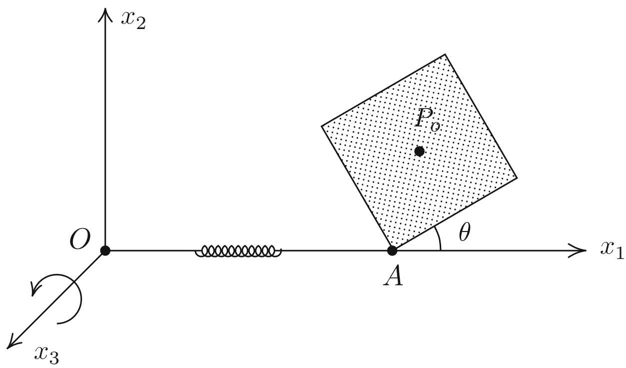

Fig. 4.2c.

Introduce a fixed inertial triad $\Sigma = \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with origin at O and a moving triad $S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ with $\mathbf{u}_3 = \mathbf{e}_3$, as in **Figure 4.2c**. The system has two degrees of freedom, and as Lagrangian coordinates choose the abscissa x of A on the guide and the angle θ. With these choices the center of mass P_o of the plate in the coordinates of S is

$$P_o = \Big[(x + \frac{\ell}{\sqrt{2}} \cos\big(\theta + \frac{\pi}{4}\big)\Big]\mathbf{u}_1 + \Big[\frac{\ell}{\sqrt{2}} \sin\big(\theta + \frac{\pi}{4}\big)\Big]\mathbf{u_2}.$$

The velocity $\mathbf{v}_S(P_o)$ and the acceleration $\mathbf{a}_S(P_o)$ of P_o with respect to S are

$$\begin{aligned}
\mathbf{v}_S(P_o) &= \Big[\dot{x} - \frac{\ell}{\sqrt{2}}\dot{\theta} \sin\big(\theta + \frac{\pi}{4}\big)\Big]\mathbf{u}_1 + \Big[\frac{\ell}{\sqrt{2}}\dot{\theta} \cos\big(\theta + \frac{\pi}{4}\big)\Big]\mathbf{u}_2, \\
\mathbf{a}_S(P_o) &= \Big[\ddot{x} - \frac{\ell}{\sqrt{2}}\ddot{\theta} \sin\big(\theta + \frac{\pi}{4}\big) - \frac{\ell}{\sqrt{2}}\dot{\theta}^2 \cos\big(\theta + \frac{\pi}{4}\big)\Big]\mathbf{u}_1 \\
&\qquad + \frac{\ell}{\sqrt{2}}\Big(\ddot{\theta} \cos\big(\theta + \frac{\pi}{4}\big) - \dot{\theta}^2 \sin\big(\theta + \frac{\pi}{4}\big)\Big)\mathbf{u}_2.
\end{aligned}$$

The plate moves with rigid motion with respect to S with characteristics $\mathbf{v}_S(P_o)$ and $\dot{\theta}\mathbf{e}_3$. Therefore the velocity of a point P of the plate with respect to S is

$$\mathbf{v}_S(P) = \mathbf{v}_S(P_o) + \dot{\theta}\mathbf{e}_3 \wedge (P - P_o).$$

The deflection velocity of P as part of the rigid motion of S with respect to Σ is $\omega\mathbf{e}_3 \wedge (P - O)$. Therefore the velocity of the generic point P of the plate with respect to the fixed triad Σ is

$$\begin{aligned}
\mathbf{v}_\Sigma(P) &= \mathbf{v}_S(P_o) + \dot{\theta}\mathbf{e}_3 \wedge (P - P_o) + \omega\, \mathbf{e}_3 \wedge (P - O) \\
&= \mathbf{v}_S(P_o) + \big(\omega + \dot{\theta}\big)\mathbf{e}_3 \wedge (P - P_o) + \omega\mathbf{e}_3 \wedge (P_o - O).
\end{aligned}$$

Denote by $\mathbf{a}_S(P)$, $\mathbf{a}_T(P)$, and $\mathbf{a}_C(P)$ respectively the acceleration of P with respect to S, the transport acceleration of P as part of the rigid motion of S, and the Coriolis acceleration

$$\begin{aligned}\mathbf{a}_S(P) &= \mathbf{a}_S(P_o) + \ddot{\theta}\mathbf{e}_3 \wedge (P - P_o) + \dot{\theta}^2(P - P_o),\\ \mathbf{a}_C(P) &= 2\omega\mathbf{e}_3 \wedge \mathbf{v}_S(P_o) + 2\omega\dot{\theta}(P - P_o),\\ \mathbf{a}_T(P) &= -\omega^2(P - O).\end{aligned}$$

Momentum and angular momentum, both computed with respect to P_o, are

$$\mathbf{Q}_{P_o} = m\,]\mathbf{v}_S(P_o) + \omega\mathbf{e}_3 \wedge (P_o - O)]\,, \qquad \mathbf{K}_{P_o} = \mathcal{J}(\omega + \dot{\theta})\mathbf{e}_3,$$

where $\mathcal{J}$ is the axial moment of the inertia of the plate with respect to the axis $\ell\,(P_o;\mathbf{e}_3)$. Verify that the cardinal equations reduce to identities and do not provide information on the motion. The kinetic energy and the potential U are given by

$$\begin{aligned}2T &= m\mathbf{v}_S^2(P_o) + \mathcal{J}(\omega + \dot{\theta})^2 + m\omega^2 P_o^2 + 2m\omega\mathbf{v}_S(P_o)\cdot\mathbf{e}_3 \wedge (P_o - O),\\ 2U &= -kx^2 + \text{const.}\end{aligned}$$

Verify that

$$\frac{\partial T}{\partial \dot{x}} = m\mathbf{v}_\Sigma(P_o)\cdot\mathbf{u}_1, \quad \frac{\partial T}{\partial \dot{\theta}} = m\mathbf{v}_\Sigma(P_o)\cdot\mathbf{e}_3 \wedge (P_o - A) + \mathcal{J}(\omega + \dot{\theta}),$$

$$\frac{\partial T}{\partial x} = m\omega\mathbf{v}_\Sigma(P_o)\cdot\mathbf{u}_2, \quad \frac{\partial T}{\partial \theta} = -m(\omega + \dot{\theta})\mathbf{v}_\Sigma(P_o)\cdot(P_o - A).$$

4.8c Two Hinged Rods with Constrained Extremities

Two rigid homogeneous equal rods ΩM and MN, each of mass m and length ℓ, are hinged at their common extremity M as in **Figure 4.3c**, and are constrained in a vertical plane. The first extremity Ω of the first rod is kept fixed at Ω, and the second extremity N of the second rod is constrained on the line segment line $(C - \Omega)$ of length 2ℓ and forming an angle $\frac{\pi}{4}$ with respect to the horizontal $\mathbf{e}_1$. The point N is attracted by C by an elastic force of Hooke's constant k. The hinges in Ω and M and the constraint on N are workless. The system has one degree of freedom, and as Lagrangian coordinate we take the angle φ formed by $(M - \Omega)$ and $\mathbf{e}_1$. Let also $\psi = \varphi + \frac{\pi}{4}$ denote the angle between $(M - \Omega)$ and $(C - \Omega)$. We have

$$\begin{aligned}P_1 - \Omega &= \tfrac{1}{2}\ell\,(\cos\varphi\mathbf{e}_1 - \sin\varphi\,\mathbf{e}_3)\,,\\ P_2 - \Omega &= \tfrac{1}{2}\ell\big[(2\cos\varphi - \sin\varphi)\mathbf{e}_1 + (\cos\varphi - 2\sin\varphi)\mathbf{e}_3\big],\\ N - \Omega &= \sqrt{2}\ell\cos\psi\big(\mathbf{e}_1 + \mathbf{e}_3\big) \qquad C - \Omega = \sqrt{2}\ell\big(\mathbf{e}_1 + \mathbf{e}_3\big).\end{aligned}$$

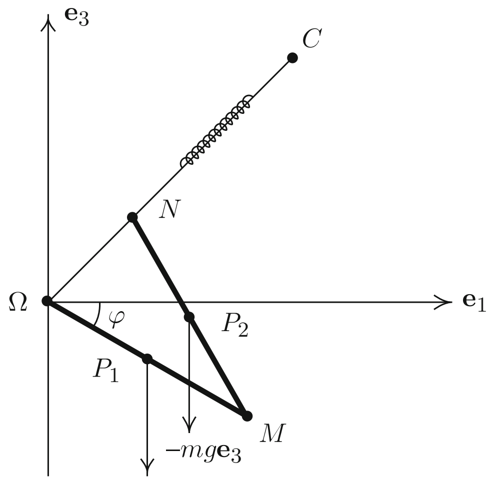

Fig. 4.3c.

4.8.1c The Energy

The sum of the weights and the elastic force is a conservative field with potential

$$\begin{aligned} U &= -mg\{(P_1 - \Omega) + (P_2 - \Omega)\} \cdot \mathbf{e}_3 - \tfrac{1}{2}k\|C - N\|^2 + \text{const} \\ &= \tfrac{1}{2}mg\ell(3\sin\varphi - \cos\varphi) - 2k\ell^2(1 - \cos\psi)^2 + \text{const.} \end{aligned}$$

The kinetic energy $T_{\Omega M}$ of the rod ΩM is $T_{\Omega M} = \frac{1}{6}m\ell^2\dot{\varphi}^2$. The kinetic energy T_{MN} of the rod MN is computed by König's theorem (3.5):

$$\begin{aligned} T_{MN} &= \tfrac{1}{24}m\ell^2\dot{\varphi}^2 + \tfrac{1}{2}m\dot{P}_2^2 \\ &= \tfrac{1}{24}m\ell^2\dot{\varphi}^2 + \tfrac{1}{2}m\ell^2\dot{\varphi}^2\left(\tfrac{5}{4} - \sin 2\varphi\right). \end{aligned}$$

Combining these calculations gives the kinetic energy of the system in the form

$$T = \tfrac{1}{2}m\ell^2\dot{\varphi}^2\left(\tfrac{5}{3} - \sin 2\varphi\right).$$

The total energy $E = T - U$ is then

$$\begin{aligned} E = \tfrac{1}{2}m\ell^2\dot{\varphi}^2\left(\tfrac{5}{3} - \sin 2\varphi\right) &- \tfrac{1}{2}mg\ell(3\sin\varphi - \cos\varphi) \\ &+ 2k\ell^2(1 - \cos\psi)^2 + \text{const.} \end{aligned}$$

4.8.2c Momentum and Angular Momentum

The center of mass P_o has coordinates

$$P_o - \Omega = \tfrac{1}{4}\ell\left[(3\cos\varphi - \sin\varphi)\mathbf{e}_1 + (\cos\varphi - 3\sin\varphi)\mathbf{e}_3\right].$$

By Theorem 2.2 on the center of mass, $\mathbf{Q} = 2m\dot{P}_o$. Therefore

$$\mathbf{Q} = -\tfrac{1}{2}m\ell\dot{\varphi}\left[(3\sin\varphi + \cos\varphi)\mathbf{e}_1 + (3\cos\varphi + \sin\varphi)\mathbf{e}_3\right].$$

The angular momentum of the rod ΩM with respect to Ω is $\mathbf{K}_{\Omega M}(\Omega) = \frac{1}{3}m\ell^2\dot{\varphi}\mathbf{e}_2$. The rod MN rotates about it center of mass P_2 with angular velocity $-\dot{\varphi}\mathbf{e}_2$. Therefore the angular momentum of MN with respect to P_2 is $\mathbf{K}_{MN}(P_2) = -\frac{1}{12}m\ell^2\dot{\varphi}\mathbf{e}_2$. The angular momentum of MN with respect to Ω is computed from

$$\mathbf{K}_{MN}(\Omega) = \mathbf{K}_{MN}(P_2) + (P_2 - \Omega) \wedge \mathbf{Q}_{MN},$$

where $\mathbf{Q}_{MN}$ is the momentum of the rod MN. Expanding these calculations, one computes

$$\mathbf{K}(\Omega) = \mathbf{K}_{\Omega M}(\Omega) + \mathbf{K}_{MN}(\Omega) = m\ell^2\dot{\varphi}\mathbf{e}_2.$$

4.8.3c Cardinal Equations

The first cardinal equation is

$$\begin{aligned}\tfrac{1}{2}m\ell\big\{&\ddot{\varphi}\left[(3\sin\varphi + \cos\varphi)\mathbf{e}_1 + (3\cos\varphi + \sin\varphi)\mathbf{e}_3\right]\\ &+ \dot{\varphi}^2\left[(3\cos\varphi - \sin\varphi)\mathbf{e}_1 + (\cos\varphi - 3\sin\varphi)\mathbf{e}_3\right]\big\}\\ &= -2mg\mathbf{e}_3 - k(N - C) + \mathbf{R}_\Omega + \mathbf{R}_N,\end{aligned}$$

where the last two terms are the reactions exerted by the constraints at Ω and N. The second cardinal equation with respect to the pole Ω takes the form

$$m\ell^2\ddot{\varphi}\mathbf{e}_2 = \tfrac{1}{2}mg\ell(3\cos\varphi - \sin\varphi)\mathbf{e}_2 + (N - \Omega) \wedge \mathbf{R}_N.$$

If the motion were resolved, e.g., if one knew the Lagrangian function $t \to \varphi(t)$, these equations would provide the reactions due to the constraints.

4.8.4c Resolving the Motion

Since the constraints are workless, the energy E is conserved. Taking the time derivative of E gives

$$\begin{aligned}\frac{1}{\dot{\varphi}}\frac{d}{dt}E =& m\ell^2\ddot{\varphi}\left(\tfrac{5}{3} - \sin 2\varphi\right) + m\ell^2\dot{\varphi}^2\cos 2\varphi\\ &- \tfrac{1}{2}mg\ell(3\cos\varphi + \sin\varphi) + 4k\ell^2(1 - \cos\psi)\sin\psi = 0.\end{aligned}$$

This differential equation in the unknown $t \to \varphi(t)$ is independent of the reactions due to the constraints. Solving it, starting from some prescribed initial configuration, for example $N(0) = C$, resolves the motion.

4.9c Triangle Rotating about an Axis

Let $\mathit{\Delta}(ABC)$ be the triangle of §6.2c of the Complements of Chapter 4. The triangle rotates about the axis $\mathbf{u}_3$, while O remains fixed. The system has one degree of freedom and as Lagrangian coordinate take the angle $\varphi = \widehat{\mathbf{u}_1\mathbf{e}_1}$. Momentum, kinetic energy, and angular momentum with respect to the pole O in terms of φ are given by

$$\mathbf{Q} = -\tfrac{1}{3}mh\dot{\varphi}\big(\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2\big), \qquad T = \tfrac{1}{12}mh^2\dot{\varphi}^2,$$
$$\mathbf{K}_O = \sigma_S\boldsymbol{\omega} = \dot{\varphi}\,(I_{13}\mathbf{e}_1 + I_{33}\mathbf{e}_3).$$

If $\mathit{\Delta}(ABC)$ is a right triangle with the right-angle vertex at $B = O$, then

$$\mathbf{K}_O = \tfrac{1}{12}mh\dot{\varphi}\,(a\mathbf{e}_1 + 2h\mathbf{e}_3).$$

6

THE LAGRANGE EQUATIONS

1 Kinetic Energy in Terms of Lagrangian Coordinates

Let $\{\mathcal{M}; d\mu\}$ be a material system whose mechanical state is described by N Lagrangian coordinates $q = (q_1, \dots, q_N)$. Since every point $P \in \{\mathcal{M}; d\mu\}$ is identified along its motion by the map $(q,t) \to P(q,t)$, the configuration of the system is determined, instant by instant, by the map $t \to q(t) : \mathbb{R} \to \mathbb{R}^N$. The latter can be regarded as the motion of some abstract point in some N-dimensional space, called *configuration space*. Since N is the least number of parameters needed to identify uniquely the position of each point P of the system, each of the maps $\{\mathcal{M}; d\mu\} \ni P \to \|\partial P/\partial q_h\|$, $h = 1, \dots, N$, is not identically zero. Equivalently, we have the following lemma.

Lemma 1.1 *Let $\xi \in \mathbb{R}^N$ be fixed. Then $\nabla_q P \cdot \xi = 0$ for all $P \in \{\mathcal{M}; d\mu\}$ if and only if $\xi = 0$.*

The actual and virtual displacements dP and δP of $P \in \{\mathcal{M}; d\mu\}$ are

$$dP = \frac{\partial P}{\partial q_h} dq_h + \frac{\partial P}{\partial t} dt, \qquad \delta P = \frac{\partial P}{\partial q_h} \delta q_h,$$

where dq and δq represent the actual and virtual displacement of the point q in the configuration space. The velocity $\dot{P}$ of a point P of the system and the “velocity” $\dot{q}$ of its Lagrangian representation, are related by

$$\dot{P} = \frac{\partial P}{\partial q_h} \dot{q}_h + \frac{\partial P}{\partial t}, \qquad \text{from which} \qquad \frac{\partial \dot{P}}{\partial \dot{q}_h} = \frac{\partial P}{\partial q_h}. \tag{1.1}$$

Using the expression for $\dot{P}$ in terms of q and $\dot{q}$, compute

$$\begin{aligned} T &= \frac{1}{2} \int \dot{P}^2 d\mu(P) = \frac{1}{2} \int \left(\frac{\partial P}{\partial q_h} \dot{q}_h + \frac{\partial P}{\partial t} \right)^2 d\mu(P) \\ &= \frac{1}{2} \int \left[\left(\frac{\partial P}{\partial q_h} \dot{q}_h \right)^2 + 2 \frac{\partial P}{\partial q_h} \dot{q}_h \cdot \frac{\partial P}{\partial t} + \left(\frac{\partial P}{\partial t} \right)^2 \right] d\mu(P). \end{aligned}$$

E. DiBenedetto, *Classical Mechanics: Theory and Mathematical Modeling*, Cornerstones, DOI 10.1007/978-0-8176-4648-6_6,

Set

$$A_{hk}(q;t) = \int \frac{\partial P}{\partial q_h}\frac{\partial P}{\partial q_k} d\mu(P), \quad B_h(q;t) = \int \frac{\partial P}{\partial q_h}\frac{\partial P}{\partial t} d\mu(P), \tag{1.2}$$

and compute

$$\begin{aligned}\int \left(\frac{\partial P}{\partial q_h}\dot q_h\right)^2 d\mu(P) &= \int \left(\frac{\partial P}{\partial q_h}\dot q_h\right)\left(\frac{\partial P}{\partial q_k}\dot q_k\right) d\mu(P) \\ &= \sum_{h,k=1}^{N} A_{hk}(q;t)\dot q_h \dot q_k.\end{aligned}$$

Then the expression of the kinetic energy in terms of the Lagrangian velocity $\dot q$ can be given the form

$$T = T_0 + T_1 + T_2, \tag{1.3}$$

where

$$T_0(q;t) = \frac{1}{2}\int \left(\frac{\partial P}{\partial t}\right)^2 d\mu(P), \qquad T_1(q,\dot q;t) = B_h(q;t)\dot q_h, \tag{1.4}$$

and

$$T_2(q,\dot q;t) = \frac{1}{2}\sum_{h,k=1}^{N} A_{hk}(q;t)\dot q_h \dot q_k. \tag{1.5}$$

The term T_2 is a homogeneous function of degree two in the variables $\dot q$, whereas T_1 is a homogeneous function of degree one in $\dot q$, and T_0 is a homogeneous function of degree zero in $\dot q$. For time-independent constraints, $T_0 = T_1 = 0$ and the kinetic energy reduces to T_2.

Proposition 1.1 *$T_2(q,\xi;t)$ is a positive definite quadratic form in the variables $(\xi_1,\dots,\xi_N)$.*

Proof. From the definition of T_2 and A_{hk} compute

$$T_2(q,\xi;t) = \int \frac{\partial P}{\partial q_h}\xi_h \frac{\partial P}{\partial q_k}\xi_k d\mu(P) = \int |\nabla_q P\cdot \xi|^2 \, d\mu(P).$$

Therefore $T_2(q,\xi;t) \ge 0$ for all $\xi \in \mathbb{R}^N$. Moreover, for a fixed ξ $T_2(q,\xi;t)$ vanishes if and only if $\nabla_q P\cdot\xi = 0$ for all $P \in \{\mathcal{M}; d\mu\}$. By Lemma 1.1 this is impossible unless $\xi = 0$. ∎

Corollary 1.1 $\det(A_{hk}) \ne 0$.

Corollary 1.2 *The kinetic energy of a mechanical system subject only to holonomic and fixed constraints is a positive definite quadratic form of the Lagrangian velocities.*

1.1 Kinetic Energy for Rigid-Body Motion

Choose a fixed triad Σ and a moving triad $S = \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ whose axes are principal axes of inertia for the material system $\{\mathcal{M}; d\mu\}$. As Lagrangian coordinates take the coordinates of the center of mass P_o in Σ and the Euler's angles φ, ψ, θ of S with respect to Σ. Then[1]

$$2T = m\dot{P}_o^2 + \boldsymbol{\omega}^t \sigma \boldsymbol{\omega}, \qquad m = \int d\mu(P), \tag{1.6}$$

where σ is the inertia tensor of the system with respect to S and $\boldsymbol{\omega}$ is the angular velocity of the rigid motion written in the coordinates of S. The first term is a quadratic form of the coordinates of P_o, whereas the second term is a quadratic form of the Lagrangian velocities $\dot{\varphi}$, $\dot{\psi}$, $\dot{\theta}$. Indeed,

$$\begin{aligned} \boldsymbol{\omega}^t \sigma \boldsymbol{\omega} = \mathcal{I}_1 (&\dot{\varphi} \sin\psi \sin\theta + \dot{\theta} \cos\psi)^2 \\ &+ \mathcal{I}_2 (\dot{\varphi} \cos\psi \sin\theta - \dot{\theta} \sin\psi)^2 \\ &+ \mathcal{I}_3 (\dot{\varphi} \cos\theta + \dot{\psi})^2, \end{aligned} \tag{1.7}$$

where $\mathcal{I}_i$ are the axial moments of inertia of the system with respect to the principal axes of S. If two of the axial moments of inertia coincide, say, for example, $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}$, then the kinetic energy takes the form

$$2T = \mathcal{I}(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) + \mathcal{I}_3 (\dot{\varphi} \cos\theta + \dot{\psi})^2. \tag{1.8}$$

A rigid body with such a property is a *gyroscope*, and the axis of $\mathbf{u}_3$ is the *gyroscopic axis.*

2 The Principle of Virtual Work

The system $\{\mathcal{M}; d\mu\}$ is subject to constraints and is acted upon by a distribution of forces

$$(P, \dot{P}; t) \longrightarrow [\mathbf{f}(P, \dot{P}; t) + \mathbf{r}(P, \dot{P}; t)] d\mu(P).$$

The function $\mathbf{f}$ is the pointwise distribution of internal and external forces per unit mass, whose functional form is assumed to be known. The function $\mathbf{r}$ is the pointwise distribution, per unit mass, of the reactions due to the constraints,

[1] Corollary 4.2 of Chapter 5. The components of $\boldsymbol{\omega}$ in S, in terms of $\dot{\varphi}$, $\dot{\psi}$, and $\dot{\theta}$, are computed in (9.4) of Chapter 1.

and its functional form is in general unknown. The elementary work done by the forces $\mathbf{f}$ and the reactions $\mathbf{r}$ for an elemental virtual displacement δq is

$$\begin{aligned}\delta L &= \int \left[\mathbf{f}(P,\dot{P};t) + \mathbf{r}(P,\dot{P};t)\right] \cdot \delta P d\mu(P) \\ &= \int \left[\mathbf{f}(P,\dot{P};t) + \mathbf{r}(P,\dot{P};t)\right] \cdot \frac{\partial P}{\partial q_h}\delta q_h d\mu(P) \\ &= \delta q_h \int \mathbf{f} \cdot \frac{\partial P}{\partial q_h} d\mu(P) + \delta q_h \int \mathbf{r} \cdot \frac{\partial P}{\partial q_h} d\mu(P).\end{aligned}$$

The function

$$(q,\dot{q};t) \to \Phi_h(q,\dot{q};t) = \int \mathbf{f} \cdot \frac{\partial P}{\partial q_h} d\mu(P) \tag{2.1}$$

is the hth component, in configuration space, of the Lagrangian *resultant* of the distribution of forces $\mathbf{f}(P,\dot{P};t)$ acting on the system. The quantity

$$\delta\Lambda = \delta q_h \int \mathbf{r} \cdot \frac{\partial P}{\partial q_h} d\mu(P)$$

is the virtual work due to the reactions $\mathbf{r}(P,\dot{P};t)$. With this notation

$$\delta L = \Phi_h \delta q_h + \delta\Lambda. \tag{2.2}$$

If $\mathbf{f}(P,\dot{P};t)$ is conservative, there exists a function $P \to U(P)$ such that $\mathbf{f} = \nabla U$. In such a case Φ_h takes the form

$$\Phi_h = \int \nabla U \cdot \frac{\partial P}{\partial q_h} d\mu(P) = \frac{\partial}{\partial q_h} V, \tag*{(2.1)$_V$}$$

where the function

$$(q;t) \longrightarrow V(q;t) = \int U d\mu(P)$$

is the *potential* of the system. Therefore for conservative distributions of forces $\mathbf{f}(P,\dot{P};t)$, the Lagrangian components Φ_h of the force are the derivatives of the potential with respect to q_h. Equivalently,

$$\Phi = (\Phi_1,\ldots,\Phi_N) = \nabla_q V \quad \text{and} \quad \Phi_h \delta q_h = \nabla_q V \cdot \delta q = \delta V.$$

With this notation, (2.2) can be written as

$$\delta(L - V) = \delta\Lambda. \tag*{(2.2)$_V$}$$

The principle of virtual work stipulates that instant by instant, the work $\delta\Lambda$ done globally by all the reactions due to the constraints is zero for *every* virtual displacement δq compatible with the constraints [101].[2]

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[2]More generally, to include the case of unilateral constraints, one might require that $\delta\Lambda$ have a sign.

Remark 2.1 The principle of virtual work is a condition on the nature of the constraints and not on the particular motion of the system. Indeed, it is required that $\delta\Lambda$ be zero for every virtual displacement δq, not being restricted to a possible Lagrangian trajectory of the motion. Constraints satisfying such a requirement are *smooth*.

Consider now d'Alembert's principle in the form [36]

$$\left[\ddot{P} - \mathbf{f}(P, \dot{P}; t) - \mathbf{r}(P, \dot{P}; t)\right] d\mu(P) = 0 \qquad \forall P \in \{\mathcal{M}; d\mu\}.$$

Multiplying this by δP and integrating in $d\mu(P)$ gives

$$\delta q_h \int (\ddot{P} - \mathbf{f}) \cdot \frac{\partial P}{\partial q_h} d\mu(P) = \delta q_h \int \mathbf{r} \frac{\partial P}{\partial q_h} d\mu(P).$$

The last integral is the elemental work $\delta\Lambda$ done by the reactions. If the constraints are smooth, the right-hand side is zero for all virtual displacement δq. Thus

$$\delta q_h \int (\ddot{P} - \mathbf{f}) \cdot \frac{\partial P}{\partial q_h} d\mu(P) = 0.$$

Since δq is arbitrary, this gives the N equations

$$\int (\ddot{P} - \mathbf{f}) \cdot \frac{\partial P}{\partial q_h} d\mu(P) = 0, \qquad h = 1, \ldots, N. \tag{2.3}$$

These are necessary conditions to be satisfied by the motion of any system subject to holonomic constraint that obey the principle of virtual work.

3 The Lagrange Equations

For such a system, start from the kinetic energy

$$T = \frac{1}{2} \int \dot{P}^2(q, \dot{q}; t) d\mu(P)$$

and take the time derivative with respect to $\dot{q}_h$. Taking into account the second equation of (1.1),

$$\frac{\partial T}{\partial \dot{q}_h} = \int \dot{P} \cdot \frac{\partial \dot{P}}{\partial \dot{q}_h} d\mu(P) = \int \dot{P} \cdot \frac{\partial P}{\partial q_h} d\mu(P).$$

Next take the time derivative with respect to time and apply the principle of virtual work in the form (2.3) to obtain

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_h} &= \int \ddot{P} \cdot \frac{\partial P}{\partial q_h} d\mu(P) + \int \dot{P} \cdot \frac{\partial \dot{P}}{\partial q_h} d\mu(P) \\
&= \int \mathbf{f} \cdot \frac{\partial P}{\partial q_h} d\mu(P) + \frac{\partial}{\partial q_h} \frac{1}{2} \int \dot{P}^2 d\mu(P) \\
&= \Phi_h(q, \dot{q}; t) + \frac{\partial}{\partial q_h} T,
\end{aligned}$$

where Φ_h is the hth component in configuration space of the Lagrangian resultant of all the forces, internal and external, acting on the system. These N equations, rewritten in the form [101]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_h} - \frac{\partial T}{\partial q_h} = \Phi_h, \qquad h = 1, \ldots, N, \tag{3.1}$$

are the *equations of Lagrange*. They form a system of N ordinary differential equations of the second order, in the unknown Lagrangian coordinates $(q_1, \ldots, q_N)$. The system is of rank N in the variables $\ddot{q}_h$. Indeed, using in (3.1), the expression (1.3)–(1.5) of the kinetic energy gives, for all $h = 1, \ldots, N$,

$$A_{hk}\ddot{q}_k = -\dot{A}_{hk}\dot{q}_k - \dot{B}_h + \frac{\partial T}{\partial q_h} + \Phi_h \stackrel{\text{def}}{=} g_h(q, \dot{q}; t),$$

where g_h are known functions of $(q, \dot{q}; t)$ and are independent of $\ddot{q}$. Since $\det(A_{hk}) \neq 0$ and $A_{hk}(q; t)$ is independent of $\dot{q}$ and $\ddot{q}$, this in turn implies

$$\ddot{q}_h = f_h(q, \dot{q}; t), \qquad h = 1, \ldots, N, \tag{3.1$'$}$$

for some known functions f_h of the arguments $(q, \dot{q}; t)$ only.

Since the rank of the system (3.1) equals the number of degrees of freedom, the Lagrange equations determine unambiguously the motion of the system, starting from some prescribed initial data.

3.1 Fixed Constraints and the Energy Integral

If the constraints are time-independent, the kinetic energy does not have an explicit dependence on time and reduces to $T_2(q; \dot{q})$. By taking the derivative with respect to time, we obtain

$$\frac{d}{dt}T = \frac{\partial T}{\partial q_h}\dot{q}_h + \frac{\partial T}{\partial \dot{q}_h}\ddot{q}_h.$$

From (1.5),

$$2T = A_{hk}(q)\dot{q}_h\dot{q}_k = \frac{\partial T}{\partial \dot{q}_h}\dot{q}_h. \tag{3.2}$$

Taking the time derivative yields

$$2\frac{d}{dt}T = \dot{q}_h\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_h} + \ddot{q}_h\frac{\partial T}{\partial \dot{q}_h}.$$

From this last formula subtract the previously obtained expression for $\frac{d}{dt}T$. Taking into account the Lagrange equations (3.1) gives us

$$\frac{d}{dt}T = \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_h} - \frac{\partial T}{\partial q_h}\right)\frac{dq_h}{dt} = \Phi_h\frac{dq_h}{dt}.$$

The differential form of this equality is $dT = dL$, where $dL = \Phi_h dq_h$ is the elemental work done by the Lagrangian resultant Φ of the external and internal forces during a displacement dq in configuration space. In integral form, one obtains the energy integral $T - L = \text{const}$.

3.2 Motion along Geodesics

A point P not subject to any external forces moves on a smooth, workless surface $\mathcal{S}$ of parametric equations $(u, v) \to P(u, v)$. Choosing $q = (u, v)$ as Lagrangian coordinates, one computes

$$\dot{P}^2 = \dot{q}^t \begin{pmatrix} A & B \\ B & C \end{pmatrix} \dot{q} = \Delta(q, \dot{q}),$$

where the functions $q \to A(q), B(q), C(q)$ are the coefficients of the first fundamental form of the surface $\mathcal{S}$ (§§3, 3c, of Chapter 2). Since there are no external forces, the energy coincides with the kinetic energy, up to a constant, and the Lagrange equations (3.1) take the form

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\Delta - \frac{\partial}{\partial q_i}\Delta = 0, \qquad i = 1, 2. \tag{3.3}$$

Dividing these by $2\sqrt{\Delta}$ gives

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\sqrt{\Delta} - \frac{\partial}{\partial q_i}\sqrt{\Delta} = -\frac{1}{4\Delta^{3/2}}\frac{\partial}{\partial \dot{q}_i}\Delta \frac{d}{dt}\Delta, \qquad i = 1, 2.$$

However, by the energy integral, $\Delta(q, \dot{q})$ is constant along the motion. Therefore q is a solution of (3.3) if and only if it is a solution of the system

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\sqrt{\Delta} - \frac{\partial}{\partial q_i}\sqrt{\Delta} = 0, \qquad i = 1, 2. \tag{3.4}$$

These are the parametric equations of the geodesics on the surface $\mathcal{S}$. Thus a force-free point on a surface $\mathcal{S}$ moves along a geodesic (§4 of Chapter 2 and §1.4c of Chapter 9).

4 Lagrangian Function for Conservative Fields

If the distribution of forces $\mathbf{f}(P, \dot{P}; t)$ is conservative, the components Φ_h are given by (2.1) and $(2.1)_V$. Since V is independent of $\dot{q}$, the Lagrange equations (3.1) may be rewritten in the form [101]

$$\frac{d}{dt}\frac{\partial(T+V)}{\partial \dot{q}_h} - \frac{\partial(T+V)}{\partial q_h} = 0, \qquad h = 1, \dots, N. \tag{3.1$_V$}$$

The function

$$(q, \dot{q}; t) \longrightarrow \mathcal{L}(q, \dot{q}; t) \stackrel{\text{def}}{=} T(q, \dot{q}; t) + V(q; t)$$

is the *Lagrangian* function of the mechanical system. The differential system $(3.1)_V$ is of rank N, since

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_h \dot{q}_k}\right) = \det\left(\frac{\partial^2 T}{\partial \dot{q}_h \partial \dot{q}_k}\right) = \det(A_{hk}) \neq 0.$$

By taking the derivative of $\mathcal{L}$ with respect to time and taking into account $(3.1)_V$, we obtain

$$\frac{d}{dt}\mathcal{L} = \dot{q}_h \frac{\partial \mathcal{L}}{\partial q_h} + \ddot{q}_h \frac{\partial \mathcal{L}}{\partial \dot{q}_h} + \frac{\partial \mathcal{L}}{\partial t}, \qquad 0 = \dot{q}_h \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_h} - \dot{q}_h \frac{\partial \mathcal{L}}{\partial q_h}.$$

Add these two relations to obtain

$$\frac{d}{dt}\mathcal{L} = \frac{d}{dt}\left(\dot{q}_h \frac{\partial \mathcal{L}}{\partial \dot{q}_h}\right) + \frac{\partial \mathcal{L}}{\partial t}, \qquad \text{i.e.,} \qquad \frac{d}{dt}\mathcal{H} = -\frac{\partial \mathcal{L}}{\partial t}, \tag{4.1}$$

where we have set

$$\mathcal{H}(q, \dot{q}; t) = \dot{q}_h \frac{\partial \mathcal{L}}{\partial \dot{q}_h} - \mathcal{L}. \tag{4.2}$$

If the constraints are time independent, both $\mathcal{L}$ and $\mathcal{H}$ are explicitly independent of t. In such a case (4.1) implies

$$\mathcal{H}(q, \dot{q}) = \text{const} \qquad \text{along the motion (fixed constraints).} \tag{4.3}$$

Thus $\mathcal{H}(q, \dot{q})$ is an integral of motion. Such an integral, however, does not give new information on the motion of the system. Indeed from (3.2),

$$\mathcal{H} = \frac{\partial T}{\partial \dot{q}_h}\dot{q}_h - (T + V) = 2T - T - V = T - V.$$

Thus (4.3) coincides precisely with the energy integral.

4.1 Further Integrals of Motion and Kinetic Momenta

A Lagrangian coordinate q_h is called *cyclic* or *ignorable* if the Lagrangian $\mathcal{L}$ is independent of q_h. Then $\mathcal{L}_{q_h} = 0$, and $(3.1)_V$ gives

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_h}(q, \dot{q}; t) = \text{const} \qquad \text{along the motion.}$$

This then is an integral of motion. The quantities

$$p_h = \frac{\partial \mathcal{L}}{\partial \dot{q}_h} = \frac{\partial T}{\partial \dot{q}_h} = A_{hk}(q; t)\dot{q}_k + B_h(q; t), \qquad h = 1, \ldots, N, \tag{4.4}$$

constants or not, are called *kinetic momenta* and have the dimensions of a momentum. In the case of a single unconstrained point $\{P; m\}$, the kinetic momentum is precisely the momentum of $\{P; m\}$. Indeed, by taking the Lagrangian coordinates as the Cartesian coordinates, one has $A_{hk} = \delta_{hk}$ and $p_h = m\dot{x}_h$. In terms of kinetic momenta, the hth Lagrange equation in $(3.1)_V$ can be written as $\dot{p}_h = \mathcal{L}_{q_h}$. Thus if q_h is cyclic or ignorable, then $p_h = \text{const}$ is an integral of the motion.

5 Lagrangian and Hamiltonian

The function $\mathcal{H}$ introduced in (4.2) is Hamilton's function or the *Hamiltonian* of the system. Even though it has been derived from the Lagrangian $\mathcal{L}$ and the Lagrange equation of motion, its relation to the Lagrangian is of a mathematical nature and is independent of the motion. First regard the Lagrangian as a function of two N-tuples of independent variables

$$(q,\xi) \longrightarrow \mathcal{L}(q,\xi;t), \qquad q,\xi \in \mathbb{R}^N, \quad t \text{ fixed.}$$

Then introduce two new N-tuples of independent variables (q,η), by setting

$$\eta_h = \frac{\partial \mathcal{L}(q,\xi;t)}{\partial \xi_h} = A_{hk}(q;t)\xi_k + B_h(q;t), \qquad h = 1,\dots,N. \tag{5.1}$$

The variables η are uniquely determined by the two N-tuples (q,ξ). Conversely, since $\det(A_{hk}) \neq 0$, the equations (5.1) permit one to compute the variables ξ in terms of the two N-tuples (q,η). In this way (5.1) can be regarded as a mutual transformation between two pairs of N-tuples of independent variables (q,ξ) and (q,η), each of which can be taken as a set of $2N$ independent variables. Choose (q,η) as independent variables and regard ξ as functions of these. The Hamiltonian is defined, independently of motion, as [72]

$$\mathbb{R}^{2N} \ni (q,\eta) \longrightarrow \mathcal{H}(q,\eta;t) = \eta\cdot\xi - \mathcal{L}(q,\xi;t),$$

where the variables ξ are meant to be computed in terms of (q,η) by the transformations (5.1). From this definition, taking into account the independence of η and q, compute

$$\begin{aligned}
\frac{\partial \mathcal{H}(q,\eta;t)}{\partial q_h} &= \eta_k \frac{\partial \xi_k}{\partial q_h} - \frac{\partial \mathcal{L}(q,\xi;t)}{\partial q_h} - \frac{\partial \mathcal{L}(q,\xi;t)}{\partial \xi_k}\frac{\partial \xi_k}{\partial q_h},\\
\frac{\partial \mathcal{H}(q,\eta;t)}{\partial \eta_h} &= \frac{\partial \eta_k}{\partial \eta_h}\xi_k + \eta_k \frac{\partial \xi_k}{\partial \eta_h} - \frac{\partial \mathcal{L}(q,\xi;t)}{\partial \xi_k}\frac{\partial \xi_k}{\partial \eta_h}.
\end{aligned}$$

5.1 Canonical Form of the Lagrange Equations

In the framework of mechanics the variables ξ are identified with the Lagrangian velocities $\dot q$, and the variables η are the kinetic momenta p. Along the motion, the Hamiltonian $\mathcal{H}(q,p;t)$ is a function of the Lagrangian coordinates q and the momenta p. Since q satisfy the Lagrange equations $(3.1)_V$, the previous differentiation formulas can be rewritten as

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial q_h} &= p_k\frac{\partial \dot q_k}{\partial q_h} - \frac{\partial \mathcal{L}}{\partial q_h} - \frac{\partial \mathcal{L}}{\partial \dot q_k}\frac{\partial \dot q_k}{\partial q_h} = p_k\frac{\partial \dot q_k}{\partial q_h} - \dot p_h - p_k\frac{\partial \dot q_k}{\partial q_h} = -\dot p_h,\\
\frac{\partial \mathcal{H}}{\partial p_h} &= \frac{\partial p_k}{\partial p_h}\dot q_k + p_k\frac{\partial \dot q_k}{\partial p_h} - \frac{\partial \mathcal{L}}{\partial \dot q_k}\frac{\partial \dot q_k}{\partial p_h} = \dot q_h + \dot p_k\frac{\partial \dot q_k}{\partial p_h} - \frac{\partial \mathcal{L}}{\partial \dot q_k}\frac{\partial \dot q_k}{\partial p_h} = \dot q_h.
\end{aligned}$$

Therefore the Lagrange equations $(3.1)_V$ in terms of the variables (q, p) take the equivalent form

$$\dot{q}_h = \frac{\partial \mathcal{H}(q, p; t)}{\partial p_h}, \qquad \dot{p}_h = -\frac{\partial \mathcal{H}(q, p; t)}{\partial q_h}, \qquad h = 1, \dots, N. \tag{5.2}$$

This is the *canonical form* of the Lagrangian system $(3.1)_V$. The variables p and q are called *conjugate* or *canonical variables* [74].

5.2 Partial and Total Time Derivative of the Hamiltonian

Multiply the first equation of (5.2) by $\dot{p}_h$ and the second by $\dot{q}_h$ and subtract the expression so obtained. This gives

$$0 = \frac{\partial \mathcal{H}}{\partial q_k}\dot{q}_k + \frac{\partial \mathcal{H}}{\partial p_k}\dot{p}_k = \frac{d\mathcal{H}}{dt} - \frac{\partial \mathcal{H}}{\partial t}.$$

Therefore, along the motion, the total derivative of $\mathcal{H}\big(q(t), p(t); t\big)$ and the partial derivative of $\mathcal{H}(q, p; t)$ with respect to time are the same. For time-independent constraints, the Lagrangian and the Hamiltonian are both explicitly independent of time. In such a case,

$$\frac{d}{dt}\mathcal{H} = \frac{\partial}{\partial t}\mathcal{H} = 0 \quad \Longrightarrow \quad t \to \mathcal{H}\big(q(t), p(t)\big) = \text{const.}$$

This gives back the energy integral, since for fixed constraints, the Hamiltonian coincides with the total energy of the system.

6 On Motion in Phase Space

As a summary, consider the motion of a system subject only to holonomic constraints, satisfying the principle of virtual work. Having introduced N Lagrangian parameters q, the movement is determined by the system (3.1) starting from a set of initial data. If in particular the external forces are conservative, denoting by $\mathcal{L}(q, \dot{q}; t)$ the Lagrangian of the system, the movement is determined by the N differential equations of the second order

$$\frac{d}{dt}\frac{\partial \mathcal{L}(q, \dot{q}; t)}{\partial \dot{q}_h} - \frac{\partial \mathcal{L}(q, \dot{q}; t)}{\partial q_h} = 0, \qquad h = 1, \dots, N, \tag{6.1}$$

starting from some given initial conditions. In the Hamiltonian formalism, one regards the kinetic momenta p_h as N new independent variables. Having established that the two sets of variables $(q, \dot{q})$ and (q, p) are mutually equivalent, the Hamiltonian

$$\mathcal{H}(q, p; t) = p_h \dot{q}_h - \mathcal{L}(q, \dot{q}; t), \qquad p_h = \frac{\partial \mathcal{L}}{\partial \dot{q}_h}, \tag{6.2}$$

satisfies the canonical system (5.2). The system of N ordinary differential equations of the second order (6.1) is equivalent to the system of $2N$ ordinary differential equations of the first order (5.2). The space $\mathbb{R}^{2N}$, where the first N-tuple of variables represent the Lagrangian configurations and where the second N-tuple of variables are the kinetic momenta, is called *phase space*, and the solutions of (5.2) can be regarded as curves $t \to [q(t), p(t)]$ in phase space. From a mathematical point of view, determining the motion of the system reduces to the integration of (6.1) or equivalently (5.2), starting from some given initial data.

6.1 Integrals of Motion

The *general* integral of (5.2) is a set of $2N$ functions $t \to q(t;\chi)$, $p(t;\pi)$ depending upon $2N$ parameters (χ, π) and satisfying (5.2) identically as the parameters range within their domain of definition. An *integral* of the system (5.2) is a nontrivial relation $f\big(q(t), p(t); t\big) = \text{const}$, satisfied identically by *all* solutions of (5.2). Equivalently, an integral is a smooth function $f(q, p; t)$, that remains constant along *any* orbits of a solution of the system (5.2). The constant depends, in general, on the particular orbit along which f is being computed. Such a relation is nontrivial in the sense that $\|\nabla_{q,p} f(q, p; t)\| > 0$ within its domain of definition.

For fixed constraints, the energy is an integral of motion, since it remains constant along solutions of (5.2). However, the value of such a constant depends on the particular orbit, or equivalently it depends on the initial conditions.

If the Lagrangian $\mathcal{L}$ does not depend on one of the coordinates q_h, then also the corresponding Hamiltonian is independent of q_h. If so, the variable q_h is called *cyclic* or ignorable. If q_h is cyclic, (5.2) implies that the corresponding kinetic momentum is constant along the motion. Thus $p_h = \text{const}$ is an integral of motion.

7 Equilibrium Configurations

The problem of equilibrium for a mechanical system consists in finding those configurations q_o for which the system (3.1) with initial conditions $q(0) = q_o$ and $\dot{q}(0) = 0$ admits the only solution $q = q_o$.

Proposition 7.1 *Assume that the constraints are fixed and satisfy the principle of virtual work. Then* $(q_o; t_o)$ *is an equilibrium configuration for* $\{\mathcal{M}; d\mu\}$ *if and only if* $\Phi_h(q_o; t) = 0$*, for all* $h = 1, \dots, N$.

Proof (Sufficiency). Since the constraints are time-independent, the equilibrium condition for (3.1) takes the form

$$\begin{cases} A_{hk}\ddot{q}_k = \Phi_h(q; t) + \frac{1}{2} A_{\ell k; q_h} \dot{q}_k \dot{q}_\ell - A_{hk; q_\ell} \dot{q}_k \dot{q}_\ell, \\ q(t_o) = q_o, \quad \dot{q}(t_o) = 0. \end{cases}$$

This has a unique solution, and since $q = q_o$ solves the system, it must be the only solution. ■

Proof (Necessity). If the system is in equilibrium in the position q_o, then $\dot{P}$ and $\ddot{P}$ are identically zero for all $P \in \{\mathcal{M}; d\mu\}$. If δP is a virtual displacement, from (2.2)$_V$ and the principle of virtual work in the form (2.3),

$$\int \mathbf{f} \cdot \delta P d\mu(P) = -\delta \Lambda = 0, \tag{7.1}$$

where $\delta \Lambda$ is the infinitesimal virtual work done by the reactions due to the constraints. In terms of the correspondent Lagrangian coordinates, this becomes

$$\begin{aligned} \int \mathbf{f}(P;t) \cdot \delta P d\mu(P) &= \delta q_h \int \mathbf{f}(P;t) \cdot \frac{\partial P}{\partial q_h} d\mu(P) \\ &= \delta q_h \Phi_h(q;t) = 0 \end{aligned}$$

for all arbitrary virtual δq. ■

Remark 7.1 The interest of (7.1) is that it can be seperated from the reactions $\mathbf{r}$, e.g., the equilibrium configurations can be determined without actually knowing the forces reacting to the constraints.

Remark 7.2 If the distribution $\mathbf{f}$ of the forces applied to the system is conservative with potential $U(q)$, then the equilibrium positions, if any, are determined by $\nabla_q U = 0$.

Remark 7.3 For a single point P acted upon by a force $\mathbf{F}$ and otherwise unconstrained, the equilibrium is realized if $\mathbf{F} \cdot \delta P = 0$ for all δP. Therefore P is in equilibrium if and only if $\mathbf{F} = 0$. If P is constrained on a smooth fixed surface, the equilibrium is realized if $\mathbf{F} \cdot \delta P = 0$ for all δP compatible with the constraint. Therefore P is in equilibrium if and only if $\mathbf{F}(P)$ is normal to the surface constraining P.

7.1 Equilibrium for Rigid Systems

Let $\{\mathcal{M}; d\mu\}$ be rigid and let S be a triad fixed with the rigid body and moving with it. A virtual displacement of a point $P \in \{\mathcal{M}; d\mu\}$ is of the form

$$\delta P = \delta O + \delta \boldsymbol{\omega} \wedge (P - O),$$

where O is an arbitrary point of S. Putting this in (7.1) gives

$$\delta O \cdot \int \mathbf{f}(P,t) d\mu(P) = -\delta \boldsymbol{\omega} \cdot \int (P - O) \wedge \mathbf{f}(P,t) d\mu(P).$$

If the rigid motion is unconstrained, there are no reactions and the virtual displacements δO and $\delta\boldsymbol{\omega}$ are arbitrary. Therefore

$$\int \mathbf{f}(P,t)d\mu(P) = 0, \qquad \int (P-O)\wedge \mathbf{f}(P,t)d\mu(P) = 0. \tag{7.2}$$

These characterize the equilibrium configurations of a rigid *unconstrained* body.

8 Canonical Form of the n-Body Problem

Consider a system of n point masses $\{P_i; m_i\}$, $i = 0, 1, \ldots, (n-1)$, subject to their mutual gravitational forces and otherwise unconstrained. The system has $3n$ degrees of freedom and the Hamiltonian $\mathcal{H}$ depends on some choice of $3n$ independent Lagrangian coordinates. There is no natural or preferred choice of such coordinates, each exhibiting advantages and shortcomings. In this and the next two sections we make three different choices of Lagrangian coordinates, and in terms of them write the Hamiltonian and the corresponding canonical system.

First, as Lagrangian coordinates choose the Cartesian coordinates of each P_i, with respect to a fixed inertial triad $\Sigma = \{\Omega; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, e.g., $\mathbf{q}_j = P_j - \Omega$, for $j = 0, 1, \ldots, n-1$. The kinetic energy and the potential V in terms of these Lagrangian parameters are given by

$$2T = \sum_{j=0}^{n-1} m_j \dot{\mathbf{q}}_j^2, \qquad 2V = \sum_{\substack{i,j=0\\ j\neq i}}^{n-1} \gamma \frac{m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|}. \tag{8.1}$$

Having introduced the Lagrangian $\mathcal{L} = T + V$, the equations $(3.1)_V$ yield

$$m_j \ddot{\mathbf{q}}_j = -\sum_{\substack{i=0\\ i\neq j}}^{n-1} \gamma \frac{m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|^2} \frac{\mathbf{q}_j - \mathbf{q}_i}{\|\mathbf{q}_j - \mathbf{q}_i\|}, \qquad j = 0, 1, \ldots, n-1. \tag{8.2}$$

The kinetic momenta p are the components of the vectors $\mathbf{p}_j = \partial\mathcal{L}/\partial\dot{\mathbf{q}}_j$, and in terms of these, the Hamiltonian is computed from

$$\mathcal{H} = \sum_{j=0}^{n-1} \mathbf{p}_j \cdot \dot{\mathbf{q}}_j - \mathcal{L} = T - V = \frac{1}{2}\sum_{j=0}^{n-1} \frac{1}{m_j}\mathbf{p}_j^2 - \frac{1}{2}\sum_{\substack{i,j=0\\ j\neq i}}^{n-1} \gamma \frac{m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|}.$$

Thus the canonical form of (8.2) is

$$\begin{aligned} \dot{\mathbf{q}}_j &= \frac{1}{m_j}\mathbf{p}_j, \\ \dot{\mathbf{p}}_j &= \sum_{\substack{i=0\\ i\neq j}}^{n-1} \gamma \frac{m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|^2} \frac{\mathbf{q}_j - \mathbf{q}_i}{\|\mathbf{q}_j - \mathbf{q}_i\|}, \end{aligned} \qquad j = 0, 1, \ldots, n-1.$$

Summing the second of these over the subscript j gives the integral of motion

$$\frac{d}{dt}\sum_{j=0}^{n-1}\dot{\mathbf{p}}_j = 0 \quad \Longrightarrow \quad \sum_{j=0}^{n-1} m_j \dot{P}_j = \mathbf{Q} = \text{const.}$$

This is precisely the conservation of momentum. Next we interpret such an integral of motion in terms of cyclic variables, for a suitable choice of Lagrangian coordinates.

9 Lagrangian Coordinates Relative to $\{P_o; m_o\}$

As Lagrangian parameters now choose the coordinates of $\{P_o; m_o\}$ with respect to the inertial triad Σ and the coordinates of the remaining points P_i relative to P_o, i.e.,

$$\mathbf{q}_o = P_o - \Omega, \quad \mathbf{q}_j = P_j - P_o, \qquad j = 1, \dots, n-1. \tag{9.1}$$

In terms of these, the kinetic energy and the potential are given by

$$\begin{aligned} 2T &= \sum_{j=0}^{n-1}\left[m_j(\dot{P}_j - \dot{P}_o)^2 + m\dot{P}_o^2 + 2m_j(\dot{P}_j - \dot{P}_o)\cdot\dot{P}_o\right] \\ &= \dot{\mathbf{q}}_o^2 \sum_{j=1}^{n-1} m_j + \sum_{j=1}^{n-1} m_j \mathbf{q}_j^2 + 2\dot{\mathbf{q}}_o \sum_{j=1}^{n-1} m_j \dot{\mathbf{q}}_j, \\ 2V &= \sum_{\substack{i,j=0 \\ j\neq i}}^{n-1} \gamma \frac{m_i m_j}{\|(P_j - P_o) - (P_i - P_o)\|} \\ &= \sum_{j=1}^{n-1} \gamma \frac{m_o m_j}{\|\mathbf{q}_j\|} + \sum_{\substack{i,j=1 \\ j\neq i}}^{n-1} \gamma \frac{m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|}. \end{aligned}$$

It follows that $\mathcal{L}$ is independent of the first three coordinates of $\mathbf{q}_o$. Therefore these are cyclic variables and give rise to the integrals of motion $\nabla_{\dot{\mathbf{q}}_o}\mathcal{L} = 0$. To interpret such integrals of motion, compute the kinetic momenta relative to the Lagrangian coordinates in (9.1). This gives

$$\begin{aligned} \nabla_{\dot{\mathbf{q}}_j}\mathcal{L} &= \mathbf{p}_j = m_j \dot{P}_j \qquad \text{for } j = 1, \dots, n-1, \\ \nabla_{\dot{\mathbf{q}}_o}\mathcal{L} &= \mathbf{p}_o = \sum_{i=0}^{n-1} m_i \dot{P}_i \quad \text{for } j = 0. \end{aligned} \tag{9.2}$$

Thus for $j = 1, \dots, n-1$, the components of the kinetic momenta relative to the Lagrangian variables $P_j - P_o$ coincide with the kinetic momenta one would obtain by taking as Lagrangian coordinates the absolute coordinates of the points P_j with respect to the inertial triad Σ. However, for $j = 0$, the components of the kinetic momenta relative to the variables P_o are the components of the momentum of the system. Thus the integral of motion $\nabla_{\dot{\mathbf{q}}_o}\mathcal{L} = 0$ is precisely the conservation of momentum. To write the Hamiltonian in terms of the variables $\mathbf{q}$ and $\mathbf{p}$, first observe that

$$m_o\dot{\mathbf{q}}_o = \left(\sum_{j=0}^{n-1} m_j \dot{P}_j\right) - \sum_{j=1}^{n-1} m_j \dot{P}_j = \mathbf{p}_o - \sum_{j=1}^{n-1} \mathbf{p}_j.$$

Then compute

$$\begin{aligned}\mathcal{H} = &\frac{1}{2m_o}\left(\mathbf{p}_o - \sum_{j=1}^{n-1} \mathbf{p}_j\right)^2 + \sum_{j=1}^{n-1} \frac{1}{2m_j}\mathbf{p}_j^2 \\ &- \frac{1}{2}\sum_{j=1}^{n} \gamma\frac{m_o m_j}{\|\mathbf{q}_j\|} - \frac{1}{2}\sum_{\substack{i,j=0\\ j\neq i}}^{n-1} \gamma\frac{m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|}.\end{aligned} \tag{9.3}$$

From this, one verifies that $\mathcal{H}$ is independent of $\mathbf{q}_o$, and writes down the corresponding Hamiltonian system

$$\begin{aligned}\dot{\mathbf{q}}_o &= \frac{1}{m_o}\left(\mathbf{p}_o - \sum_{j=1}^{n} \mathbf{p}_j\right),\\ \dot{\mathbf{q}}_i &= -\frac{1}{m_o}\left(\mathbf{p}_o - \sum_{j=1}^{n} \mathbf{p}_j\right) + \frac{1}{m_i}\mathbf{p}_i,\\ \dot{\mathbf{p}}_o &= 0,\\ \dot{\mathbf{p}}_i &= \frac{\gamma m_o m_i}{\|\mathbf{q}_i\|^2}\frac{\mathbf{q}_i}{\|\mathbf{q}_i\|^2} - \sum_{\substack{j=0\\ j\neq i}}^{n} \frac{\gamma m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|^2}\frac{\mathbf{q}_j - \mathbf{q}_i}{\|\mathbf{q}_j - \mathbf{q}_i\|},\end{aligned} \tag{9.4}$$

where in the second and last equations, the index i ranges from 1 to $n-1$. The choice (9.1) of Lagrangian coordinates is due to Poincaré, and (9.4) is called the *Poincaré canonical form* of the n-body problem [127].[3]

10 Inertial Systems and Further Integrals of Motion

Having fixed the inertial triad Σ, choose a new triad Σ_o whose axes are parallel to those of Σ and with origin at the center of mass Ω_o of the n bodies. By conservation of momentum, the center of mass translates with constant velocity with respect to any inertial reference system. It follows that Σ_o is inertial and the center of mass is at rest with respect to Σ_o. This, in turn, implies that since there are no external forces acting on the system, the angular momentum $\mathbf{K}$, taken in Σ_o and with respect to any point, fixed or moving, is conserved (§2 of Chapter 5). Computing it with respect to the central body P_o gives

$$\mathbf{K} = \sum_{j=0}^{n-1} (P_j - P_o) \wedge m_j \dot{P}_j = \mathbf{const}.$$

[3] The choice (9.1) is motivated by a class of linear canonical transformations. See §5.8.1c of the Complements of Chapter 10.

The system has also the energy integral $\mathcal{H} = \text{const}$. Denote by q the Lagrangian coordinates introduced in (9.1) and by p the corresponding kinetic momenta, and identify (q, p) as a point in phase space $\mathbb{R}^{2N}$, where $N = 3n$. With this symbolism, we rewrite the energy integral and the integral of angular momentum as

$$\mathcal{H}(q,p) = (\text{const})_o, \qquad K_j(q,p) = (\text{const})_j, \quad j = 1, 2, 3,$$

where K_j are the components of $\mathbf{K}$ in Σ_o. These identify locally and implicitly four surfaces in phase space, and the motion takes place on the intersection of these surfaces. Along the motion,

$$\nabla\mathcal{H} \cdot \big(\dot{q}, \dot{p}\big) = 0, \qquad \nabla K_j \cdot \big(\dot{q}, \dot{p}\big) = 0, \quad j = 1, 2, 3.$$

Thus at all times, the vectors $\nabla\mathcal{H}$ and ∇K_j are coplanar and lie on the hyperplane normal to the trajectory of (q, p) in phase space. It follows that there exist real-valued parameters λ_j such that

$$\nabla\mathcal{H} = \lambda_j \nabla K_j \qquad \text{along the motion.} \tag{10.1}$$

This is the method of Lagrange multipliers, where the parameters λ_j are, in general, functions of time. Summarizing, with respect to the inertial system Σ_o, the n-body problem has the following integrals of motion:

$$\begin{array}{ll} \mathbf{Q} = 0, & \text{conservation of momentum,} \\ \dot{\mathbf{K}} = 0, & \text{conservation of angular momentum,} \\ \dot{\Omega}_o = 0, & \text{the center of mass is at rest in } \Sigma_o, \\ \dot{\mathcal{H}} = 0, & \text{conservation of energy.} \end{array}$$

The first three equations represent nine scalar integrals of motion. Complemented with the last one, the n-body problem has ten integrals of motion. It turns out that these are the only integrals possible, e.g., any integral of motion of the n-body problem is a linear combination of these [16, 121, 127].

11 The Planar 3-Body Problem (Lagrange [99])

Assume that $n = 2$ and that the three points P_o, P_1, P_2 move in the inertial plane $x_3 = 0$. The inertial reference system Σ_o being fixed, we choose Lagrangian coordinates as in (9.1). Let (x_o, y_o) denote the coordinates of P_o with respect to Σ, and let $(P_i - P_o) = (x_i, y_i)$, $i = 1, 2$, be the coordinates of P_i relative to the central body $\{P_o; m_o\}$. The Lagrangian coordinates q and the corresponding kinetic momenta p are

$$q = (x_o, y_o, x_1, y_1, x_2, y_2), \qquad p = (m_j \dot{P}_j,\, m_1 \dot{P}_1,\, m_2 \dot{P}_2).$$

The conservation of angular momentum in this context takes the form

$$\sum_{i=1,2} \left\{ x_i (m_i \dot{P}_i) \cdot \mathbf{e}_2 - y_i (m_i \dot{P}_i) \cdot \mathbf{e}_1 \right\} = \text{const.} \tag{11.1}$$

This and (10.1) imply that for $\omega = \lambda_3$,

$$\frac{\partial \mathcal{H}}{\partial [(m_i \dot{P}_i) \cdot \mathbf{e}_1]} = -\omega y_i, \qquad \frac{\partial \mathcal{H}}{\partial [(m_i \dot{P}_i) \cdot \mathbf{e}_2]} = \omega x_i, \qquad i = 1, 2.$$

Therefore the differential equations of the canonical form (9.4), relative to the variables $(P_i - P_o)$, are

$$\dot{x}_i + \omega y_i = 0, \qquad \dot{y}_i - \omega x_i = 0, \qquad i = 1, 2. \tag{11.2}$$

Thus P_1 and P_2, spin about P_o with constant angular velocity ω. Subtracting yields

$$\begin{aligned} (x_1 - x_2)' + \omega (y_1 - y_2) &= 0, \\ (y_1 - y_2)' - \omega (x_1 - x_2) &= 0. \end{aligned} \tag{11.3}$$

Multiply the first equation of (11.2) by x_i and the second by y_i and add the resulting equations. Also multiply the first equation of (11.3) by $(x_1 - x_2)$ and the second by $(y_1 - y_2)$ and add the resulting equations. These operations give

$$\frac{d}{dt} \|P_i - P_o\|^2 = 0, \quad i = 1, 2, \quad \text{and} \quad \frac{d}{dt} \|P_1 - P_2\|^2 = 0.$$

Thus, along the motion, the mutual distance of the three bodies remains constant. As a consequence, their mutual geometric configuration remains constant.

12 Configuration of the Three Bodies

From (11.1) also using (10.1), compute

$$\frac{\partial \mathcal{H}}{\partial x_i} = \omega (m_i \dot{P}_i) \cdot \mathbf{e}_2, \qquad \frac{\partial \mathcal{H}}{\partial y_i} = -\omega (m_i \dot{P}_i) \cdot \mathbf{e}_1, \qquad i = 1, 2. \tag{12.1}$$

Moreover, having chosen Σ_o with the origin at the center of mass of the system, the momentum is identically zero. Therefore

$$\begin{aligned} m_o \dot{P}_o &= -m_1 \dot{P}_1 - m_2 \dot{P}_2 \\ &= -m_1 (\dot{P}_1 - \dot{P}_o) - m_2 (\dot{P}_2 - \dot{P}_o) - (m_1 + m_2) \dot{P}_o. \end{aligned}$$

From this,

$$\dot{P}_o = -\frac{m_1}{m} (\dot{P}_1 - \dot{P}_o) - \frac{m_2}{m} (\dot{P}_2 - \dot{P}_o), \qquad m = (m_o + m_1 + m_2).$$

Return to the first of (12.1) and write it for the index $i = 1$:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial x_1} &= \omega[m_1(\dot{P}_1 - \dot{P}_o) + m_1\dot{P}_o] \cdot \mathbf{e}_2 \\ &= \omega m_1 \dot{y}_1 - \omega m_1 \left[\frac{m_1}{m}(\dot{P}_1 - \dot{P}_o) + \frac{m_2}{m}(\dot{P}_2 - \dot{P}_o)\right] \cdot \mathbf{e}_2 \\ &= \omega m_1 \dot{y}_1 - \frac{\omega m_1}{m}\left(m_1\dot{y}_1 + m_2\dot{y}_2\right).\end{aligned}$$

Compute the last term by putting $\omega\dot{y}_i = -\ddot{x}_i$, which follows from (11.2). Using also the conservation of momentum,

$$\begin{aligned}\frac{\omega m_1}{m}\left(m_1\dot{y}_1 + m_2\dot{y}_2\right) &= -\frac{m_1}{m}\left(m_1\ddot{x}_1 + m_2\ddot{x}_2\right) \\ &= -\frac{m_1}{m}\frac{d}{dt}\left[m_1(\dot{P}_1 - \dot{P}_o) + m_2(\dot{P}_2 - \dot{P}_o)\right] \cdot \mathbf{e}_1 \\ &= -\frac{m_1}{m}\frac{d}{dt}\left(\sum_{j=0}^{2} m_i\dot{P}_i - m\dot{P}_o\right) \cdot \mathbf{e}_1 = m_1\ddot{x}_o.\end{aligned}$$

Moreover, from (9.3),

$$\frac{\partial \mathcal{H}}{\partial x_1} = \gamma\frac{m_o m_1 x_1}{\|P_1 - P_o\|^3} + \gamma\frac{m_1 m_2(x_1 - x_2)}{\|P_1 - P_2\|^3}.$$

Therefore, from this and the second equation of (11.2),

$$\gamma\frac{m_o x_1}{\|P_1 - P_o\|^3} + \gamma\frac{m_2(x_1 - x_2)}{\|P_1 - P_2\|^3} = \omega^2 x_1 - \ddot{x}_o.$$

Finally, from (8.2) for the index $j = 0$,

$$\ddot{x}_o = \gamma\frac{m_1 x_1}{\|P_1 - P_o\|^3} + \gamma\frac{m_2 x_2}{\|P_2 - P_o\|^3}.$$

Putting this equation in the preceding one and factoring out x_1 and x_2 gives

$$A_1 x_1 + A_2 x_2 = 0, \tag{12.2}$$

where

$$\begin{aligned}A_1 &= \frac{\gamma(m_o + m_1)}{\|P_1 - P_o\|^3} + \frac{\gamma m_2}{\|P_1 - P_2\|^3} - \omega^2, \\ A_2 &= \frac{\gamma m_2}{\|P_2 - P_o\|^3} - \frac{\gamma m_2}{\|P_1 - P_2\|^3}.\end{aligned}$$

In a similar way compute

$$B_1 x_1 + B_2 x_2 = 0, \tag{12.3}$$

where

$$B_1 = \frac{\gamma m_1}{\|P_1 - P_o\|^3} - \frac{\gamma m_1}{\|P_1 - P_2\|^3},$$
$$B_2 = \frac{\gamma(m_o + m_2)}{\|P_2 - P_o\|^3} + \frac{\gamma m_1}{\|P_1 - P_2\|^3} - \omega^2.$$

By the symmetry of the variables x and y, or by direct calculation we also have

$$A_1 y_1 + A_2 y_2 = 0, \qquad B_1 y_1 + B_2 y_2 = 0, \tag{12.4}$$

with the same coefficients A_i and B_i.

12.1 Configuration of an Equilateral Triangle

Let (x_1, x_2) be a solution of the algebraic system (12.2)–(12.3), and similarly, let (y_1, y_2) be a solution of the system (12.4). If these solutions are linearly independent, then $(x_1 y_2 - x_2 y_1) \neq 0$. This implies that the linear homogeneous system

$$A_1 x_1 + A_2 x_2 = 0, \qquad A_1 y_1 + A_2 y_2 = 0,$$

regarded in the unknowns A_i, has only the trivial solution $A_1 = A_2 = 0$. Similarly, we must also have $B_1 = B_2 = 0$. It follows from the definitions of the parameters A_j and B_j that the three points P_o, P_1, P_2 are the vertices of an equilateral triangle. The motion of the three bodies then is a uniform rotation about the center of mass Ω_o with angular velocity

$$\|\boldsymbol{\omega}\| = \sqrt{\frac{\gamma m}{\|P_1 - P_2\|^3}}.$$

The possible motions are then ∞^2, corresponding the choice of $\|\boldsymbol{\omega}\|$ and the orientation of $\boldsymbol{\omega}$, with respect to the inertial plane $x_3 = 0$.

12.2 Configuration of a Segment

If the solutions (x_1, x_2) of (12.2)–(12.3) and (y_1, y_2) of (12.4) are linearly dependent, the three bodies P_o, P_1, P_2 remain collinear during the motion, and at constant mutual distance. Apart from a possible reordering, we may assume that P_o is between P_1 and P_2. In the plane $x_3 = 0$ introduce a Cartesian system with the x-axis on the line of the collinear points P_o, P_1, P_2 with origin at P_o and positive orientation from P_o to P_1. With this notation,

$$\|P_1 - P_o\| = x_1, \quad \|P_2 - P_o\| = -x_2, \qquad y_1 = y_2 = 0.$$

The the system (12.2)–(12.3) takes the form

$$\begin{aligned}\omega^2 x_1 &= \frac{\gamma(m_o+m_1)}{x_1^2} - \frac{\gamma m_2}{x_2^2} + \frac{\gamma m_2}{(x_1-x_2)^2},\\ \omega^2 x_2 &= \frac{\gamma m_1}{x_1^2} - \frac{\gamma m_1}{(x_1-x_2)^2} - \frac{\gamma(m_o+m_2)}{x_2^2}.\end{aligned} \tag{12.5}$$

Eliminating ω^2 and setting $X = -x_2/x_1$ gives

$$\begin{aligned}&(m_o+m_1)X^5 + (2m_o+3m_1)X^4 + (m_o+3m_1)X^3\\ &- (m_o+3m_2)X^2 - (2m_o+3m_2)X - (m_o+m_2) = 0.\end{aligned}$$

This algebraic equation has a single positive root.[4] Such a root permits one to compute the ratio $X = -x_2/x_1$. Then, having fixed an arbitrary $x_1 > 0$, either equation of (12.5) permits us to compute $\|\boldsymbol{\omega}\|$. Also in such a case the possible motions are ∞^2, corresponding to the choice of x_1 and the orientation of $\boldsymbol{\omega}$.

13 Collapse of the n Bodies

The n-body problem originates from the motion of planets. It is then natural to ask whether at some future time, the n bodies might collapse to a point. Choose an inertial triad $\Sigma_o\{\Omega_o; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with origin at the center of mass Ω_o as indicated in §10, and as Lagrangian coordinates take the coordinates of the points P_j, as indicated in §8. A total collapse will occur if there exists a time, finite or infinite, such that

$$\sum_{i,j=0}^{n-1} m_i m_j \|P_i - P_j\|^2 = \sum_{i,j=0}^{n-1} m_i m_j \|\mathbf{q}_i - \mathbf{q}_j\|^2 = 0.$$

The polar moment of inertia of the system with respect to Ω_o is

$$J = \sum_{j=0}^{n-1} m_j \|P_j - \Omega_o\|^2 = \sum_{j=0}^{n-1} m_j \mathbf{q}_j^2.$$

The notion of total collapse can be expressed in terms of J by computing

$$\begin{aligned}\sum_{i,j=0}^{n-1} m_i m_j \|\mathbf{q}_i - \mathbf{q}_j\|^2 &= \sum_{i=0}^{n-1} m_i \sum_{j=0}^{n-1} m_j \|\mathbf{q}_i - \mathbf{q}_j\|^2\\ &= \sum_{i=0}^{n-1} m_i \sum_{j=0}^{n-1} m_j \left(\|\mathbf{q}_i\|^2 - 2\mathbf{q}_i\cdot\mathbf{q}_j + \|\mathbf{q}_j\|^2\right)\\ &= 2\sum_{i=0}^{n-1} m_i \|\mathbf{q}_i\|^2 \sum_{j=0}^{n-1} m_j = 2mJ,\end{aligned}$$

where m is the total mass of the system. Thus a total collapse occurs at some time t if and only if at that time $J = 0$.

[4] The existence of a *unique* positive root follows from Lagrange's method of alternation of the signs of the coefficients [102].

13.1 The Lagrange–Jacobi Identity

Computing the second time derivative of J gives

$$\begin{aligned}\ddot{J} &= 2\sum_{j=0}^{n-1} m_j \dot{\mathbf{q}}_j^2 + 2\sum_{j=0}^{n-1} m_j \mathbf{q}_j \cdot \ddot{\mathbf{q}}_j \\ &= 4T + 2\sum_{\substack{i,j=0\\ i\neq j}}^{n-1} \gamma \frac{m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|^3}\left(\mathbf{q}_i \cdot \mathbf{q}_j - \mathbf{q}_i^2\right),\end{aligned}$$

where we have used the expression (8.1) of the kinetic energy and (8.2). From the identity

$$2(\mathbf{q}_i \cdot \mathbf{q}_j - \mathbf{q}_i^2) = (\mathbf{q}_j^2 - \mathbf{q}_i^2) - \|\mathbf{q}_j - \mathbf{q}_i\|^2$$

compute

$$2\sum_{\substack{i,j=0\\ i\neq j}}^{n-1} \gamma \frac{m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|^3}(\mathbf{q}_i \cdot \mathbf{q}_j - \mathbf{q}_i^2) = -\sum_{\substack{i,j=0\\ i\neq j}}^{n-1} \gamma \frac{m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|} = -V.$$

Combining these remarks proves the *Lagrange–Jacobi identity*

$$\tfrac{1}{2}\ddot{J} = 2T - V = T + E. \tag{13.1}$$

13.2 The Sundman Inequality [142]

Let $\mathbf{M}$ be the angular momentum of the system with respect to Ω_o, e.g.,

$$\mathbf{M} = \sum_{j=0}^{n-1} m_j (P_j - \Omega_o) \wedge (\dot{P}_j - \dot{\Omega}_o) = \sum_{j=0}^{n-1} m_j \mathbf{q}_j \wedge \dot{\mathbf{q}}_j.$$

By the Cauchy–Schwarz inequality,

$$\|\mathbf{M}\|^2 \leq \sum_{j=0}^{n-1} m_j \|\mathbf{q}_j\|^2 \sum_{j=0}^{n-1} m_j \|\dot{\mathbf{q}}_j\|^2 = 2JT.$$

Putting into this the expression of the kinetic energy T computed from the Lagrange–Jacobi identity proves the *Sundman inequality* [142]

$$\|\mathbf{M}\|^2 \leq J(\ddot{J} - 2E). \tag{13.2}$$

13.3 A Collapse Would Occur in Only Finite Time

Suppose that a collapse occurs as $t \to \infty$, e.g.,

$$\lim_{t\to\infty} J = \lim_{t\to\infty} \frac{1}{2m} \lim_{t\to\infty} \sum_{i,j=0}^{n-1} m_i m_j \|\mathbf{q}_i - \mathbf{q}_j\|^2 = 0.$$

From the expression of V given by the second equation of (8.1),

$$\lim_{t\to\infty} V = \lim_{t\to\infty} \frac{1}{2} \sum_{\substack{i,j=0\\ i\neq j}}^{n-1} \gamma \frac{m_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|} = \infty.$$

Since the energy $T - V$ is conserved, $T \to \infty$, and by the Lagrange–Jacobi identity also $\ddot{J} \to \infty$ as $t \to \infty$. Therefore $\ddot{J} \geq 1$ for t sufficiently large, and

$$J \geq \frac{1}{2}t^2 + at + b \qquad \text{for } t \text{ sufficiently large}$$

for some real constants a and b. As a consequence, $J \to \infty$, contradicting the collapse.

13.4 A Total Collapse Would Occur Only If $\mathbf{M} = 0$

If there is a total collapse at some finite time t_*, then

$$\begin{aligned} \lim_{t\to t_*} J &= \lim_{t\to t_*} \frac{1}{2m} \sum_{i,j=0}^{n-1} m_i m_j \|\mathbf{q}_i - \mathbf{q}_j\|^2 = 0, \\ \lim_{t\to t_*} \ddot{J} &= \lim_{t\to t_*} T = \lim_{t\to t_*} V = \infty. \end{aligned}$$

Therefore there exists $t_o \in [0, t_*)$ such that $\ddot{J} > 0$ for all $t \in [t_o, t_*]$. From this,

$$0 = J(t_*)\dot{J}(t_*) \geq J(t)\dot{J}(t) + \int_t^{t_*} \dot{J}^2 d\tau \geq J(t)\dot{J}(t)$$

for all $t \in [t_o, t_*]$. Therefore if a collapse occurs at some finite time t_*, there exists an interval $[t_o, t_*]$ such that $\dot{J} \leq 0$ for all $t \in [t_o, t_*]$. Multiply the Sundman inequality by $\dot{J}/J$ for $t \in [t_o, t_*]$. Since $\dot{J} \leq 0$,

$$\frac{\dot{J}}{J}\|\mathbf{M}\|^2 \geq \frac{1}{2}\big(\dot{J}\ddot{J} - 2\dot{J}E\big).$$

Integrate this inequality over the interval $[t_o, t] \subset [t_o, t_*)$ to obtain

$$\|\mathbf{M}\|^2 \ln \frac{J(t)}{J(t_o)} \geq \frac{1}{2}\big(\dot{J}^2(t) - \dot{J}(t_o)^2\big) - 2E\big(J(t) - J(t_o)\big) \geq -\frac{1}{2}\dot{J}^2(t_o).$$

From this,

$$\|\mathbf{M}\|^2 \leq \frac{\dot{J}^2(t_o)}{2\big[\ln J(t_o) - \ln J(t)\big]} \qquad \text{for all } \ t \in [t_o, t).$$

Thus $\|\mathbf{M}\| \to 0$ as $t \to t_*$.

Problems and Complements

1c Kinetic Energy in Lagrangian Coordinates

1.1c Homogeneous Functions

A function $\eta \to F(\eta)$ continuous in $\mathbb{R}^N - \{0\}$ is *homogeneous* of degree λ if

$$F(\tau\eta) = \tau^\lambda F(\eta) \quad \text{for all } \eta \in \mathbb{R}^N \text{ and for all } \tau > 0.$$

As an example, let $a_{ij} \in \mathbb{R}$ for $i, j = 1, \dots, N$. Then

$$F(\eta) = a_{ij}\eta_i\eta_j \qquad \text{is homogeneous of degree 2,}$$

$$F(\eta) = \sqrt{|a_{ij}\eta_i\eta_j|} \quad \text{is homogeneous of degree 1.}$$

More generally, if F is homogeneous of degree 2, then $\sqrt{|F|}$ is homogeneous of degree 1. Every function of the ratio $\eta/\|\eta\|$ is homogeneous of degree zero.

A remarkable fact observed by Euler is that the mere homogeneity of F implies its differentiability along rays through the origin. For a fixed nonzero vector $\eta \in \mathbb{R}^N$, denote by F_η the derivative of F along η, whenever it exists.

Theorem 1.1c (Euler). *Let F be homogeneous of degree λ. Then for every nonzero $\eta \in \mathbb{R}^N$, the directional derivative F_η exists and $\|\eta\|F_\eta(\eta) = \lambda F(\eta)$. Moreover, setting $\mathbf{u} = \eta/\|\eta\|$,*

$$\lambda F(\eta) = \eta \cdot \nabla_{\mathbf{u}} F(\eta). \tag{1.1c}$$

If F is differentiable, then this relation characterizes homogeneous functions in the following sense.

Proposition 1.1c *A function $F \in C^1(\mathbb{R}^N)$ is homogeneous of degree λ if and only if*

$$\lambda F(\eta) = \eta \cdot \nabla F, \qquad \forall \eta \in \mathbb{R}^N. \tag{1.2c}$$

If F is homogeneous and regular, then its directional derivatives are also homogeneous. This is made precise in the following proposition.

Proposition 1.2c *Let $F \in C^1(\mathbb{R}^N)$ be homogeneous of degree λ. Then the partial derivatives F_{η_i} are homogeneous of degree $(\lambda - 1)$.*

2c The Principle of Virtual Work

2.1c An Example of Nonconservative Lagrangian Components of Forces

Two point masses $\{P_i; m_i\}$, $i = 1, 2$, move in a horizontal plane, constrained to be at constant mutual distance ℓ and subject to the *nonconservative* field

$$\mathbf{f}(P) = \frac{\mathbf{u} \wedge (P - O)}{\|\mathbf{u} \wedge (P - O)\|^2},$$

where $\mathbf{u}$ is the unit vector perpendicular to the plane of motion. Having introduced a reference system with origin at O and $\mathbf{e}_3 = \mathbf{u}$, we describe P_1 by its polar coordinates (ρ, φ). We describe P_2 by the angle ψ between $(P_2 - P_1)$ and $(P_1 - O)$. Therefore the Lagrangian coordinates are (ρ, φ, ψ) and

$$\begin{aligned} P_1 - O &= \rho \cos\varphi \mathbf{e}_1 + \rho \sin\varphi \mathbf{e}_2, \\ P_2 - O &= \{\rho\cos\varphi + \ell\cos(\varphi - \psi)\}\mathbf{e}_1 \\ &\qquad + [\rho \sin\varphi + \ell \sin(\varphi - \psi)]\mathbf{e}_2. \end{aligned}$$

The measure $d\mu(P)$ in (2.1) is a combination of Dirac masses acting on the points P_i. Therefore the Lagrangian components of the resultant of forces along the "directions" (ρ, φ, ψ) are

$$\begin{aligned} \Phi_\rho &= \int \mathbf{f}(P)\frac{\partial P}{\partial \rho} d\mu(P) = \mathbf{f}(P_1)\frac{\partial P_1}{\partial \rho} + \mathbf{f}(P_2)\frac{\partial P_2}{\partial \rho}; \\ \Phi_\varphi &= \int \mathbf{f}(P)\frac{\partial P}{\partial \varphi} d\mu(P) = \mathbf{f}(P_1)\frac{\partial P_1}{\partial \varphi} + \mathbf{f}(P_2)\frac{\partial P_2}{\partial \varphi}; \\ \Phi_\psi &= \int \mathbf{f}(P)\frac{\partial P}{\partial \psi} d\mu(P) = \mathbf{f}(P_1)\frac{\partial P_1}{\partial \psi} + \mathbf{f}(P_2)\frac{\partial P_2}{\partial \psi}. \end{aligned}$$

These permit one to compute such components explicitly.

4c Lagrangian Function for Conservative Fields

4.1c Rigid Rod with Extremities Sliding on a Circle

The extremities A and B of a rigid rod of length R and mass m slide without friction on a circle of mass M and radius R. The extremities are attracted to a point P fixed on the circle by two springs of elasticity constant $k/\sqrt{3} > 0$. The circle, in turn, is free to rotate about its center O, remaining in a fixed horizontal plane (**Figure 4.1c**). The system has two degrees of freedom. In the plane of the system fix a line through O and let $P_o = (R, 0)$. Then as

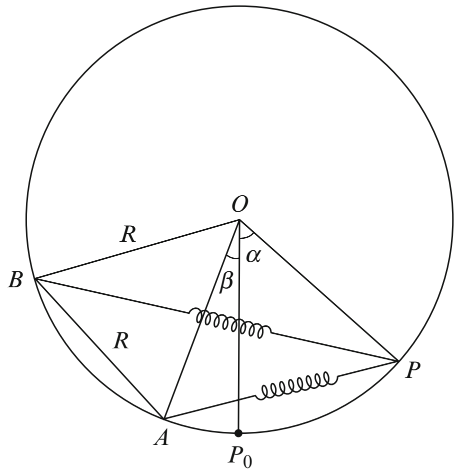

Fig. 4.1c.

Lagrangian parameters choose the angles $\alpha = \widehat{POP_o}$ and $\beta = \widehat{P_oOA}$. Then $\widehat{AOB} = \pi/3$ and

$$\|P - A\| = \sqrt{2}R\sqrt{1 - \cos(\alpha + \beta)},$$
$$\|P - B\| = \sqrt{2}R\sqrt{1 - \cos(\alpha + \beta + \pi/3)}.$$

The kinetic energy T and the potential V are

$$T = \tfrac{1}{2}R^2(M\dot{\alpha}^2 + \tfrac{5}{6}m\dot{\beta}^2), \qquad V = kR^2\cos(\alpha + \beta + \tfrac{1}{6}\pi) + \text{const.}$$

The Lagrangian is $\mathcal{L} = T + V$, and the Lagrange equations $(3.1)_V$ are

$$\ddot{\alpha} + \frac{k}{M}\sin(\alpha + \beta + \tfrac{1}{6}\pi) = 0, \qquad \ddot{\beta} + \frac{6k}{5m}\sin(\alpha + \beta + \tfrac{1}{6}\pi) = 0.$$

These imply that $\alpha = (5m/6M)\beta$ and reduce to the equation of a pendulum,

$$\ddot{\theta} + \nu^2\sin\theta = 0, \qquad \theta = \frac{(5m + 6M)}{5m}\alpha + \frac{\pi}{6}, \quad \nu^2 = \frac{k(5m + 6M)}{5mM}.$$

Resolve the motion starting from the rest position $A = P_o = P$, i.e.,

$$\alpha(0) = \beta(0) = \dot{\alpha}(0) = \dot{\beta}(0) = 0 \quad \text{or} \quad \theta(0) = \tfrac{1}{6}\pi, \quad \dot{\theta}(0) = 0.$$

Compute the reactions due to the constraints at A, B, and O.

4.2c Points Sliding on Rectilinear Guides

Two points P_1 and P_2, of equal mass m, slide on frictionless rectilinear horizontal guides ℓ_i, $i = 1, 2$, at mutual distance d. The two points P_i are attracted by points $O_i \in \ell_i$, fixed at the same ordinate $y = 0$, by springs of elasticity constant k. They are also mutually attracted by an elastic force of constant k. The points have coordinates $P_1 = (0, y_1)$ and $P_2 = (d, y_2)$, and the variables y_i can be taken as Lagrangian coordinates. The Lagrangian is

$$2\mathcal{L} = m(\dot{y}_1^2 + \dot{y}_2^2) - k\left[y_1^2 + y_2^2 + (y_2 - y_1)^2\right] + \text{const.}$$

The two Lagrange equations are

$$m\ddot{y}_1 + k(2y_1 - y_2) = 0, \qquad m\ddot{y}_2 + k(2y_2 - y_1) = 0.$$

If the initial conditions are homogeneous, the trivial solution is the only one. Assume then

$$\dot{y}_1(0) = v_o \neq 0, \qquad \dot{y}_2(0) = y_1(0) = y_2(0) = 0.$$

Introducing the new variables $u = y_1 + y_2$ and $v = y_1 - y_2$, and the constant $\nu^2 = k/m$, the system is transformed into

$$\begin{aligned} &\ddot{u} + \nu^2 u = 0, \;\; \dot{u}(0) = v_o, \; u(0) = 0, \\ &\ddot{v} + 3\nu^2 v = 0, \; \dot{v}(0) = v_o, \; v(0) = 0. \end{aligned}$$

This has solutions

$$u = \frac{v_o}{\nu}\sin \nu t, \qquad v = \frac{v_o}{\sqrt{3}\nu}\sin\sqrt{3}\nu t.$$

Assume now that the guides ℓ_i are parallel and vertical so that the P_i are subject also to their weight. Write down the Lagrangian and solve the resulting Lagrangian equations.

4.3c Pulleys and Weights

Two equal pulleys of radius R and mass M are connected as in **Figure 4.2c** by perfectly flexible, not extensible, weightless ropes, not allowed to slide in the grooves. The two material points $\{P_i; m_i\}$, $i = 1, 2$, are subject to their weight, and P_1 is also acted upon by the elastic force $\mathbf{F} = -k(P_1 - A)$, for a given $k > 0$.

Denoting by z_P the vertical coordinate of a point P:

$$T = \tfrac{1}{2}m_1\dot{z}_{P_1}^2 + \tfrac{1}{2}m_2\dot{z}_{P_2}^2 + \tfrac{1}{4}MR^2(\dot{\varphi}^2 + \dot{\psi}^2) + \tfrac{1}{2}M\dot{z}_{O_1}^2.$$

Moreover,

$$\begin{aligned} \dot{z}_{P_1} &= -\dot{z}_{O_1}, \; \dot{z}_{P_2} = 2\dot{z}_{O_1}, \\ \dot{z}_{P_1} &= -R\dot{\varphi}, \;\; \dot{z}_{O_1} = R\dot{\psi}, \end{aligned} \qquad \text{which implies} \quad \dot{\varphi} = \dot{\psi}.$$

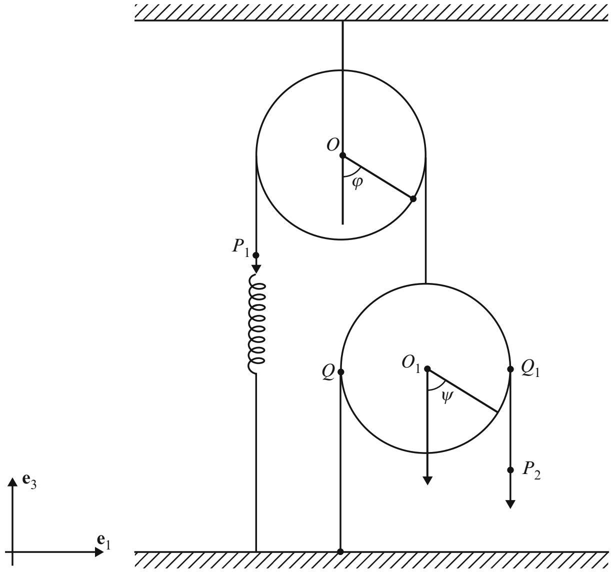

Fig. 4.2c.

Choosing $z = z_{P_1}$ as the only Lagrangian coordinate, we obtain

$$T = \tfrac{1}{2}(m_1 + 4m_2 + 2M)\dot{z}^2.$$

The total of gravitational and elastic potentials is

$$V = -m_1 g z_{P_1} - m_2 g z_{P_2} - M g z_{O_1} - \tfrac{1}{2}k z_{P_1}^2 + \text{const}.$$

Moreover,

$$\begin{aligned} z_{P_1} + z_{O_1} &= \text{const}, \\ 2z_{O_1} - z_{P_2} &= \text{const}, \end{aligned} \qquad \text{which implies} \qquad 2z_{P_1} + z_{P_2} = \text{const}.$$

Therefore

$$V = (-m_1 + 2m_2 + M)gz - \tfrac{1}{2}kz^2 + \text{const}.$$

The energy integral is then

$$(m_1 + 4m_2 + 2M)\dot{z}^2 + 2(m_1 - 2m_2 - M)gz + kz^2 = \text{const}.$$

Take the time derivative of this and divide by $\dot{z}$, to get

$$(m_1 + 4m_2 + 2M)\ddot{z} + kz = (2m_2 - m_1 + M)g.$$

Therefore the system exhibits harmonic oscillations of period T about its equilibrium configuration z_o, where

$$z_o = \frac{g}{k}(2m_2 - m_1 + M), \qquad T = 2\pi\sqrt{\frac{m_1 + 4m_2 + M}{k}}.$$

Write down the Lagrangian $\mathcal{L}$ and verify that the corresponding Lagrangian equation is precisely the previous one. The equilibrium configuration can also be found by setting $V_z = 0$ (see also Remark 7.2).

4.4c Homogeneous Material Frame

A homogeneous square material frame of edge 2ℓ and mass m constrained in a horizontal plane π can rotate about the midpoint O of one of its edges by a fixed cylindrical, workless hinge. On the opposite side, a point P of mass M can slide with no friction, and it is attracted by an elastic force of constant k by the center Q of that side (**Figure 4.3c**). Write down the integrals of the energy and angular momentum. Compute the Lagrangian, write down the Lagrangian system, and put it in canonical form. The system has two degrees

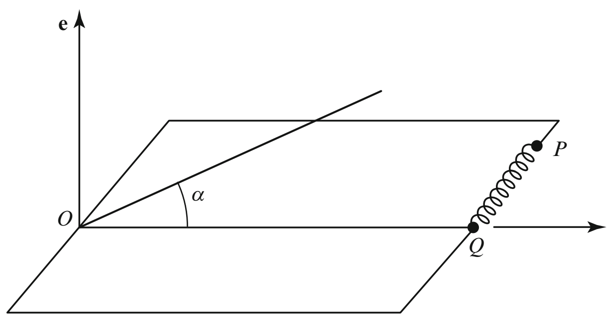

Fig. 4.3c.

of freedom, and as Lagrangian coordinates take the angle φ formed by $(Q-O)$ with a fixed direction of the plane π and the distance x from P to Q. Denote by $\mathbf{e}$ the ascending unit normal to π. The kinetic energy is

$$T = \tfrac{1}{2}\mathcal{I}\dot{\varphi}^2 + M\dot{P}^2, \quad \mathcal{I} = \tfrac{7}{3}m\ell^2, \quad \dot{P}^2 = (2\ell\dot{\varphi} - \dot{x})^2 + x^2\dot{\varphi}^2.$$

Here $\mathcal{I}$ is the moment of inertia of the frame with respect to the vertical axis through O, and it can be computed by Huygens's theorem, whereas $\dot{P}$ can be computed by the formula of relative velocities. Therefore

$$T = \tfrac{7}{6}m\ell^2\dot{\varphi}^2 + \tfrac{1}{2}M\left[(2\ell\dot{\varphi} - \dot{x})^2 + x^2\dot{\varphi}^2\right].$$

The potential of the elastic force is $2V = -kx^2 + \text{const}$. Therefore the energy integral takes the form

$$7m\ell^2\dot{\varphi}^2 + 3M\left[(2\ell\dot{\varphi} - \dot{x})^2 + x^2\dot{\varphi}^2\right] + 3kx^2 = \text{const}.$$

The angular momentum with respect to O is the sum of the angular momentum $\mathbf{K}_1$ of the frame and the angular momentum $\mathbf{K}_2 = M(P-O)\wedge\dot{P}$ of the point P. One computes

$$\mathbf{K}_1 = \mathcal{I}\boldsymbol{\omega} = -\tfrac{7}{3}m\ell^2\dot{\varphi}\mathbf{e}, \qquad \mathbf{K}_2 = M\left[2\ell\dot{x} - (4\ell^2 + x^2)\dot{\varphi}\right]\mathbf{e}.$$

Therefore the integral of the angular momentum takes the form

$$(\tfrac{7}{3}m\ell^2 + 4M\ell^2 + Mx^2)\dot{\varphi} + 2M\ell\dot{x} = \text{const.}$$

The Lagrangian of the system is

$$\mathcal{L} = \tfrac{7}{6}m\ell^2\dot{\varphi}^2 + \tfrac{1}{2}M\left[(2\ell\dot{\varphi} - \dot{x})^2 + x^2\dot{\varphi}^2\right] - \tfrac{1}{2}kx^2 + \text{const.}$$

From this we obtain

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{x}} &= -M(2\ell\dot{\varphi} - \dot{x}), \qquad & \frac{\partial\mathcal{L}}{\partial x} &= Mx\dot{\varphi}^2 - kx,\\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} &= -M(2\ell\ddot{\varphi} - \ddot{x}), \qquad & \frac{\partial\mathcal{L}}{\partial\varphi} &= 0,\\
\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} &= \tfrac{7}{3}m\ell^2\dot{\varphi} + M\left[2\ell(2\ell\dot{\varphi} - \dot{x}) + x^2\dot{\varphi}\right], \\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} &= \tfrac{7}{3}m\ell^2\ddot{\varphi} + M\left[2\ell(2\ell\ddot{\varphi} - \ddot{x}) + 2x\dot{x}\dot{\varphi} + x^2\ddot{\varphi}\right].
\end{aligned}$$

Thus the Lagrange equations are

$$\begin{aligned}
M(2\ell\ddot{\varphi} - \ddot{x}) + Mx\dot{\varphi}^2 - kx &= 0,\\
\tfrac{7}{3}m\ell^2\ddot{\varphi} + M\left[2\ell(2\ell\ddot{\varphi} - \ddot{x}) + 2x\dot{x}\dot{\varphi} + x^2\ddot{\varphi}\right] &= 0.
\end{aligned}$$

For the Hamilton canonical form, introduce the variables

$$q_1 = x, \quad q_2 = \varphi, \quad p_1 = \frac{\partial\mathcal{L}}{\partial\dot{x}}, \quad p_2 = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}},$$

and compute

$$2\mathcal{H} = \frac{3\,(p_2 + 2\ell p_1)^2}{(7m\ell^2 + 3Mq_1^2)} + \frac{p_1^2}{M} + kq_1^2.$$

4.5c Disk Rolling on a Slanted Guide

Consider the Problem 3.1c of Chapter 5, and continue to assume that the disk rolls without slipping on the guide. A point mass $\{P; m\}$ is placed on a smooth circular guide centered at P_o with radius $0 < \rho < R$, grooved on the disk. Compute the total kinetic and potential energy of the system. Write down the Lagrangian and the corresponding Lagrangian equations. Determine the equilibrium configurations, if any, in terms of $\alpha \in [0, \pi/2)$. Determine those initial configurations, if any, starting from which the angle between $(P - P_o)$ and $\mathbf{e}_1$ remains constant. If such configurations exist, determine the corresponding reaction force exerted by the constraint on P.

Let φ be the angle formed by $\mathbf{e}_1$ and $(P - P_o)$. Then

$$\dot{P} = \frac{d}{dt}(P - P_o) + \dot{P}_o = (\dot{x} - \rho\dot{\varphi}\sin\varphi)\,\mathbf{e}_1 + \rho\dot{\varphi}\cos\varphi\mathbf{e}_3.$$

From this, the kinetic energy T_P of the point P is

$$2T_P = m\left(\dot{x}^2 + \rho^2\dot{\varphi}^2 - 2\rho\dot{x}\dot{\varphi}\sin\varphi\right).$$

Therefore the total kinetic energy of the system is given by

$$T = \left(\tfrac{3}{4}M + \tfrac{1}{2}m\right)\dot{x}^2 + \tfrac{1}{2}m\rho^2\dot{\varphi}^2 - m\rho\dot{x}\dot{\varphi}\sin\varphi.$$

The total potential is

$$V = -(M + m)gx\sin\alpha - mg\rho\sin(\varphi - \alpha) + \text{const.}$$

From the Lagrangian $\mathcal{L} = T + V$, one find the equations of motion

$$\begin{aligned} \left(\tfrac{3}{2}M + m\right)\ddot{x} - m\rho\ddot{\varphi}\sin\varphi - m\rho\dot{\varphi}^2\cos\varphi + (M + m)g\sin\alpha &= 0, \\ m\rho^2\ddot{\varphi} - m\rho\ddot{x}\sin\varphi + m\rho\dot{x}\dot{\varphi}\cos\varphi + mg\rho\cos(\varphi - \alpha) &= 0. \end{aligned}$$

Equilibrium occurs for $\dot{x} = \ddot{x} = \dot{\varphi} = \ddot{\varphi} = 0$. From the Lagrange equations, this occurs only for $\sin\alpha = 0$ and $\cos(\varphi - \alpha) = 0$, i.e., only if $\alpha = 0$ and $\varphi = \pm\pi/2$.

If φ remains constant, then $\dot{\varphi} = 0$ and the Lagrange equations give

$$\ddot{x} = kg\sin\alpha, \qquad \tan\alpha\tan\varphi = \frac{-1}{k+1}; \quad k = \frac{2(M+m)}{3M+2m}.$$

Thus such a motion is possible if initially, and hence for all later times, φ is given to satisfy such a relation. The reaction is then computed from

$$m\ddot{P} = -mg\mathbf{u} + \mathbf{R}, \qquad \mathbf{u} = -\sin\alpha\mathbf{e}_1 + \cos\alpha\mathbf{e}_3.$$

5c Lagrangian and Hamiltonian

5.1c The Legendre Transform of the Lagrangian

For q and t fixed the function $\xi \to \mathcal{L}(q, \xi; t)$ is convex and

$$\lim_{\|\xi\|\to\infty} \frac{\mathcal{L}(q,\xi;t)}{\|\xi\|} = \lim_{\|\xi\|\to\infty} \left(\frac{1}{2}A_{hk}(q;t)\frac{\xi_h\xi_k}{\|\xi\|} + \frac{V(q;t)}{\|\xi\|}\right) = \infty. \qquad (*)$$

The Legendre transform of $\mathcal{L}$ with respect to ξ is

$$\mathcal{L}^*(q,\eta;t) = \sup_{\xi\in\mathbb{R}^N} \big[\eta\cdot\xi - \mathcal{L}(q,\xi;t)\big].$$

By the structure of $\mathcal{L}$ as a function of ξ, and the growth condition $(*)$, the supremum is achieved at a unique ξ, at finite distance from the origin, and satisfying

$$\eta_h = \frac{\partial\mathcal{L}(q,\xi;t)}{\partial\xi_h}, \qquad h = 1,2,\dots,N.$$

This permits expressing $\xi = \xi(q,\eta;t)$. For such a ξ,

$$\mathcal{L}^*(q,\eta;t) = \eta\cdot\xi - \mathcal{L}(q,\xi;t) = \mathcal{H}(q,\eta;t).$$

Therefore the Hamiltonian $\eta \to \mathcal{H}(q,\eta;t)$ is the Legendre transform of the Lagrangian $\xi \to \mathcal{L}(q,\xi;t)$. Prove that the Hamiltonian is a convex function of η, and it satisfies a growth condition similar to $(*)$ as $\|\eta\| \to \infty$. Thus one might take the Legendre transform $\xi \to \mathcal{H}^*(q,\xi;t)$ of $\eta \to \mathcal{H}(q,\eta;t)$. Prove that $\mathcal{H}^* = \mathcal{L}$. Therefore $\mathcal{L}^{**} = \mathcal{L}$ and $\mathcal{H}^{**} = \mathcal{H}$. It is said that the Lagrangian and the Hamiltonian are in involution with respect to the Legendre transform.

7c Equilibrium Configurations

A material smooth semicircle of mass M, radius R, and center C is constrained to move in a vertical plane π as in **Figure 7.1c**. One of the extremes O is fixed by a cylindrical hinge, and the second extreme Q is attracted to a point P on the horizontal through O and at a distance $2R$ from it by a spring of elasticity constant k. Along the arc a ring A of mass m can slide without friction. Determine the equilibrium configurations of the system. Denote by $\mathbf{e}_1$ the unit vector along $(P-O)$ and by $\mathbf{e}_2$ the unit ascending vertical. As Lagrangian coordinates take the angle φ between $(Q-O)$ and $\mathbf{e}_1$ and the angle ψ between

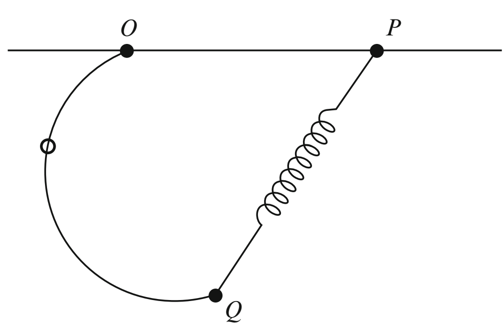

Fig. 7.1c.

$(A - C)$ and $\mathbf{e}_1$. Denoting by P_o the center of mass of the semicircle, we have $\|P_o - C\| = 2R/\pi$. Then in terms of φ and ψ,

$$\begin{aligned} P_o &= R\Big(\cos\varphi - \frac{2}{\pi}\sin\varphi\Big)\mathbf{e}_1 - R\Big(\sin\varphi + \frac{2}{\pi}\cos\varphi\Big)\mathbf{e}_2, \\ Q &= 2R\cos\varphi\,\mathbf{e}_1 - 2R\sin\varphi\,\mathbf{e}_2, \\ A &= R\,(\cos\varphi + \cos\psi)\,\mathbf{e}_1 - R\,(\sin\varphi + \sin\psi)\,\mathbf{e}_2, \\ (P - Q) &= 2R\,[(1 - \cos\varphi)\,\mathbf{e}_1 + \sin\varphi\mathbf{e}_2]\,. \end{aligned}$$

Taking virtual variations yields

$$\begin{aligned} \delta P_o &= -R\Big[\Big(\sin\varphi - \frac{2}{\pi}\cos\varphi\Big)\mathbf{e}_1 + \Big(\cos\varphi - \frac{2}{\pi}\sin\varphi\Big)\mathbf{e}_2\Big]\delta\varphi, \\ \delta Q &= -2R(\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2)\delta\varphi, \\ \delta A &= -R[(\sin\varphi\delta\varphi + \sin\psi\delta\psi)\mathbf{e}_1 + (\cos\varphi\delta\varphi + \cos\psi\delta\psi)\mathbf{e}_2]. \end{aligned}$$

The external forces are

$$-Mg\mathbf{e}_2 \ \text{ on } \ P_o, \quad -mg\mathbf{e}_2 \ \text{ on } \ A, \quad -k(Q - P) \ \text{ on } \ Q.$$

Therefore the elemental work is computed as

$$\begin{aligned} \delta L &= -Mg\mathbf{e}_2 \cdot \delta P_o - mg\mathbf{e}_2 \cdot \delta A - k(Q - P) \cdot \delta Q \\ &= R\,[g\pi(M + m)\cos\varphi - (2Mg + 4k\pi R)\sin\varphi]\,\delta\varphi/\pi \\ &\quad + mgR\cos\psi\delta\psi. \end{aligned}$$

For the equilibrium, $\delta L = 0$ for all virtual variations. This is realized for

$$\psi = \frac{\pi}{2}, \qquad \tan\varphi = \frac{\pi g(m + M)}{2gM + 4k\pi R}.$$

7

PRECESSIONS AND GYROSCOPES

1 The Euler Equations

Let $\{\mathcal{M}; d\mu\}$ be a rigid system in precession about a pole O. Introduce a fixed inertial triad Σ and a moving triad S, both with origin at O, so that S is in rigid motion with respect to Σ with angular characteristic $\boldsymbol{\omega}$. The latter is the unknown of the motion. The system is acted upon by external forces that generate a resultant moment $\mathbf{M}^{(e)}$ with respect to the pole O. The constraint that keeps O fixed and other possible constraints give rise to reactions of resultant moment $\mathcal{M}$ with respect to O. It is assumed that the moment $\mathbf{M}^{(e)}$ and $\mathcal{M}$ are known functions of $\boldsymbol{\omega}$, or equivalently of the Euler angles. For such a system the cardinal equations (3.1)–(3.2) of Chapter 5 are necessary and sufficient to resolve the motion.

The moving triad S will be chosen so that its coordinate axes are principal axes of inertia for the system. For such a choice, the inertia tensor σ with respect to such a triad takes the form

$$\sigma = \begin{pmatrix} \mathcal{I}_1 & 0 & 0 \\ 0 & \mathcal{I}_2 & 0 \\ 0 & 0 & \mathcal{I}_3 \end{pmatrix},$$

where $\mathcal{I}_i$ are the moments of inertia of $\{\mathcal{M}; d\mu\}$ with respect to the principal coordinate axes. From the formula of time differentiation relative to S,

$$\dot{\mathbf{K}} = \left(\frac{d\sigma\boldsymbol{\omega}}{dt}\right)_S + \boldsymbol{\omega} \wedge \sigma\boldsymbol{\omega} = \sigma(\dot{\boldsymbol{\omega}})_S + \boldsymbol{\omega} \wedge \sigma\boldsymbol{\omega}.$$

Therefore the second cardinal equation (3.2), written in the coordinates of S, takes the form

$$\sigma\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \wedge \sigma\boldsymbol{\omega} = \mathbf{M}^{(e)} + \mathcal{M}. \tag{1.1}$$

E. DiBenedetto, *Classical Mechanics: Theory and Mathematical Modeling*, Cornerstones, DOI 10.1007/978-0-8176-4648-6_7,

Equivalently, in term of the components in S,

$$\begin{aligned} \mathcal{I}_1\dot{\omega}_1 &= (\mathcal{I}_2 - \mathcal{I}_3)\omega_2\omega_3 + M_1^{(e)} + \mathcal{M}_1, \\ \mathcal{I}_2\dot{\omega}_2 &= (\mathcal{I}_3 - \mathcal{I}_1)\omega_1\omega_3 + M_2^{(e)} + \mathcal{M}_2, \\ \mathcal{I}_3\dot{\omega}_3 &= (\mathcal{I}_1 - \mathcal{I}_2)\omega_1\omega_2 + M_3^{(e)} + \mathcal{M}_3. \end{aligned} \tag{1.2}$$

These are called the *Euler equations* of the precession.

2 Precessions by Inertia or Free Rotators

A special case is $\mathbf{M}^{(e)} = \mathcal{M} = 0$. This would occur, for example, for a rigid body constrained to move about its center of mass, by a workless constraint. Another example is the rotation of Earth about its longitudinal axis. In such a case, the system (1.2) becomes

$$\begin{aligned} \mathcal{I}_1\dot{\omega}_1 &= (\mathcal{I}_2 - \mathcal{I}_3)\omega_2\omega_3, \\ \mathcal{I}_2\dot{\omega}_2 &= (\mathcal{I}_3 - \mathcal{I}_1)\omega_1\omega_3, \\ \mathcal{I}_3\dot{\omega}_3 &= (\mathcal{I}_1 - \mathcal{I}_2)\omega_1\omega_2, \end{aligned} \tag{2.1}$$

and the motion is called either precession by inertia or precession of a free rotator or Poinsot precession. The first integral of motion follows from the second cardinal equation and gives $\mathbf{K} = \mathbf{K}_o$ for a given vector $\mathbf{K}_o$ constant in Σ. The second integral is that of the energy, and it is derived from (2.1) by multiplying the ith equation by ω_i and summing over $i = 1, 2, 3$. This gives

$$0 = \frac{1}{2}\frac{d}{dt}\mathcal{I}_i\omega_i^2 = \frac{1}{2}\frac{d}{dt}\boldsymbol{\omega}^t\sigma\boldsymbol{\omega} = \frac{1}{2}\frac{d}{dt}\mathcal{I}_{\boldsymbol{\omega}}\|\boldsymbol{\omega}\|^2 = \frac{dT}{dt}.$$

Therefore the kinetic energy remains constant along the motion. The two integrals

$$\begin{aligned} &\mathbf{K} = \mathbf{K}_o \qquad \text{constant vector in } \Sigma, \\ &T = T_o \qquad \text{given nonnegative constant,} \end{aligned} \tag{2.2}$$

permit a remarkable geometrical description of the Poinsot precessions. If T and $\mathbf{K}$ are zero, the system (2.1) admits only the trivial solution. Assume henceforth that $T > 0$ and $\|\mathbf{K}\| > 0$, set

$$\lambda = \sqrt{2T}, \tag{2.3}$$

and construct the corresponding inertia ellipsoid

$$\mathcal{E}_\lambda = \left\{ x \in \mathbb{R}^3 \;\middle|\; x^t\sigma x = \lambda^2 \right\}.$$

From the origin O draw the semiaxis directed as $\boldsymbol{\omega}$, and denote by P its intersection with the inertia ellipsoid $\mathcal{E}_\lambda$. Consider now the plane π tangent to $\mathcal{E}_\lambda$ at P. The next proposition asserts that such a plane remains constant, and therefore it is a geometric integral of the motion.

Proposition 2.1 (Poinsot [129]) *Along the motion,* $\mathbf{K}$ *remains normal to* π*. Moreover,* $\pi = \pi_o$ *for a given constant plane* π_o*.*

Proof. The gradient of the function $f(x) = x^t\sigma x$ is $2\sigma x$, and it is normal to the tangent plane to the surface $x^t\sigma x = \lambda^2$ at x. Therefore $\sigma\boldsymbol{\omega} = \mathbf{K}$ is normal to π and points outside the ellipsoid. Since $\mathbf{K} = \mathbf{K}_o$, the plane π has, along the motion, a constant normal vector. To establish that indeed π is a constant plane, it will suffice to show that the distance from the center O of $\mathcal{E}_\lambda$ to π is constant. Denote by H the intersection of the normal from O to π and continue to denote by P the point at which π is tangent to the ellipsoid $\mathcal{E}_\lambda$. The angle α formed by $P - O$ and $H - O$ is the same as the angle between $\boldsymbol{\omega}$ and $\mathbf{K}$ regarded as vectors starting at O. The angle α is acute, since

$$\mathbf{K} \cdot \boldsymbol{\omega} = \boldsymbol{\omega}^t\sigma\boldsymbol{\omega} = 2T > 0. \tag{2.4}$$

Moreover,

$$\|H - O\| = \|P - O\| \cos\alpha, \qquad \cos\alpha = \frac{\boldsymbol{\omega}\cdot\mathbf{K}}{\|\boldsymbol{\omega}\|\,\|\mathbf{K}\|}.$$

Since $P \in \mathcal{E}_\lambda$,

$$\|P - O\|^2 I_{\boldsymbol{\omega}} = (P - O)^t\sigma(P - O) = \lambda^2, \ \text{ i.e., } \ \|P - O\| = \frac{\lambda}{\sqrt{I_{\boldsymbol{\omega}}}}.$$

Therefore

$$\begin{aligned} \|H - O\| &= \frac{\lambda}{\sqrt{I_{\boldsymbol{\omega}}}} \frac{\boldsymbol{\omega}^t\sigma\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|\,\|\mathbf{K}\|} = \frac{\lambda}{\sqrt{I_{\boldsymbol{\omega}}}} \frac{I_{\boldsymbol{\omega}}\|\boldsymbol{\omega}\|^2}{\|\boldsymbol{\omega}\|\,\|\mathbf{K}\|} \\ &= \frac{\lambda}{\|\mathbf{K}\|}\sqrt{I_{\boldsymbol{\omega}}\|\boldsymbol{\omega}\|^2} = \frac{\lambda\sqrt{2T}}{\|\mathbf{K}\|} = \frac{2T}{\|\mathbf{K}\|}. \end{aligned} \tag{2.5}$$

■

Remark 2.1 By (2.4), the components of $\boldsymbol{\omega}$, regarded as a vector starting at O, satisfy the equation of $\mathcal{E}_\lambda$. Therefore $\boldsymbol{\omega} \in \mathcal{E}_\lambda$ and $\boldsymbol{\omega} = P - O$.

Remark 2.2 The axis through O and parallel to $\boldsymbol{\omega}$ is the axis of motion of the rigid motion of S with respect to Σ. Therefore its intersection P with $\mathcal{E}_\lambda$, as part of the rigid motion of S, has instantaneous zero velocity (§7 of Chapter 1).

Since the inertial ellipsoid $\mathcal{E}_\lambda$ is fixed with S, one might think of it as a rigid material ellipsoid and identify its rigid motion, as the rigid motion of S. Along the motion, $\mathcal{E}_\lambda$ remains tangent to a fixed plane π normal to $\mathbf{K}$. Since the point of tangency P has instantaneous zero velocity, the ellipsoid rolls without slipping on π.

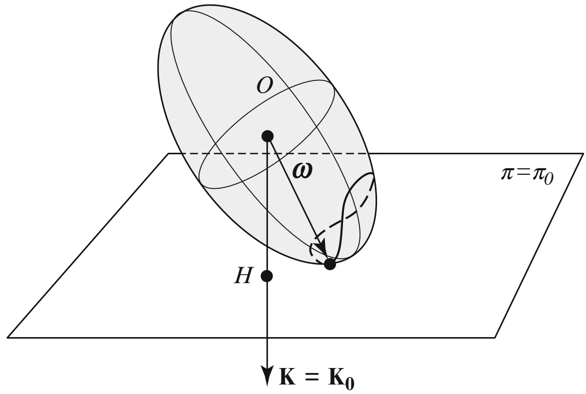

Fig. 2.1.

3 Moving and Fixed Polhodes

The geometrical locus of the contact points of $\mathcal{E}_\lambda$ and π can be described either in S or in Σ. When interpreted in S it is a curve traced on $\mathcal{E}_\lambda$ called a *moving polhode*. When regarded in Σ it is a curve traced in the fixed plane π, called a *fixed polhode*. This double description is closely related to that of moving and fixed axodes $\mathbb{G}_S$ and $\mathbb{G}_\Sigma$ (§10 of Chapter 1). For a precession, the Poinsot cones are generated by the axis of motion of the intrinsic parametric equation

$$(\lambda, t) \longrightarrow P(\lambda; t) = O + \lambda\boldsymbol{\omega}.$$

When regarded in Σ this generates a fixed cone, and when interpreted in S it generates a moving cone. For $\lambda > 0$, this equation gives the positive Poinsot semicone generated by the half-lines from O and directed as $\boldsymbol{\omega}$. The moving and fixed polhodes are the intersections of the positive Poinsot semicones with the inertia ellipsoid $\mathcal{E}_\lambda$ (moving polhode) and the fixed plane π (fixed polhode). Conversely, given the moving and fixed polhodes and projecting on them the origin O generates the moving and fixed positive Poinsot semicones.

3.1 Equations of the Moving Polhode

Let x be a point of the moving polhode. The tangent plane to $\mathcal{E}_\lambda$ at x is

$$(y - x) \cdot \mathbf{K} = (y - x)^t \sigma\boldsymbol{\omega} = \mathcal{I}_i x_i (y_i - x_i) = 0,$$

and the distance h from such a plane to O is

$$h = \frac{\mathcal{I}_i x_i^2}{\sqrt{\mathcal{I}_i^2 x_i^2}} = \frac{\lambda^2}{\sqrt{\mathcal{I}_i^2 x_i^2}}. \qquad (3.1)$$

The equations of the moving polhode are then

$$\begin{aligned} \mathcal{I}_1 x_1^2 + \mathcal{I}_2 x_2^2 + \mathcal{I}_3 x_3^2 &= \lambda^2, \\ \mathcal{I}_1(\mathcal{I}_1 - \mathbf{I})x_1^2 + \mathcal{I}_2(\mathcal{I}_2 - \mathbf{I})x_2^2 + \mathcal{I}_3(\mathcal{I}_3 - \mathbf{I})x_3^2 &= 0, \end{aligned} \tag{3.2}$$

where we have set

$$\mathbf{I} = \left(\frac{\lambda}{h}\right)^2. \tag{3.3}$$

The first equation of (3.2) expresses that $x \in \mathcal{E}_\lambda$. The second is derived by rewriting (3.1) in the form

$$\mathcal{I}_i^2 x_i^2 = \left(\frac{\mathcal{I}_i x_i^2}{h}\right)^2 = \mathbf{I}\,\mathcal{I}_i x_i^2.$$

The number $\mathbf{I}$ defined by (3.3) has the dimensions of a moment of inertia. For a fixed λ, the largest (smallest) value of $\mathbf{I}$ along the motion occurs when h equals the length of the smallest (largest) of the semiaxes of the ellipsoid $\mathcal{E}_\lambda$.

3.2 Geometry of Moving Polhodes

Assume first that the axial moments of inertia $\mathcal{I}_i$ are all distinct and have, for example, the ordering

$$\mathcal{I}_1 < \mathcal{I}_2 < \mathcal{I}_3. \tag{3.4}$$

Then the parameters h and $\mathbf{I}$ range over the intervals

$$h \in \left[\frac{\lambda}{\sqrt{\mathcal{I}_3}}, \frac{\lambda}{\sqrt{\mathcal{I}_1}}\right], \qquad \mathbf{I} \in [\mathcal{I}_1, \mathcal{I}_3]\,.$$

The parameters λ, h, and $\mathbf{I}$ are mutually and uniquely determined by the initial data T_o and $\mathbf{K}_o$, through (2.2), (2.5), and (3.3). Thus the possible geometrical configurations of the moving polhodes, for different values of these parameters, corresponds to the possible motions originating from different initial data.

3.2.1 Either $\mathcal{I}_1 < \mathbf{I} < \mathcal{I}_2 < \mathcal{I}_3$ or $\mathcal{I}_1 < \mathcal{I}_2 < \mathbf{I} < \mathcal{I}_3$

If the first of these occurs, the second of (3.2) implies

$$\mathcal{I}_1(\mathbf{I} - \mathcal{I}_1)x_1^2 = \mathcal{I}_2(\mathcal{I}_2 - \mathbf{I})x_2^2 + \mathcal{I}_3(\mathcal{I}_3 - \mathbf{I})x_3^2.$$

This is the equation of a cone whose axis is along the major semiaxis of $\mathcal{E}_\lambda$. Therefore the moving polhodes are closed curves traced on $\mathcal{E}_\lambda$ and surrounding its major semiaxis. Such an axis bears the least moment of inertia $\mathcal{I}_1$. In the second occurrence the moving polhodes are closed curves traced on $\mathcal{E}_\lambda$ and surrounding its minor semiaxis. Such an axis bears the largest moment of inertia $\mathcal{I}_3$.

3.2.2 The Degenerate Cases $\mathcal{I}_1 = \mathbf{I}$ and $\mathbf{I} = \mathcal{I}_3$

If $\mathbf{I} = \mathcal{I}_1$, then the only solutions of (3.2) are the two points $(\pm\lambda/\sqrt{\mathcal{I}_1}, 0, 0)$, and the moving polhodes degenerate in either of them. In such a case h equals the length of the major axis of $\mathcal{E}_\lambda$. Such an axis remain fixed along the motion and $\mathcal{E}_\lambda$ rotates about it. A similar degeneracy occurs if $\mathbf{I} = \mathcal{I}_3$.

3.2.3 The Degenerate Case $\mathcal{I}_2 = \mathbf{I}$

The parameter h equals the length of the intermediate axis of $\mathcal{E}_\lambda$, and the second equation of (3.2) degenerates into the two planes

$$\sqrt{\mathcal{I}_1(\mathcal{I}_2 - \mathcal{I}_1)}x_1 \pm \sqrt{\mathcal{I}_3(\mathcal{I}_3 - \mathcal{I}_2)}x_3 = 0.$$

Each of these traces an ellipse on the ellipsoid $\mathcal{E}_\lambda$. Each of these ellipses is, in turn, divided into two arcs by the two points $(0, \pm\lambda/\sqrt{\mathcal{I}_2}, 0)$. Each of these four arcs is a *degenerate limiting* moving polhode. Moreover, the two points $(0, \pm\lambda/\sqrt{\mathcal{I}_2}, 0)$ are two distinct, independent, degenerate polhodes. Indeed, by the remarks of §3.4, rotations about the principal axes are permanent.

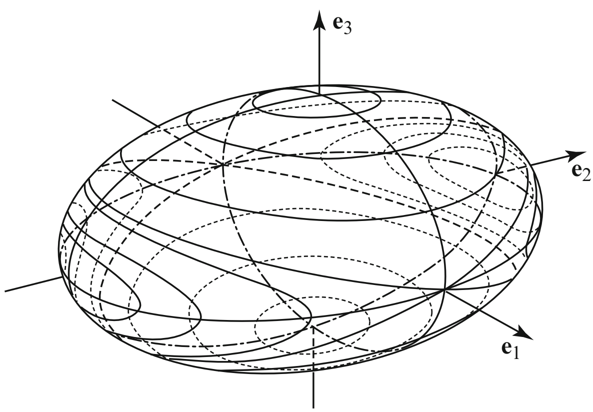

Fig. 3.1.

3.3 Ellipsoids of Rotation

If two of the moments of inertia $\mathcal{I}_i$ coincide, $\mathcal{E}_\lambda$ is an ellipsoid of rotation. The geometric axis of rotation is the *gyroscopic* axis, and the material system to which $\mathcal{E}_\lambda$ corresponds is a *gyroscope.*

Proposition 3.1 *Assume, for example, that* $\mathcal{E}_\lambda$ *is an ellipsoid of rotation about* $\mathbf{u}_3$ *and that* $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}$. *Then* $\|\boldsymbol{\omega}\|$ *is constant, and the Poinsot cones* *are right circular cones, possibly degenerate. The moving polhode is a circle, possibly degenerate, traced on $\mathcal{E}_\lambda$, and the fixed polhode is a circle, possibly degenerate, traced in the fixed plane π.*

Proof. By (2.1) the third component ω_3 of $\boldsymbol{\omega}$ is constant. From the assumptions,

$$\mathbf{K} = \sigma\boldsymbol{\omega} = \mathcal{I}(\omega_1\mathbf{u}_1 + \omega_2\mathbf{u}_2) + \mathcal{I}_3\omega_3\mathbf{u}_3 = \mathcal{I}\boldsymbol{\omega} + (\mathcal{I}_3 - \mathcal{I})\omega_3\mathbf{u}_3.$$

Taking the scalar product by $\boldsymbol{\omega}$ gives

$$\|\boldsymbol{\omega}\|^2 = \frac{\mathbf{K}\cdot\boldsymbol{\omega}}{\mathcal{I}} + \Big(1 - \frac{\mathcal{I}_3}{\mathcal{I}}\Big)\omega_3^2 = \frac{2T_o}{\mathcal{I}} + \Big(1 - \frac{\mathcal{I}_3}{\mathcal{I}}\Big)\omega_3^2 = \text{const.} \qquad \blacksquare$$

3.4 Principal Axes of Inertia as Axes of Permanent Rotation

Let $\boldsymbol{\omega}$ be a nontrivial solution of (2.1), and recall that such a system corresponds to (1.1) with the right-hand side identically zero. Assume that at some instant, say for example $t = 0$, the vector $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_o$ is directed as one of the coordinate axes of S. Since the triad S is principal of inertia, one has $\sigma\boldsymbol{\omega}_o = \eta\boldsymbol{\omega}_o$ for some $\eta \in \mathbb{R}$ and therefore $\omega_o \wedge \sigma\boldsymbol{\omega}_o = 0$. It follows that the only solution of

$$\sigma\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \wedge \sigma\boldsymbol{\omega} = 0 \qquad \text{with } \boldsymbol{\omega}(0) = \boldsymbol{\omega}_o$$

is $\boldsymbol{\omega} = \boldsymbol{\omega}_o$ identically. Thus if at some instant, $\boldsymbol{\omega}$ is directed as one of the principal axes of inertia, it remains constant in that configuration and the rotation about that axis is permanent. For this reason the principal axes of inertia are also referred to as the axes of *permanent rotation*. By analogous considerations, if $\boldsymbol{\omega}$ is constant, it must be directed as one of the principal axes of inertia.

4 Integrating Euler's Equations of Free Rotators (Jacobi [88])

The initial data in (2.1) are given in terms of the first integrals

$$\mathcal{I}_i\omega_i^2 = 2T, \qquad \mathcal{I}_i^2\omega_i^2 = \|\mathbf{K}\|^2, \qquad \|\mathbf{K}\|^2 = 2T\mathbf{I}. \tag{4.1}$$

The first is the energy integral; the last one follows from (2.5), by choosing the parameter λ from (2.3), and by using the definition (3.3) of the "moment of inertia" $\mathbf{I}$. The second follows from (3.1) with x replaced by $\boldsymbol{\omega}$, by Remark 2.1. The function ω_2, the second component of $\boldsymbol{\omega}$, will be computed by assuming that the moments of inertia $\mathcal{I}_i$ are ordered as in (3.4). Multiply the first equation of (4.1) by $\mathcal{I}_1$ and by $\mathcal{I}_3$, subtract the resulting expressions, and take into account the last equation of (4.1) to obtain

$$\begin{aligned}\mathcal{I}_1(\mathcal{I}_1-\mathcal{I}_3)\omega_1^2 &= 2T_o(\mathbf{I}-\mathcal{I}_3)-\mathcal{I}_2(\mathcal{I}_2-\mathcal{I}_3)\omega_2^2,\\ \mathcal{I}_3(\mathcal{I}_3-\mathcal{I}_1)\omega_3^2 &= 2T_o(\mathbf{I}-\mathcal{I}_1)-\mathcal{I}_2(\mathcal{I}_2-\mathcal{I}_1)\omega_2^2.\end{aligned}$$

These are rewritten in the form

$$\omega_1^2 = A_1^2\left(\nu_1^2-\omega_2^2\right), \qquad \omega_3^2 = A_3^2\left(\nu_3^2-\omega_2^2\right), \tag{4.2}$$

where

$$\begin{aligned} A_1^2 &= \frac{\mathcal{I}_2(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_1(\mathcal{I}_1-\mathcal{I}_3)}, & A_3^2 &= \frac{\mathcal{I}_2(\mathcal{I}_2-\mathcal{I}_1)}{\mathcal{I}_3(\mathcal{I}_3-\mathcal{I}_1)},\\ \nu_1^2 &= \frac{2T_o}{\mathcal{I}_2}\,\frac{\mathbf{I}-\mathcal{I}_3}{\mathcal{I}_2-\mathcal{I}_3}, & \nu_3^2 &= \frac{2T_o}{\mathcal{I}_2}\,\frac{\mathbf{I}-\mathcal{I}_1}{\mathcal{I}_2-\mathcal{I}_1}. \end{aligned} \tag{4.2$'$}$$

Putting (4.2) into the second equation of (2.1) gives

$$\dot\omega_2 = \pm A\sqrt{\nu_1^2-\omega_2^2}\sqrt{\nu_3^2-\omega_2^2}, \tag{4.3}$$

where

$$A = \sqrt{\frac{(\mathcal{I}_3-\mathcal{I}_2)(\mathcal{I}_2-\mathcal{I}_1)}{\mathcal{I}_1\,\mathcal{I}_3}}. \tag{4.3$'$}$$

4.1 Integral of (4.3) when $\mathbf{I}\neq\mathcal{I}_i$, $i=1,2,3$

Assume, for example, that $\mathcal{I}_1<\mathbf{I}<\mathcal{I}_2$ and set

$$\eta = \frac{\omega_2}{\nu_3}, \qquad k = \frac{\nu_3}{\nu_1}.$$

The definitions of ν_i, $i=1,3$, imply $|\eta|<1$ and $k\in(0,1)$. Then (4.3) can be rewritten in terms of η as

$$\dot\eta = \pm A\nu_1\sqrt{1-\eta^2}\,\sqrt{1-k^2\eta^2}.$$

This is the equation of a mathematical pendulum of period (§8 of Chapter 3)

$$\frac{4}{A\nu_1}\int_0^1 \frac{ds}{\sqrt{1-s^2}\,\sqrt{1-k^2s^2}}.$$

4.2 Integral of (4.3) when $\mathbf{I}=\mathcal{I}_i$ for Some $i=1,2,3$

If $\mathbf{I}=\mathcal{I}_1$, then $\nu_3=0$. The polhodes are degenerate, and the system spins about the axis $\mathbf{u}_1$ with constant angular velocity $\boldsymbol{\omega}$. Indeed, (2.1) and (4.2)–(4.2)$'$ imply $\omega_1=$ const and $\omega_2=\omega_3=0$. The case $\mathbf{I}=\mathcal{I}_3$ is analogous. If $\mathbf{I}=\mathcal{I}_2$, then

$$\nu_1^2=\nu_3^2=\frac{2T_o}{\mathcal{I}_2}=\nu^2, \qquad k=1,$$

and (4.3) takes the form

$$\dot{\omega}_2 = \pm A \left(\nu^2 - \omega_2^2\right), \qquad \omega_2(0) = \omega_{2,o}. \tag{4.4}$$

If $\omega_{2,o} = \pm\nu$, one has the only solution $(0, \pm\nu, 0)$ resulting in a constant rotation about the intermediate axis $\mathbf{u}_2$. If $|\omega_{2,o}| < \nu$, then (4.4) can be integrated explicitly as

$$\frac{\nu + \omega_2}{\nu - \omega_2} = \frac{\nu + \omega_{2,o}}{\nu - \omega_{2,o}} \, e^{\pm 2A\nu\, t}.$$

These solutions tend to $\pm\nu$ as $t \to \pm\infty$.

5 Rotations about a Fixed Axis

The rigid system $\{\mathcal{M}; d\mu\}$ is constrained to spin, without slipping, about a fixed axis ℓ of unit direction $\mathbf{u}$. Such a constraint can be realized by a spherical hinge placed on a point $O \in \ell$ and a cylindrical hinge placed on another point $Q \in \ell$. The constraints generate reactions $\mathbf{R}_O$ and $\mathbf{R}_Q$ as vectors applied in O and Q respectively. By the nature of the constraint, $\mathbf{R}_Q$ is normal to ℓ, whereas $\mathbf{R}_O$ could take any direction. Fixed and moving triads are taken as

$$S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 = \mathbf{u}\}, \qquad \Sigma = \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{u}\}.$$

For such a choice, the axis ℓ need not be principal of inertia and the corresponding inertia matrix $\sigma = (I_{ij})$ need not be diagonal. As the only Lagrangian coordinate one may choose the angle $\varphi = \widehat{\mathbf{e}_1 \mathbf{u}_1}$, so that $\boldsymbol{\omega} = (0, 0, \dot{\varphi})^t$. With these choices, (1.1), written in components, takes the form

$$\begin{aligned} I_{13}\ddot{\varphi} - I_{23}\dot{\varphi}^2 &= \mathbf{M}^{(e)} \cdot \mathbf{u}_1 - \|Q - O\|\, \mathbf{R}_Q \cdot \mathbf{u}_2, \\ I_{23}\ddot{\varphi} + I_{13}\dot{\varphi}^2 &= \mathbf{M}^{(e)} \cdot \mathbf{u}_2 + \|Q - O\|\, \mathbf{R}_Q \cdot \mathbf{u}_1, \\ I_{33}\ddot{\varphi} &= \mathbf{M}^{(e)} \cdot \mathbf{u}_3 + \mu, \end{aligned} \tag{5.1}$$

where μ is the moment along ℓ generated by the possible presence of friction. Here $\mathbf{M}^{(e)}$ and μ are known functions of $(\varphi, \dot{\varphi})$. The last of these permits the determination of φ starting from some given initial data, whence φ is known as a function of time; it can be put in the first two equations to determine the reactions due to the constraints.

5.1 Rotations about a Principal Axis of Inertia

If ℓ is a principal axis of inertia, then $I_{i3} = 0$, $i = 1, 2.$ and (5.1) become

$$\begin{aligned} \mathbf{M}^{(e)} \cdot \mathbf{u}_1 &= \|Q - O\|\, \mathbf{R}_Q \cdot \mathbf{u}_2, \\ \mathbf{M}^{(e)} \cdot \mathbf{u}_2 &= -\|Q - O\|\, \mathbf{R}_Q \cdot \mathbf{u}_1, \\ I_{33}\ddot{\varphi} &= \ \mathbf{M}^{(e)} \cdot \mathbf{u}_3 + \mu. \end{aligned} \tag{5.2}$$

Thus the role of the reactions $\mathbf{R}_Q$ of the constraint is to generate a moment with respect to O that will balance the component normal to ℓ of the resultant moment $\mathbf{M}^{(e)}$ of the external forces acting on the system. This occurrence characterizes the principal axes of inertia as formalized by the following theorem.

Proposition 5.1 *Let $\{\mathcal{M}; d\mu\}$ be in rigid rotation about a fixed axis ℓ, and let $\mathbf{M}_{\perp\ell}$ denote the component, normal to ℓ, of the resultant moment of the external forces and the reaction due to constraints acting on $\{\mathcal{M}; d\mu\}$. Then*

$$\{\ell \textit{ principal of inertia}\} \quad \Longleftrightarrow \quad \{\mathbf{M}_{\perp\ell} = 0\}\,.$$

Proof. It suffices to prove the implication $\Longleftarrow$. By the assumption the first two equations of (5.1) must hold with the right-hand sides identically zero. Multiplying the first by I_{23} and the second by I_{13} and subtracting the expressions so obtained gives

$$\left(I_{23}^2 + I_{13}^2\right)\dot{\varphi}^2 = 0. \qquad \blacksquare$$

Remark 5.1 Suppose ℓ is not a principal axis of inertia, and that the external forces generate a moment $\mathbf{M}^{(e)}$ whose component normal to ℓ is zero. Then the material system $\{\mathcal{M}; d\mu\}$ spontaneously generates reaction moments normal to ℓ. For this reason the moments I_{ij}, $i \neq j$, are called *deflecting moments.*

6 Gyroscopes

For a material system $\{\mathcal{M}; d\mu\}$ let $S = \{P_o; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be its central principal triad of inertia (§5.1 of Chapter 4). The system is a *gyroscope* if the ellipsoid of inertia $\mathcal{E}_\lambda$ with respect to S is of rotation, say for example if $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}$. In such a case the axis $\mathbf{u}_3$ is called a *gyroscopic axis.* Assume that the center of mass P_o is fixed and that the gyroscope is in precession about P_o. One might perturb the motion of the system by applying a force $\mathbf{F} = F\mathbf{u}_1$, for some $F \in \mathbb{R}$, to a point of the gyroscopic axis other than P_o. Such a force generates a moment $\mathbf{M} = M\mathbf{u}_2$, for some $M \in \mathbb{R}$. In this setting, the Euler equations (1.2) take the form

$$\begin{aligned} \dot{\omega}_1 &= -\nu\omega_2, \\ \dot{\omega}_2 &= \nu\omega_1 + \frac{M}{\mathcal{I}}, \qquad\qquad \nu = \frac{(\mathcal{I}_3 - \mathcal{I})}{\mathcal{I}}\omega_{o,3}, \\ \omega_3 &= \omega_{o,3}. \end{aligned} \tag{6.1}$$

By taking the time derivative we obtain

$$\ddot{\omega}_1 + \nu^2\omega_1 = -\nu\frac{M}{\mathcal{I}}, \qquad \ddot{\omega}_2 + \nu^2\omega_2 = 0.$$

This can be integrated explicitly starting from given initial data

$$\begin{aligned}\omega_1(0) &= \omega_{o,1}, \quad \dot\omega_1(0) = -\nu\omega_{o,2},\\ \omega_2(0) &= \omega_{o,2}, \quad \dot\omega_2(0) = \nu\omega_{o,1} + \frac{M}{\mathcal{I}}.\end{aligned}$$

The explicit solution is

$$\omega_1 + \frac{M}{\mathcal{I}\nu} = A\cos(\nu t + \alpha), \qquad \omega_2 = A\sin(\nu t + \alpha), \tag{6.2}$$

where

$$A = \sqrt{\Big(\omega_{o,1} + \frac{M}{\mathcal{I}\nu}\Big)^2 + \omega_{o,2}^2}, \qquad \tan\alpha = \frac{\mathcal{I}\nu\omega_{o,2}}{\mathcal{I}\nu\omega_{o,1} + M}. \tag{6.2$'$}$$

Assume that the rotation about $\mathbf{u}_3$ is large with respect to the remaining parameters of the motion, in the sense that

$$\Big|\frac{M}{\mathcal{I}\nu\omega_{o,3}}\Big| \ll 1, \qquad \Big|\frac{\omega_i}{\omega_{o,3}}\Big| \ll 1, \qquad i = 1,2. \tag{6.3}$$

It follows from (6.2)–(6.2)$'$ that if (6.3) holds at some time, say for example for $t = 0$, then it continues to hold at all times. In particular, if $\omega_{o,i}$, $i = 1,2$, are negligible with respect to the rotation $|\omega_{o,3}|$ about the gyroscopic axis, then they continue to be negligible at all times.

The velocity of the point $\mathbf{u}_3 = (0,0,1)$ as part of the rigid motion of S is

$$\dot{\mathbf{u}}_3 = \boldsymbol{\omega} \wedge \mathbf{u}_3 = \omega_2\mathbf{u}_1 - \omega_1\mathbf{u}_2 \qquad \text{and} \qquad \|\dot{\mathbf{u}}_3\| = \sqrt{\omega_1^2 + \omega_2^2}.$$

Thus if (6.3) holds at some time, then at all times the motion of the gyroscopic axis, as detected by $\mathbf{u}_3$, remains negligible with respect to $\omega_{o,3}$. This effect is called *tenacity* of the gyroscopic axis, in the sense that it resists perturbations. The functions ω_i for $i = 1,2$ are periodic with period $2\pi/|\nu|$ and frequency $|\nu|$ proportional to $|\omega_{o,3}|$. Thus $\mathbf{u}_3$ exhibits periodic oscillations with frequency proportional to $|\omega_{o,3}|$. This is the *quivering* of the gyroscopic axis. From (6.2),

$$\dot{\mathbf{u}}_3 = A\sin(\nu t + \alpha)\mathbf{u}_1 - A\cos(\nu t + \alpha)\mathbf{u}_2 + \frac{M}{\mathcal{I}\nu}\mathbf{u}_2. \tag{6.4}$$

Integrating over a period yields

$$\frac{1}{T}\int_0^T \dot{\mathbf{u}}_3 dt = \frac{M}{\mathcal{I}\nu}\mathbf{u}_2.$$

Therefore the average of the velocity of $\mathbf{u}_3$ over a period exhibits a deflection along $\mathbf{u}_2$. Recall that the perturbation $\mathbf{F} = F\mathbf{u}_1$ has been applied along $\mathbf{u}_1$ and the resultant moment $\mathbf{M} = M\mathbf{u}_2$ acts along $\mathbf{u}_2$. Thus it might appear that the gyroscope would deflect toward the moments rather than toward the perturbing force. This is called the *paradox of parallelism* (see also Remark 8.1 and §6c of the Complements).

7 Spinning Top Subject to Gravity (Lagrange)

Let $\{\mathcal{M}; d\mu\}$ be a gyroscope subject to gravity and in precession about a pole O on the gyroscopic axis at distance $d > 0$ from its center of mass P_o, as in **Figure 7.1**. The fixed triad $\Sigma = \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is chosen with $\mathbf{e}_3$ ascending vertically. The moving triad $S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is chosen with $\mathbf{u}_3$ along the gyroscopic axis. Therefore S is principal of inertia, although not central, and the inertia tensor σ with respect to such a triad takes the form

$$\begin{pmatrix} \mathcal{I} & 0 & 0 \\ 0 & \mathcal{I} & 0 \\ 0 & 0 & \mathcal{I}_3 \end{pmatrix}.$$

The Lagrangian coordinates are chosen as the Euler angles. The kinetic energy in terms of these angles is (§1.1 of Chapter 6)

$$2T = \mathcal{I}(\dot\varphi^2 \sin^2\theta + \dot\theta^2) + \mathcal{I}_3(\dot\varphi\cos\theta + \dot\psi)^2.$$

Denoting by m the mass of $\{\mathcal{M}; d\mu\}$, the potential of the gravitational force is

$$V(\varphi, \psi, \theta) = -mgd\cos\theta + \text{const.}$$

Therefore the Lagrangian is given, up to a constant, by

$$2\mathcal{L}(\varphi, \psi, \theta) = \mathcal{I}(\dot\varphi^2 \sin^2\theta + \dot\theta^2) + \mathcal{I}_3(\dot\varphi\cos\theta + \dot\psi)^2 - 2mgd\cos\theta. \tag{7.1}$$

Since the constraints are fixed, the energy $E = T - V$ is conserved and gives the first integral

$$\begin{aligned} 2E &= \mathcal{I}(\dot\varphi^2 \sin^2\theta + \dot\theta^2) + \mathcal{I}_3(\dot\varphi\cos\theta + \dot\psi)^2 + 2mgd\cos\theta \\ &= \mathcal{I}(\dot\varphi_o^2 \sin^2\theta_o + \dot\theta_o^2) + \mathcal{I}_3(\dot\varphi_o\cos\theta_o + \dot\psi_o)^2 + 2mgd\cos\theta_o, \end{aligned} \tag{7.2}$$

where for a generic function h of time, we have set $h(0) = h_o$. Since the Lagrangian is independent of φ and ψ, these are cyclic or ignorable Lagrangian coordinates, and give rise to the two first integrals (§4.1 of Chapter 6)

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\dot\varphi} &= \mathcal{I}\dot\varphi\sin^2\theta + \mathcal{I}_3(\dot\varphi\cos\theta + \dot\psi)\cos\theta && (7.3) \\ &= \mathcal{I}\dot\varphi_o\sin^2\theta_o + \mathcal{I}_3(\dot\varphi_o\cos\theta_o + \dot\psi_o)\cos\theta_o, \\ \frac{\partial\mathcal{L}}{\partial\dot\psi} &= \mathcal{I}_3(\dot\varphi\cos\theta + \dot\psi) = \mathcal{I}_3(\dot\varphi_o\cos\theta_o + \dot\psi_o). && (7.4) \end{aligned}$$

7.1 Choosing the Initial Data: Sleeping Top

Assume that initially the gyroscopic axis $\mathbf{u}_3$ has no nutation with respect to the fixed vertical axis $\mathbf{e}_3$, that is, $\dot\theta_o = 0$. Gravity tends to pull down the

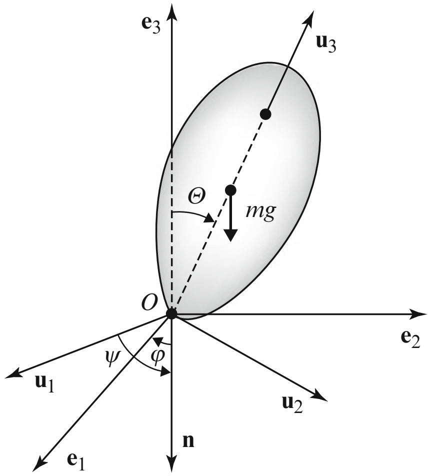

Fig. 7.1.

system by generating a nutation $\dot{\theta} \neq 0$. The initial angles φ_o and ψ_o in (7.2)–(7.4) can be chosen arbitrarily. Choosing them both zero amounts to having the moving axis $\mathbf{u}_1$ and the fixed axis $\mathbf{e}_1$ both coincide, initially, with the nodal axis. The initial vector $\boldsymbol{\omega}_o$ is computed from (9.4) of Chapter 1 with $\dot{\theta}_o = \varphi_o = \psi_o = 0$,

$$\boldsymbol{\omega}_o = \dot{\varphi}_o \sin\theta_o \mathbf{u}_2 + (\dot{\varphi}_o \cos\theta_o + \dot{\psi}_o)\mathbf{u}_3.$$

Putting the integral (7.4) into (7.3) and (7.2), for such a choice of initial data, gives

$$\begin{aligned} \mathcal{I}\dot{\varphi}\sin^2\theta &= \mathcal{I}\dot{\varphi}_o \sin^2\theta_o + \mathcal{I}_3\omega_{o,3}(\cos\theta_o - \cos\theta), \\ \mathcal{I}\dot{\theta}^2 &= 2mgd(\cos\theta_o - \cos\theta) + \mathcal{I}(\dot{\varphi}_o^2 \sin^2\theta_o - \dot{\varphi}^2\sin^2\theta), \\ \omega_{o,3} &= (\dot{\varphi}\cos\theta + \dot{\psi}) = (\dot{\varphi}_o\cos\theta_o + \dot{\psi}_o). \end{aligned} \tag{7.5}$$

If $\theta_o = 0$, the second of these equations has the only solution $\theta = 0$, so that the gyroscopic axis remains vertical. In such a case the nodal axis and the angles φ and ψ are not defined. The first equation of (7.5) holds identically, regardless of the meaning of these angles. The last one can be interpreted as $\boldsymbol{\omega} = \omega_{o,3}\mathbf{e}_3$. Since along the motion the gyroscopic axis remains vertical, gravity does not generate a moment with respect to the pole O, and the motion reduces to the Poinsot precession of a free rotator. For such a motion the remarks of §3.4 are in force, and the precession reduces to a constant rotation about the gyroscopic axis. For this reason, the case of the initial datum $\theta_o = 0$ is referred to as the case of the *sleeping top*. A similar analysis holds if $\theta_o = \pi$.

7.2 First Integrals and Constant Solutions

Starting from the expression (7.1) of the Lagrangian, write the Lagrange equation for the variable θ, and use (7.3)–(7.4). This gives

$$\begin{aligned}\ddot{\theta} =&\ \sin\theta\Big(\frac{mgd}{\mathcal{I}} + \dot{\varphi}^2\cos\theta - \dot{\varphi}\frac{\mathcal{I}_3}{\mathcal{I}}\omega_{o,3}\Big),\\ \theta(0) =&\ \theta_o, \quad \dot{\theta}(0) = \dot{\theta}_o = 0.\end{aligned} \tag{7.6}$$

For the sleeping top, $\theta_o = 0$, and (7.5) has the only solution $\theta = 0$, which is compatible with (7.6). For $\theta_o \in (0, \pi)$, the system (7.5) admits the constant solution $\theta = \theta_o$, $\dot{\varphi} = \dot{\varphi}_o$, $\dot{\psi} = \dot{\psi}_o$. Such a solution is compatible with (7.6) only if

$$\dot{\varphi}_o^2\cos\theta_o - \dot{\varphi}_o\frac{\mathcal{I}_3}{\mathcal{I}}\omega_{o,3} + \frac{mgd}{\mathcal{I}} = 0. \tag{7.7}$$

In particular, the constant solution $\theta = \theta_o \in (0, \pi)$ is not possible if $\dot{\varphi}_o = 0$.

8 Precession with Zero Initial Velocity

Gravitational effects are felt only if $\theta_o \in (0, \pi)$. To single them out, assume that the initial angular velocity of nutation and precession is zero, i.e., $\dot{\theta}_o = \dot{\varphi}_o = 0$. The motion then is described by (7.5) starting from the initial data

$$\varphi(0) = \psi(0) = \dot{\theta}_o = \dot{\varphi}_o = 0, \quad \theta(0) = \theta_o \in (0, \pi), \quad \dot{\psi}(0) = \omega_{o,3}.$$

The equations of the resulting precession are ([97, Vol. XII]; also in [132])

$$\dot{\varphi} = \omega_{o,3}\frac{\mathcal{I}_3}{\mathcal{I}}\frac{(\cos\theta_o - \cos\theta)}{\sin^2\theta}, \tag{8.1}$$

$$\dot{\psi} = \omega_{o,3} - \omega_{o,3}\frac{\mathcal{I}_3}{\mathcal{I}}\frac{(\cos\theta_o - \cos\theta)}{\sin^2\theta}\cos\theta, \tag{8.2}$$

$$\begin{aligned}\dot{\theta}^2 =&\ \frac{2mgd}{\mathcal{I}}(\cos\theta_o - \cos\theta) - \omega_{o,3}^2\frac{\mathcal{I}_3^2}{\mathcal{I}^2}\Big(\frac{\cos\theta_o - \cos\theta}{\sin\theta}\Big)^2\\ \stackrel{\text{def}}{=}&\ f(\theta).\end{aligned} \tag{8.3}$$

We will seek nonconstant solutions of (8.3), since the constant solution $\theta = \theta_o$ is not compatible with (7.6). Equation (8.3) is well defined only for $f(\theta) \ge 0$, i.e., for $\theta \ge \theta_o$. The function $f(\cdot)$ vanishes for $\theta = \theta_o$, and one computes

$$f'(\theta)\,\Big|_{\theta=\theta_o} = \frac{2mgd}{\mathcal{I}}\sin\theta_o > 0.$$

Therefore $f(\theta)$ is positive in a right neighborhood of θ_o. Moreover, $f(\theta) \to -\infty$ as $\theta \to \pi$. Therefore there exists some angle $\theta_1 \in (\theta_o, \pi)$ such that $f(\theta) > 0$

for $\theta \in (\theta_o, \theta_1)$ and $f(\theta_o) = f(\theta_1) = 0$. In such an interval, by separation of variables,

$$t = \int_{\theta_o}^{\theta} \frac{d\tau}{\sqrt{f(\tau)}} \quad \text{or} \quad t = \int_{\theta_1}^{\theta} \frac{d\tau}{\sqrt{f(\tau)}} \quad \text{for} \quad \theta \in [\theta_o, \theta_1]\,. \tag{8.4}$$

These permit one to compute the integral of (8.3) in implicit form. The angle θ effects oscillations between θ_o and θ_1 and remains confined in this interval.

8.1 Oscillatory Nutation and Its Period

To establish such an oscillatory behavior and to compute its period, write

$$\begin{aligned} f(\theta) &= \frac{2mgd}{\mathcal{I}} \frac{(\cos\theta_o - \cos\theta)}{\sin^2\theta} \Big[\sin^2\theta - \frac{\omega_{o,3}^2 \mathcal{I}_3^2}{2mgd\mathcal{I}} (\cos\theta_o - \cos\theta) \Big] \\ &= \frac{2mgd}{\mathcal{I}} \frac{(\cos\theta_o - \cos\theta)}{\sin^2\theta}\, g(\theta), \end{aligned} \tag{8.5}$$

where we have set

$$g(\theta) = \sin^2\theta - a(\cos\theta_o - \cos\theta), \qquad a = \frac{\omega_{o,3}^2 \mathcal{I}_3^2}{2mgd\mathcal{I}}.$$

The angle θ_1 is the first zero, following θ_o, of the function $g(\cdot)$, and for such an angle,[1]

$$2\cos\theta_1 = a - \sqrt{a^2 - 4a\cos\theta_o + 4}.$$

Regarding θ_1 as a function of the parameter a, one computes

$$\lim_{a\to\infty} \theta_1(a) = \theta_o, \qquad \lim_{a\to 0} \theta_1(a) = \pi. \tag{8.6}$$

The parameter a involves the geometric-material parameters h, m, $\mathcal{I}$, $\mathcal{I}_3$, and the kinematic parameter $\omega_{o,3}$. Assume first that the geometric-material parameters are fixed and examine the behavior of the precession as $\omega_{o,3} \to \infty$. From $g(\theta_1) = 0$,

$$\cos\theta_o - \cos\theta_1 = \frac{2mgd\mathcal{I}}{\omega_{o,3}^2 \mathcal{I}_3^2} \sin^2\theta_1. \tag{8.6$'$}$$

Therefore, as $\omega_{o,3}$ increases, the amplitude of the nutation decreases up to vanishing as $\omega_{o,3} \to \infty$. For $\omega_{o,3} \gg 1$, the period T of the nutation can be computed from the first equation of (8.4), and we have

$$T = 2\sqrt{\frac{\mathcal{I}}{2mgd}} \int_{\theta_o}^{\theta_1} \frac{\sin\theta\, d\theta}{\sqrt{\cos\theta_o - \cos\theta}\,\sqrt{\sin^2\theta - a(\cos\theta_o - \cos\theta)}}. \tag{8.7}$$

[1] Prove that this is the only admissible solution of $g(\theta) = 0$ for all $a \ge 0$. Prove also that $f(\theta) < 0$ for $\theta > \theta_1$.

The last integral is computed by the change of variables

$$a(\cos\theta_o - \cos\theta) = \sin^2\theta \sin^2 s, \qquad s \in (0, \pi/2).$$

Taking differentials, we obtain

$$\begin{aligned} a \sin\theta \, d\theta &= 2\sin\theta\cos\theta\sin^2 s \, d\theta + 2\sin^2\theta \sin s \cos s \, ds, \\ \sin\theta \, d\theta &= \frac{2\sin^2\theta \sin s \cos s \, ds}{a - 2\cos\theta\sin^2 s} = \frac{2\sin^2\theta \sin s \cos s \, ds}{a\,(1-\eta)}, \end{aligned}$$

where we have set

$$\eta(\theta, s, a) = \frac{2\,\cos\theta\,\sin^2 s}{a}.$$

Since $a \gg 1$, one also has $\eta \ll 1$. In terms of the new variables,

$$\begin{aligned} T &= \frac{4}{\sqrt{a}}\sqrt{\frac{\mathcal{I}}{2mgd}}\int_0^{\pi/2}\frac{ds}{1-\eta} \\ &= \frac{4}{\sqrt{a}}\sqrt{\frac{\mathcal{I}}{2mgd}}\int_0^{\pi/2}\Big(1 + \sum_{i=1}^{\infty}\eta^i\Big) ds \\ &= \frac{2\pi}{\omega_{o,3}}\frac{\mathcal{I}}{\mathcal{I}_3} + o\Big(\frac{1}{\omega_{o,3}}\Big). \end{aligned}$$

Thus for $\omega_{o,3} \gg 1$ the gyroscopic axis effects oscillatory nutations of frequency of the order of $\omega_{o,3}$ and amplitude of the order of $\omega_{o,3}^{-2}$. As $\omega_{o,3} \to \infty$ the vibrations are of high frequency and low amplitude, thereby exhibiting another occurrence of the *quivering* of the gyroscopic axis.

8.2 Precession about the Fixed Vertical Axis $\mathbf{e}_3$

Whence the nutation $t \to \theta(t)$ is known, the remaining aspects of the precession are determined by (8.1). The gyroscopic axis starts from rest and begins to precess about the fixed axis $\mathbf{e}_3$ with speed $\dot\varphi$. Denoting by T_φ the period of such a precession, one computes from (8.1)

$$2\pi = \int_0^{T_\varphi}\dot\varphi dt = \omega_{o,3}\frac{\mathcal{I}_3}{\mathcal{I}}\int_0^{T_\varphi}\frac{\cos\theta_o - \cos\theta}{\sin^2\theta}dt.$$

For $\omega_{o,3} \gg 1$, by virtue of (8.6)$'$, the last integrand is of the order of a^{-1}. Therefore

$$\frac{2\pi}{T_\varphi} = \frac{2mgd}{\omega_{o,3}\mathcal{I}_3} + o\Big(\frac{1}{\omega_{o,3}}\Big). \tag{8.8}$$

The left-hand side is the average speed, over a period, of the precession of the top about the fixed vertical axis $\mathbf{e}_3$. Thus the faster the top spins about its gyroscopic axis ($\omega_{o,3} \gg 1$), the slower, on average over a period, it precesses about $\mathbf{e}_3$.

Remark 8.1 As $\omega_{o,3} \to \infty$ the amplitude of nutation is of the order of $\omega_{o,3}^{-2}$, as indicated by (8.6)′. On the other hand, the *average* of the angular velocity $\dot{\varphi}$, over a period, is of the order of $\omega_{o,3}^{-1}$. It might appear, then, that the gyroscope would deflect first in the direction of the moment of its weight, rather than along it. This is a further occurrence of the *paradox of parallelism*. Formula (8.3) resolves the paradox by revealing an incipient motion in the direction of gravity. If $\omega_{o,3} = 0$, the system exhibits harmonic oscillations about the equilibrium position $\theta = \pi$, and it behaves like a *compound pendulum* (§5.1c of the Complements). Indeed, taking the time derivative of (8.3) with $\omega_{o,3} = 0$ and setting $\alpha = \pi - \theta$ gives

$$\ddot{\alpha} + \frac{g}{L} \sin\alpha = 0, \qquad L = \frac{\mathcal{I}}{md}.$$

Remark 8.2 For $\omega_{o,3}$ fixed, one might trace the amplitude of the resulting nutation from the geometric-material parameters h, m, $\mathcal{I}$, $\mathcal{I}_3$ of the system. For example, the amplitude of the nutation tends to zero as $d \to 0$, or as $\mathcal{I}_3 \to \infty$. In the first case the center of mass tends to coincide with the pole O, and the precession tends to a Poinsot precession. In the second case the gyroscope tends to be a *very flat top*. If, on the other hand, $\mathcal{I}_3 \to 0$, the top tends to resemble, so to speak, a *material stick*, of cross section a small radius. In such a case the second equation of (8.6) implies that $\theta_1 \to \pi$, and the "stick" tends to "fall" on its vertical position. If $d \to 0$, the pole of the precession tends to coincide with the center of mass, and the motion tends to a Poinsot precession.

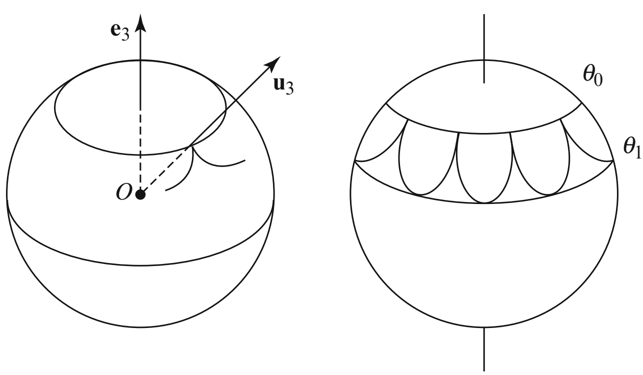

Fig. 8.1.

8.3 Visualizing the Motion

A visualization can be realized by the geometric configurations of the point $(\varphi, \theta) \to P(\varphi, \theta)$, traced by the positive gyroscopic semiaxis, on the unit sphere of Σ. For $\theta = \theta_o$, nutation and precession both have zero velocity. From (8.1) it follows that $\dot{\varphi} > 0$ for all $\theta \in (\theta_o, \theta_1)$. From (8.1) and (8.3), regarding φ as a function of θ, one also computes

$$\lim_{\theta\to\theta_o} \frac{\partial\varphi}{\partial\theta} = \lim_{\theta\to\theta_o} \frac{\dot{\varphi}}{\dot{\theta}} = 0, \qquad \lim_{\theta\to\theta_1} \frac{\partial\varphi}{\partial\theta} = \lim_{\theta\to\theta_1} \frac{\dot{\varphi}}{\dot{\theta}} = \infty.$$

Therefore as θ travels from θ_o to θ_1 and then back to θ_o, the angle of precession φ increases, undergoes an instantaneous stop at $\theta = \theta_o$, and has infinite speed at $\theta = \theta_1$. The corresponding $P(\varphi, \theta)$ always rotates counterclockwise about the fixed vertical axis $\mathbf{e}_3$; it stops whenever $\theta = \theta_o$ and has infinite speed whenever $\theta = \theta_1$. Its configurations are traced in **Figure 8.1**. Because of such a visual rendering of the motion, the gyroscopic axis is also called the *figure axis*.

9 Precession with Arbitrary Initial Velocity

We will continue to assume that of $\dot{\theta}_o = 0$, so that the equations of (7.5) are in force. If no further assumptions are made on the initial velocities, they yield

$$\dot{\varphi} = \omega_{o,3} \frac{\mathcal{I}_3}{\mathcal{I}} \frac{(\cos\theta_o - \cos\theta)}{\sin^2\theta} + \dot{\varphi}_o \Big(\frac{\sin\theta_o}{\sin\theta}\Big)^2, \tag{9.1}$$

$$\dot{\psi} = \omega_{o,3} - \omega_{o,3} \frac{\mathcal{I}_3}{\mathcal{I}} \frac{(\cos\theta_o - \cos\theta)}{\sin^2\theta} \cos\theta - \dot{\varphi}_o \Big(\frac{\sin\theta_o}{\sin\theta}\Big)^2 \cos\theta, \tag{9.2}$$

$$\begin{aligned} \dot{\theta}^2 = {} & \frac{2mgd}{\mathcal{I}}(\cos\theta_o - \cos\theta) - \omega_{o,3}^2 \frac{\mathcal{I}_3^2}{\mathcal{I}^2} \frac{(\cos\theta_o - \cos\theta)^2}{\sin^2\theta} \\ & + \dot{\varphi}_o^2 \Big(\frac{\sin\theta_o}{\sin\theta}\Big)^2 \big(\cos^2\theta_o - \cos^2\theta\big) \\ & - \frac{2\omega_{o,3}\dot{\varphi}_o\mathcal{I}_3}{\mathcal{I}} \Big(\frac{\sin\theta_o}{\sin\theta}\Big)^2 (\cos\theta_o - \cos\theta) \stackrel{\text{def}}{=} f(\theta). \end{aligned} \tag{9.3}$$

The last is well defined for $f(\theta) \ge 0$. Write

$$f(\theta) = \Big(\frac{\cos\theta_o - \cos\theta}{\sin^2\theta}\Big) g(\theta), \tag{9.4}$$

where

$$\begin{aligned} g(\theta) = {} & \frac{2mgd}{\mathcal{I}} \sin^2\theta - \omega_{o,3}^2 \frac{\mathcal{I}_3^2}{\mathcal{I}^2} (\cos\theta_o - \cos\theta) \\ & + \dot{\varphi}_o^2 \sin^2\theta_o (\cos\theta_o + \cos\theta) - 2\dot{\varphi}_o \omega_{o,3} \frac{\mathcal{I}_3}{\mathcal{I}} \sin^2\theta_o. \end{aligned} \tag{9.5}$$

In the case $\dot{\varphi}_o = 0$, it follows from (8.1) that for $\theta > \theta_o$, both $\omega_{o,3}$ and $\dot{\varphi}$ have the same sign. In particular, $\dot{\varphi}$ has constant sign and the precession occurs always in the same direction. By analogy, it is assumed that

$$\dot{\varphi}_o \qquad \text{and} \qquad \omega_{o,3} \qquad \text{are both positive.} \tag{9.6}$$

This, however, does not imply that the precession has constant direction. It will be shown in the next sections that if $g(\theta_o) \geq 0$, the precession has constant direction, whereas if $g(\theta_o) < 0$, the precession might invert its direction. The sign of $g(\theta_o)$ determines also the direction of the incipient nutation. Indeed, computing (7.6) for $t = 0$ gives

$$\ddot{\theta}(t_o) = \frac{g(\theta_o)}{2 \sin \theta_o}.$$

Therefore, since $\dot{\theta}_o = 0$, if $g(\theta_o) > 0$, the gyroscopic axis begins to move away from the vertical axis, whereas if $g(\theta_o) < 0$, it begins to approach it.

10 Precessions of Constant Direction ($g(\theta_o) \geq 0$)

Assume first $g(\theta_o) > 0$. Then $\theta \to f(\theta)$ vanishes for $\theta = \theta_o$, and one computes

$$f'(\theta) \Big|_{\theta=\theta_o} = \frac{g(\theta_o)}{\sin \theta_o} > 0.$$

Therefore $f(\theta)$ is positive in a right neighborhood of θ_o. Moreover, $f(\theta) \to -\infty$ as $\theta \to \pi$. Therefore there exists a first angle $\theta_o < \theta_1 < \pi$ such that $f(\theta)$ remains positive in the interval (θ_o, θ_1), it vanishes at θ_1, and it is negative in a right neighborhood of θ_1. In such an interval θ is computed explicitly as in (8.4) starting from (9.3), with the proper new meaning of the function f. Since $f(\theta)$ would change sign across θ_o and θ_1, the nutation angle θ remains confined in such an interval and effects oscillations of period[2]

$$T = 2 \int_{\theta_o}^{\theta_1} \frac{d\tau}{\sqrt{f(\tau)}}. \tag{10.1}$$

It follows from (9.1) and the assumption (9.6) that the angle of precession φ is nondecreasing and the precession has constant direction. In particular, the figure axis effects a counterclockwise rotation, about the fixed vertical axis $\mathbf{e}_3$. Regarding φ as a function of θ, we compute, starting from (9.1) and (9.2),

$$\lim_{\theta \to \theta_o} \frac{\partial \varphi}{\partial \theta} = \lim_{\theta \to \theta_o} \frac{\dot{\varphi}}{\dot{\theta}} = \infty, \qquad \lim_{\theta \to \theta_1} \frac{\partial \varphi}{\partial \theta} = \lim_{\theta \to \theta_1} \frac{\dot{\varphi}}{\dot{\theta}} = \infty.$$

Following the trace $\theta \to P(\theta)$ of the figure axis on the unit sphere of Σ gives a visual rendering of the motion as in **Figure 10.2**.

The limiting case $g(\theta_o) = 0$ implies

$$\frac{mgd}{\mathcal{I}} = \dot{\varphi}_o \omega_{o,3} \frac{\mathcal{I}_3}{\mathcal{I}} - \dot{\varphi}_o^2 \cos \theta_o. \tag{10.2}$$

[2] For an asymptotic analysis as $\omega_{o,3} \to \infty$ see §10c of the Complements.

This coincides precisely with the compatibility condition (7.7) for solutions of the system (7.5). From that discussion it follows that the system (9.1)–(9.2) admits the unique solution

$$\theta = \theta_o, \qquad \dot{\varphi} = \dot{\varphi}_o, \qquad \dot{\psi} = \omega_{o,3} - \dot{\varphi}_o \cos\theta_o.$$

The nutation is zero and the motion reduces to a precession of constant velocity $\dot{\varphi} = \dot{\varphi}_o$, of period $T_\varphi = 2\pi/\dot{\varphi}_o$.

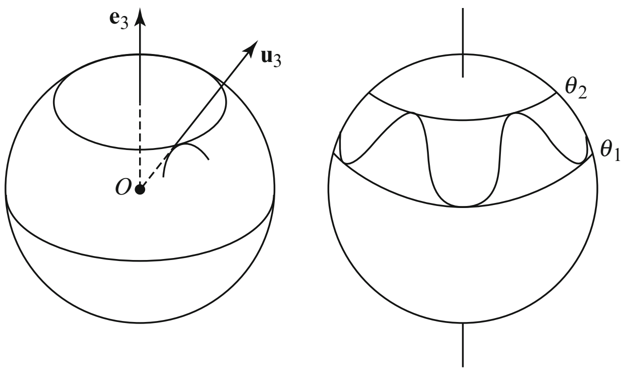

Fig. 10.2.

11 Precessions That Invert Their Direction ($g(\theta_o) < 0$)

The condition $g(\theta_o) < 0$ imposes on $\omega_{o,3}$ the restriction

$$\omega_{o,3}\frac{\mathcal{I}_3}{\mathcal{I}} > \dot{\varphi}_o \cos\theta_o + \frac{mgd}{\mathcal{I}\,\dot{\varphi}_o}.$$

Rewrite $g(\theta)$ in the form

$$\begin{aligned} g(\theta) =& \frac{2mgd}{\mathcal{I}} \sin^2\theta - \Big(\omega_{o,3}^2 \frac{\mathcal{I}_3^2}{\mathcal{I}^2} + \dot{\varphi}_o^2 \sin^2\theta_o\Big)(1-\cos\theta) \\ &+ \frac{\sin^2\theta_o}{(1+\cos\theta_o)}\Big(\omega_{o,3}\frac{\mathcal{I}_3}{\mathcal{I}} - \dot{\varphi}_o(1+\cos\theta_o)\Big)^2. \end{aligned} \tag{11.1}$$

The nature of the precession depends on various combinations of the initial data $\omega_{o,3}$ and $\dot{\varphi}_o$ with the geometric material parameters of the gyroscope. To single out the inversion of the direction of the precession we assume

$$\omega_{o,3}\frac{\mathcal{I}_3}{\mathcal{I}} > \frac{2mgd}{\mathcal{I}\dot{\varphi}_o} \geq 2\dot{\varphi}_o. \tag{11.2}$$

For such a choice, the last term of (11.1) is a positive constant. The function $f(\cdot)$ vanishes for $\theta = \theta_o$ and

$$f'(\theta)\Big|_{\theta=\theta_o} = \frac{g(\theta_o)}{\sin\theta_o} < 0.$$

Therefore $f(\theta)$ is positive in a left neighborhood of θ_o. Moreover, by (11.2), $f(\theta) \to -\infty$ as $\theta \to 0$. Thus there exists an angle $0 < \theta_1 < \theta_o$ such that $f(\theta_1) = 0$. Such an angle is a zero of $g(\cdot)$. Now g is a quadratic polynomial in $\cos\theta$, and $g(\theta_o) < 0$ and $g(0) > 0$. It follows that $g(\theta) = 0$ has the only root $\theta = \theta_1$. We conclude that there here exists $\theta_1 \in (0, \theta_o)$ such that $f(\theta) > 0$ for $\theta \in (\theta_1, \theta_o)$, $f(\theta_1) = f(\theta_o) = 0$, and $f(\cdot)$ is negative in $(0, \theta_1)$.

The gyroscopic axis, starting from θ_o, with zero velocity of nutation, moves up, so to speak, to θ_1 and effects oscillations within $[\theta_1, \theta_o]$. Amplitude and period of such oscillations, as well as their asymptotic behavior as $\omega_{o,3} \to \infty$, are computed as in §8.1 (see also §10c of the Complements).

The function $\theta \to \sin^2\theta\dot{\varphi}(\theta)$ is increasing in $(0, \theta_o)$, it is positive for $\theta = \theta_o$, and

$$\begin{aligned}\lim_{\theta\to 0} \dot{\varphi}\sin^2\theta &= -\omega_{o,3}\frac{\mathcal{I}_3}{\mathcal{I}}(1-\cos\theta_o) + \dot{\varphi}_o\sin^2\theta_o \\ &= -\left[\omega_{o,3}\frac{\mathcal{I}_3}{\mathcal{I}} - \dot{\varphi}_o(1+\cos\theta_o)\right]\frac{\sin^2\theta_o}{1+\cos\theta_o} < 0.\end{aligned}$$

Therefore there exists a unique angle $\theta_* \in (0, \theta_o)$ such that $\dot{\varphi}(\theta_*) = 0$. Such an angle is computed from

$$\mathcal{I}\dot{\varphi}_o\sin^2\theta_o = \omega_{o,3}\mathcal{I}_3\left(\cos\theta_* - \cos\theta_o\right).$$

The angle θ_* is in the interval of positivity of $f(\cdot)$. To prove this we refer to the second equation of (7.5) with $\dot{\varphi}(\theta_*) = 0$, and use (11.2). This gives

$$\mathcal{I}\,\dot{\theta}^2\Big|_{\theta=\theta_*} = (\omega_{o,3}\mathcal{I}_3\dot{\varphi}_o - 2mgd)\left(\cos\theta_* - \cos\theta_o\right) > 0.$$

When θ spans the interval $[\theta_o, \theta_*)$, the velocity of precession $\dot{\varphi}$ is positive, the angle of precession φ increases, and the figure axis rotates counterclockwise about the fixed axis $\mathbf{e}_3$. When θ spans $(\theta_*, \theta_1]$, the velocity of precession $\dot{\varphi}$ is negative, φ decreases, and the figure axis regresses, effecting a clockwise rotation about $\mathbf{e}_3$. To gain a qualitative description of the motion, compute

$$\lim_{\theta\to\theta_1}\frac{\partial\varphi}{\partial\theta} = \lim_{\theta\to\theta_1}\frac{\dot{\varphi}}{\dot{\theta}} = -\infty, \qquad \lim_{\theta\to\theta_o}\frac{\partial\varphi}{\partial\theta} = \lim_{\theta\to\theta_o}\frac{\dot{\varphi}}{\dot{\theta}} = \infty.$$

The trace of the figure axis on the fixed unit sphere is visualized in **Figure 11.1**.

12 Spinning Top Subject to Friction

Let $\{\mathcal{M}; d\mu\}$ be a gyroscope in precession about a pole O of the gyroscopic axis. The moving triad $S = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is chosen with $\mathbf{u}_3$ along the gyroscopic axis. It is assumed that the possible external forces generate a zero

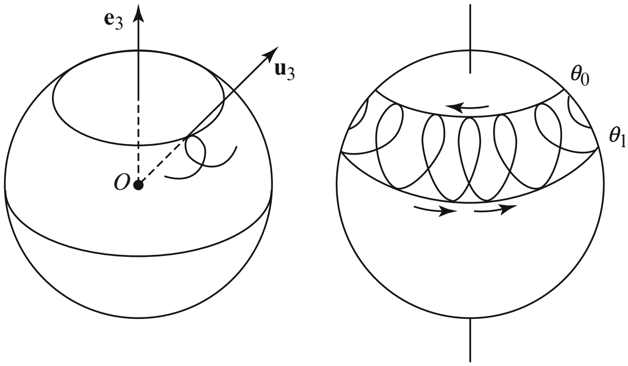

Fig. 11.1.

resultant moment and that friction opposes the motion by generating a moment directed as $\boldsymbol{\omega}$,

$$\mathcal{M} = -\lambda\boldsymbol{\omega} = -\lambda\,\omega_i\mathbf{u}_i \quad \text{for some } \lambda > 0.$$

Therefore $\mathbf{M}^{(e)} = 0$ and (1.2) take the form

$$\begin{aligned} \mathcal{I}\dot{\omega}_1 &= (\mathcal{I} - \mathcal{I}_3)\omega_2\omega_3 - \lambda\omega_1, \\ \mathcal{I}\dot{\omega}_2 &= (\mathcal{I}_3 - \mathcal{I})\omega_1\omega_3 - \lambda\omega_2, \\ \mathcal{I}_3\dot{\omega}_3 &= -\lambda\omega_3. \end{aligned} \tag{12.1}$$

The third equation is integrated explicitly as

$$\omega_3 = \omega_{o,3}e^{-(\lambda/\mathcal{I}_3)t}. \tag{12.2}$$

Put this in the first two equations and multiply the resulting equations by $\exp\{(\lambda/\mathcal{I})t\}$. This transforms the system into

$$\begin{aligned} \frac{d}{dt}\widetilde{\omega}_1 &= \frac{(\mathcal{I} - \mathcal{I}_3)}{\mathcal{I}}\omega_{o,3}e^{-(\lambda/\mathcal{I}_3)\,t}\widetilde{\omega}_2, \\ \frac{d}{dt}\widetilde{\omega}_2 &= \frac{(\mathcal{I}_3 - \mathcal{I})}{\mathcal{I}}\omega_{o,3}e^{-(\lambda/\mathcal{I}_3)t}\widetilde{\omega}_1, \end{aligned} \qquad \text{where} \quad \widetilde{\omega}_i = e^{(\lambda/\mathcal{I})t}\omega_i. \tag{12.3}$$

Multiplying the first equation by $\widetilde{\omega}_1$ and the second by $\widetilde{\omega}_2$ and adding gives

$$\widetilde{\omega}_1^2 + \widetilde{\omega}_2^2 = \omega_{o,1}^2 + \omega_{o,2}^2.$$

Therefore the solutions are

$$\begin{aligned} \omega_1 &= \sqrt{\omega_{o,1}^2 + \omega_{o,2}^2}\,e^{-(\lambda/\mathcal{I})t}\cos\alpha, \\ \omega_2 &= \sqrt{\omega_{o,1}^2 + \omega_{o,2}^2}\,e^{-(\lambda/\mathcal{I})t}\sin\alpha, \end{aligned} \tag{12.4}$$

where

$$\alpha = \frac{\mathcal{I}_3(\mathcal{I}_3 - \mathcal{I})}{\lambda \mathcal{I}} \omega_{o,3} \Big(1 - e^{-(\lambda/\mathcal{I}_3)t}\Big) + \tan^{-1}\Big(\frac{\omega_{o,2}}{\omega_{o,1}}\Big). \qquad (12.5)$$

This expression for α is derived by putting (12.4) into (12.3). This gives a differential equation for $\dot{\alpha}$, which is integrated by elementary quadratures.

From (12.2) and (12.4) one estimates

$$\|\boldsymbol{\omega}\|^2 \le \|\boldsymbol{\omega}_o\|^2 e^{-\lambda t/\max\{\mathcal{I};\mathcal{I}_3\}} \quad \text{for } t \ge 0.$$

From this, the kinetic energy is estimated as

$$T = \boldsymbol{\omega}^t \sigma \boldsymbol{\omega} \le \max\{\mathcal{I};\mathcal{I}_3\} \|\boldsymbol{\omega}_o\|^2 e^{-\lambda t/\max\{\mathcal{I};\mathcal{I}_3\}}.$$

Therefore $T \to 0$ exponentially as $t \to \infty$.

If $\lambda \to 0$, the motion tends to a Poinsot precession.

If the ellipsoid $\mathcal{E}_\lambda$ is a sphere, then $\mathcal{I}_3 = \mathcal{I}$ and (12.2)–(12.4) reduce to

$$\omega_i = \omega_{o,i} e^{-(\lambda/\mathcal{I})t}, \qquad i = 1, 2, 3.$$

Problems and Complements

1c The Euler Equations

1.1c Precession Relative to the Center of Mass

Let $\{\mathcal{M}; d\mu\}$ be in rigid motion with characteristics $\dot{P}_o$ and $\boldsymbol{\omega}$. For such a system the cardinal equations (3.1)–(3.2) of Chapter 5 are in force, where $\mathbf{F}^{(e)}$ and $\mathbf{M}^{(e)}$ are known functions of P_o, $\dot{P}_o$, and $\boldsymbol{\omega}$. If $\mathbf{F}^{(e)}$ were independent of $\boldsymbol{\omega}$, the first cardinal equation would provide the kinematics of the center of mass P_o; this in turn could be put in the second to determine $\boldsymbol{\omega}$. In general, however, $\mathbf{F}^{(e)}$ depends on $\boldsymbol{\omega}$, and the first cardinal equation is not directly integrable. One might then think of determining first $\boldsymbol{\omega}$, which put in the first cardinal equation would determine the motion of P_o. The key idea is that the characteristic $\boldsymbol{\omega}$ does not change by referring the rigid motion to a triad centered at P_o and translating with velocity $\dot{P}_o$. Thus the determination of $\boldsymbol{\omega}$ reduces to integrating the second cardinal equation, written for $P_o = \dot{P}_o = O$. This is the motion of $\{\mathcal{M}; d\mu\}$ relative to its center of mass (§3.2 of Chapter 5).

1.2c Precession of Earth

The Sun's planar apparent path over a year is called the *ecliptic*. The center of mass P_o of Earth moves in the plane of the ecliptic. Let $\Sigma = \{P_o; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a triad with $\mathbf{e}_3$ normal to the plane of the ecliptic. With respect to Σ, Earth is in precession of pole P_o and is acted upon by moments due to the presence of the Sun, Moon, and other celestial bodies. Even assuming that the Sun and the Moon are point masses, these moments are not zero, since Earth is not perfectly spherical, being slightly flat at its poles. The average equatorial radius is 6,378 km, whereas the difference between the equatorial and polar radii is 21.5 km [76, F–145].

1.2.1. Assuming the Sun and the Moon to be point masses, prove that if Earth were perfectly spherical, the resultant gravitational moment would be zero. Moreover, if the gyroscopic axis of Earth, from south to north, were normal to the plane of the ecliptic, the resultant gravitational moment due to the Sun would be zero.

1.2.2. The maximum inclination of the gyroscopic axis in the plane of the ecliptic is about 23°29″ [76, F–144]. Using this value estimate the size of the component along the gyroscopic axis of Earth, of the resultant gravitational moment due to the Sun and the Moon, and conclude that they are both negligible.

1.3c Poinsot and Astronomical Precessions of Earth

Let $S = \{P_o; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be the moving triad, clamped to Earth, oriented so that the angle $\widehat{\mathbf{u}_3\mathbf{e}_3}$ is acute. Independently of the motion of P_o, the triad S is in precession with respect to Σ, with characteristic $\boldsymbol{\omega}$. The resultant gravitational moment $\mathbf{M}$ is a function of $\boldsymbol{\omega}$ only, and it is rewritten as

$$\mathbf{M} = \mathbf{M}_{\mathbf{u}_3} + \mathbf{M}_{\perp\mathbf{u}_3}, \quad \mathbf{M}_{\mathbf{u}_3} = \mathbf{M}\cdot\mathbf{u}_3, \quad \mathbf{M}_{\perp\mathbf{u}_3} = \mathbf{M} - \mathbf{M}_{\mathbf{u}_3}.$$

The precession of Earth can be regarded, with good approximation, as the composition of two precessions with the same pole P_o (§9 of Chapter 1). The first is acted upon by $\mathbf{M}_{\mathbf{u}_3}$ only. Using **1.2.2** above, $\mathbf{M}_{\mathbf{u}_3} \approx 0$ and the resulting precession is a Poinsot precession. The second is acted upon by the moment $\mathbf{M}_{\perp\mathbf{u}_3}$ and it is called *astronomical precession.*

1.4c Poinsot Precession of Earth

While not perfectly spherical, Earth is a gyroscope. Setting $\mathbf{M}_{\mathbf{u}_3} = 0$, the equations (6.1) with $M = 0$ are in force, and their explicit integral is given by (6.2). If the equatorial component of $\boldsymbol{\omega}$ is zero, that is, if $\sqrt{\omega_{o,1}^2 + \omega_{o,2}^2} = 0$, then the precession is a constant rotation about the gyroscopic axis of Earth. However, astronomical observations reveal that while small, such a term is not zero. This implies that the axis of rotation does not coincide with the gyroscopic axis. These axes form an angle, called the *constant of nutation*, or *solar parallax*, estimated to be about 0.0087″ sexagesimal seconds [76, F–144]. The rotation is from west to east (counterclockwise) and $\boldsymbol{\omega}$ is oriented accordingly. Then the solar parallax is counted from the axis of $\boldsymbol{\omega}$ to the gyroscopic axis, oriented from south to north.

1.4.1. Using this value of the solar parallax and the average radius of Earth, give an estimate of $\sqrt{\omega_{o,1}^2 + \omega_{o,2}^2}$. The points where the positive gyroscopic semiaxis and the positive semiaxis of rotation meet the surface of Earth are the *geographic* and *boreal* poles. Verify that the distance between these poles does not exceed 50 cm.

1.5c Astronomical Precession of Earth

This is the slow counterclockwise rotation of gyroscopic axis of Earth about the fixed axis $\mathbf{e}_3$ normal to the plane of the ecliptic, whose period is estimated to be about 26,000 years. It might be described by the angle of precession φ between the fixed axis $\mathbf{e}_1$ and the nodal axis determined by the configurations of Σ and S. The intersection of the nodal positive semiaxis with the ecliptic is the *vernal equinox*, and it occurs about March 21. The intersection of the nodal negative semiaxis with the ecliptic is the *autumnal equinox*, and it occurs about September 23. For this reason the nodal axis is also called *axis of the equinoxes*.[3]

1.5.1. Prove that because of the astronomical precession, the alternation of the equinoxes does not occur at equal intervals of time. Indeed, every year the Sun reaches an equinox slightly earlier than the preceding year. Because of this phenomenon of *preceding*, these motions are called *precessions*.[4]

5c Rotations about a Fixed Axis

5.1c The Compound Pendulum (Huygens [81])

Consider a rigid body, subject to its weight, rotating without friction about a fixed horizontal axis ℓ of unit direction $\mathbf{u}$, through a fixed point O, but not containing the center of mass P_o, as in **Figure 5.1c**. As the sole Lagrangian

[3] From Latin *equi noctis*, since when the Sun is in that position, day and night have the same duration.

[4] Observed first by Hipparchus of Nicaea, (circa 190–126 BCE). Hipparchus had compiled a map of the sky including about 800 stars along with their coordinates and their relative brightness. Comparing his map with the one compiled by Timocharis of Alexandria about 50 years earlier, he noticed a difference of about 2° in the position of the same stars. He explained the difference by the precession of equinoxes, whose advance he estimated to be about 36 seconds. Modern calculations have it at about 50 seconds. Hipparchus's map remained essentially unchanged up to the fifteenth century, except for the addition of new stars by Ptolemy (second century CE). Generations of astronomers based their investigations on that map, including Copernicus (1473–1543) and Galileo (1564–1642).

parameter take the angle between $P_o - O$ and the descending vertical. From the third equation of (5.1) with $\mu = 0$,

$$I_{33}\ddot{\varphi} = -mg\|P_o - O\| \sin\varphi, \qquad m = \int d\mu(P),$$

which is rewritten as

$$\ddot{\varphi} + \frac{g}{L}\sin\varphi = 0, \qquad L = \frac{I_{33}}{m\,\|P_o - O\|}. \tag{5.1c}$$

Thus the motion reduces to that of a pendulum of length L, and for this reason the system it is called a *compound pendulum*. Set $I_{33} = I_O$ and denote

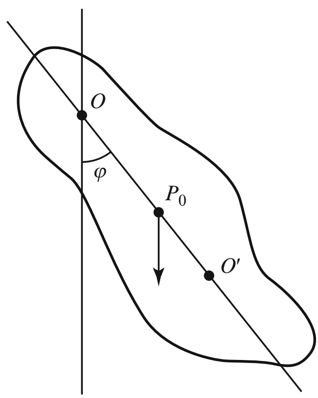

Fig. 5.1c.

by I_{P_o} the moment of inertia with respect to the axis through P_o and parallel to ℓ. By Huygens's theorem,

$$I_O = I_{P_o} + m\|P_o - O\|^2.$$

Therefore

$$L = \|P_o - O\| + \frac{I_{P_o}}{m\,\|P_o - O\|} > \|P_o - O\|.$$

Starting from O and moving a distance L along $P_o - O$, determine a point O' and denote by ℓ' an axis through O' and parallel to ℓ. The point O' and the axis ℓ' have a remarkable property established by Huygens.

Proposition 5.1c (Huygens [81]) *The pairs $\{O; \ell\}$ and $\{O'; \ell'\}$ are interchangeable in the sense that gravity-driven, frictionless precessions about ℓ with pole O and about ℓ' with pole O' have the same period.*

Proof. Let $I_{O'}$ be the axial moment of inertia with respect to ℓ' and let

$$L' = \frac{I_{O'}}{m\,\|P_o - O'\|}$$

be the length of the new compound pendulum. It will suffice to show that $L = L'$. By Huygens's theorem,

$$L' = \|P_o - O'\| + \frac{I_{P_o}}{m\,\|P_o - O'\|}.$$

From the expression of L and the definition of O',

$$\|P_o - O'\| = L - \|P_o - O\| = \frac{I_{P_o}}{m\,\|P_o - O\|}.$$

Putting this in the previous expression of L' gives

$$L' = \|P_o - O\| + \|P_o - O'\| = L. \qquad \blacksquare$$

The Lagrangian and Hamiltonian are

$$\mathcal{L} = \tfrac{1}{2} I_{33}\dot{\varphi}^2 + mg\|P_o - O\|\cos\varphi + \text{const},$$

$$\mathcal{H} = \tfrac{1}{2}\frac{p^2}{I_{33}} - mg\|P_o - O\|\cos\varphi + \text{const}.$$

The sole Lagrange equation coincides with (5.1c). The Hamilton canonical equations are

$$\dot{\varphi} = \frac{p}{I_{33}}, \qquad \dot{p} = -mg\|P_o - O\|\sin\varphi.$$

6c Gyroscopes

6.1. Write down (6.2) with initial data $\omega_{o,i} = 0$, $i = 1, 2$. Compute

$$\ddot{\mathbf{u}}_3 = \dot{\boldsymbol{\omega}} \wedge \mathbf{u}_3 + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{u}_3)$$

and show that the incipient acceleration of $(0, 0, 1)$ is parallel to $\mathbf{F} = F\mathbf{u}_1$. Show that $\ddot{\mathbf{u}}_3$ is parallel to the resultant force, at all times, provided in compounding the forces one includes those due the transport and Coriolis accelerations (§8 of Chapter 1). Show that the same occurs if $\omega_{o,i}$ are not zero.

6.2. A material homogeneous rigid ellipsoid $\mathcal{E}$ of mass m and semiaxes a_i, $i = 1, 2, 3$, is in precession about its center P_o with characteristic $\boldsymbol{\omega}$. The central principal triad S is taken with the unit vectors $\mathbf{u}_i$ clamped along the homologous axes. The ellipsoid rotates about its a_3 axis with velocity $\dot{\varphi}\mathbf{u}_3$, which, in turn, rotates about a fixed unit vector $\mathbf{e}$ with velocity $\dot{\theta}\mathbf{e}$.

Express the kinetic energy in terms of φ, θ, and their time derivatives. Write down the Euler equation of the corresponding Poinsot precession and find an explicit integral in the case $a_1 = a_2 = \sqrt{3}a_3$.

The inertia matrix is $\sigma = \mathcal{I}_i\delta_{ij}$, where the $\mathcal{I}_i$ are computed from §3.1c of the Complements of Chapter 4. Denoting by φ the angle between $\mathbf{u}_1$ and $\mathbf{e}$,

$$\boldsymbol{\omega} = \dot{\theta}\cos\varphi\,\mathbf{u}_1 - \dot{\theta}\sin\varphi\,\mathbf{u}_2 + \dot{\varphi}\,\mathbf{u}_3.$$

The kinetic energy is computed from

$$2T = \boldsymbol{\omega}^t\sigma\boldsymbol{\omega} = \mathcal{I}_1\dot{\theta}^2\cos^2\varphi + \mathcal{I}_2\dot{\theta}^2\sin^2\varphi + \mathcal{I}_3\dot{\varphi}^2.$$

There are no external forces, and the moments with respect to P_o of the reactions due to the constraints are zero. Therefore (1.1) yields

$$\begin{aligned}(a_2^2 + a_3^2)(\ddot{\theta}\cos\varphi - \dot{\theta}\dot{\varphi}\sin\varphi) &= (a_2^2 - a_3^2)\dot{\theta}\dot{\varphi}\sin\varphi,\\ (a_1^2 + a_3^2)(\ddot{\theta}\sin\varphi + \dot{\theta}\dot{\varphi}\cos\varphi) &= (a_3^2 - a_1^2)\dot{\theta}\dot{\varphi}\cos\varphi,\\ (a_1^2 + a_2^2)\ddot{\varphi} &= (a_1^2 - a_2^2)\dot{\theta}^2\sin\varphi\cos\varphi.\end{aligned}$$

Multiplying the first equation by $\dot{\theta}\cos\varphi$, the second by $\dot{\theta}\sin\varphi$, and the third by $\dot{\varphi}$ and adding gives $\dot{T} = 0$. If $a_1 = a_2 = \sqrt{3}a_3$, the third equation implies $\ddot{\varphi} = 0$, and the first two equations became

$$2\ddot{\theta}\cos\varphi - 3\dot{\theta}\dot{\varphi}\sin\varphi = 0, \qquad 2\ddot{\theta}\sin\varphi + 3\dot{\theta}\dot{\varphi}\cos\varphi = 0.$$

Multiplying the first by $\cos\varphi$ and the second by $\sin\varphi$ and adding gives $\ddot{\theta} = 0$.

7c Spinning Top Subject to Gravity

7.1c Intrinsic Equation of a Gyroscope

The trace P of the positive gyroscopic semiaxis on the unit sphere of Σ is a curve γ that can be parameterized by the arc length s. As the positive intrinsic triad of γ take $\{\mathbf{t}, \mathbf{v}, \mathbf{u}_3\}$, where

$$\mathbf{t} = \frac{d\mathbf{u}_3}{ds}, \qquad \mathbf{v} = \mathbf{u}_3 \wedge \mathbf{t}, \qquad \text{and also} \qquad \mathbf{t} = \mathbf{v} \wedge \mathbf{u}_3. \tag{7.1c}$$

Taking the s-derivative of the second equation and using the Frenet formulas (§1 of Chapter 1) gives

$$\frac{d\mathbf{v}}{ds} = \mathbf{u}_3 \wedge \kappa\mathbf{n} = \kappa\mathbf{u}_3 \wedge (\cos\beta\mathbf{u}_3 - \sin\beta\mathbf{v}) = \kappa\sin\beta\mathbf{t}, \tag{7.2c}$$

where $\beta = \widehat{\mathbf{n}\mathbf{u}_3}$. The curvature κ can be expressed in terms of β. Indeed, taking the s-derivative of the third equation of (7.1c) and using the Frenet formulas yields

$$\kappa\mathbf{n} = \mathbf{v} \wedge \mathbf{t} + (\mathbf{u}_3 \wedge \kappa\mathbf{n}) \wedge \mathbf{u}_3 = -\mathbf{u}_3 - \kappa(\mathbf{n}\cdot\mathbf{u}_3)\mathbf{u}_3 + \kappa\mathbf{n}.$$

From this we obtain

$$\kappa(\mathbf{n}\cdot\mathbf{u}_3)\mathbf{u}_3 = -\mathbf{u}_3, \qquad \text{or} \qquad \kappa = -\frac{1}{\cos\beta}.$$

Putting this in (7.2c) gives the differentiation formula

$$\frac{d\mathbf{v}}{ds} = -\tan\beta\,\mathbf{t} \quad \text{and also} \quad \dot{\mathbf{v}} = -\dot{s}\,\tan\beta\,\mathbf{t}. \tag{7.3c}$$

Next decompose $\boldsymbol{\omega}$ along the unit vectors of the intrinsic triad $\{\mathbf{t}, \mathbf{v}, \mathbf{u}_3\}$. Applying Poisson's formula $\dot{\mathbf{u}}_3 = \boldsymbol{\omega}\wedge\mathbf{u}_3$, one gets

$$\begin{aligned}\dot{s}\,\mathbf{t} = \dot{\mathbf{u}}_3 &= [(\boldsymbol{\omega}\cdot\mathbf{t})\mathbf{t} + (\boldsymbol{\omega}\cdot\mathbf{v})\mathbf{v} + (\boldsymbol{\omega}\cdot\mathbf{u}_3)\mathbf{u}_3]\wedge\mathbf{u}_3 \\ &= -(\boldsymbol{\omega}\cdot\mathbf{t})\mathbf{v} + (\boldsymbol{\omega}\cdot\mathbf{v})\mathbf{t}.\end{aligned}$$

Taking now the exterior product of both sides by $\mathbf{u}_3$ and using (7.1c) gives

$$\boldsymbol{\omega} = \dot{s}\,\mathbf{v} + \omega_3\,\mathbf{u}_3. \tag{7.4c}$$

In particular, the component of $\boldsymbol{\omega}$ along $\mathbf{t}$ is zero. Since $\mathbf{u}_3$ is directed along the gyroscopic axis, the inertia tensor σ does not change if it is computed with respect to the triad $\widetilde{S} = \{O; \mathbf{t}, \mathbf{v}, \mathbf{u}_3\}$ centered at the pole O of the precession and with axes directed along the intrinsic triad to γ. Then

$$\mathbf{K} = \sigma\boldsymbol{\omega} = \begin{pmatrix}\mathcal{I} & 0 & 0\\ 0 & \mathcal{I} & 0\\ 0 & 0 & \mathcal{I}_3\end{pmatrix}\begin{pmatrix}0\\ \dot{s}\\ \omega_3\end{pmatrix} = \mathcal{I}\dot{s}\,\mathbf{v} + \mathcal{I}_3\omega_3\,\mathbf{u}_3.$$

Taking the time derivative and using the differentiation formulas (7.3c) gives

$$\dot{\mathbf{K}} = (-\mathcal{I}\dot{s}\tan\beta + \mathcal{I}_3\omega_3)\dot{s}\,\mathbf{t} + \mathcal{I}\ddot{s}\,\mathbf{v} + \mathcal{I}_3\dot{\omega}_3\,\mathbf{u}_3.$$

From this expression for $\dot{\mathbf{K}}$, the second cardinal equation takes the form

$$[-\mathcal{I}\dot{s}\tan\beta + \mathcal{I}_3\omega_3]\dot{s} = M_{\mathbf{t}}, \qquad \mathcal{I}\ddot{s} = M_{\mathbf{v}}, \qquad \mathcal{I}_3\dot{\omega}_3 = M_{\mathbf{u}_3}, \tag{7.5c}$$

where $M_{\mathbf{t}}$, $M_{\mathbf{v}}$, and $M_{\mathbf{u}_3}$ are the components of the resultant moment of the external forces along the indicated vectors. These are the *intrinsic equations* of the gyroscope [105, art. 55], [124].

7.2c The Gyroscopic Compass or Gyrocompass

This device is a top in precession about its center of mass P_o with the figure axis constrained to remain in the fixed plane π through P_o and tangent to the surface of Earth. Such a constraint can be realized by a Cardan joint. Earth is regarded as a homogeneous sphere rotating with constant angular velocity $\omega_T\mathbf{N}$, where $\mathbf{N}$ is the unit vector of the south pole→north pole axis. Denote by $\mathbf{e}_3$ the ascending unit normal to π at P_o and let $\alpha = \widehat{\mathbf{e}_3\mathbf{N}}$. We assume that P_o is not at either pole, so that $\alpha \in (0, \pi)$. The projection of $\mathbf{N}$ on π defines a unit direction $\mathbf{e}_1$ normal to $\mathbf{e}_3$. This identifies a triad $S_T = \{P_o; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ clamped to Earth. Let $S = \{P_o; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be the moving triad clamped to the top, with the unit vector $\mathbf{u}_3$ chosen along the figure axis and regarded as applied in P_o. Thus $\mathbf{u}_3$ remains on π, and its second extreme moves along a unit circle centered in P_o and lying on π. Therefore $\mathbf{v} = \mathbf{e}_3$ and $\beta = 0$. The triad S is in precession with respect to S_T, with $\boldsymbol{\omega}$ given by (7.4c), and the intrinsic equations (7.5c) become

$$\mathcal{I}_3\omega_3\dot{s} = M_{\mathbf{t}}, \qquad \mathcal{I}\ddot{s} = M_{\mathbf{e}_3}, \qquad \mathcal{I}_3\dot{\omega}_3 = M_{\mathbf{u}_3}. \tag{7.6c}$$

The constraints that keep the figure axis on π exert on that axis reactions normal to it, so that the corresponding resultant moment, which is in general unknown, is directed as $\mathbf{t}$. Thus $M_{\mathbf{e}_3}$ and $M_{\mathbf{u}_3}$ do not involve moments of the reactions. If $M_{\mathbf{e}_3}$ and $M_{\mathbf{u}_3}$ are known as functions of (s, ω_3), then the last two equations of (7.6c) can be integrated and provide s and ω_3 as functions of time. Putting these in the first equation gives the component along $\mathbf{t}$ of the resultant moment. The remaining active forces are those due to gravity and those due to the rotation of Earth. The sum of the forces of gravity and the forces due to the centrifugal acceleration of Earth is by definition the weight of the top (§6 of Chapter 3), and as such, it does not generate a moment with respect to P_o. Therefore the functional form of the components $M_{\mathbf{e}_3}$ and $M_{\mathbf{u}_3}$ is determined only from the Coriolis forces. These generate the moment

$$\mathbf{M} = -2\int (P - P_o) \wedge [\omega_T \mathbf{N} \wedge \mathbf{v}_{S_T}(P)] d\mu(P),$$

where P is the generic point of the top, contiguous to the elemental mass $d\mu(P)$, and $\mathbf{v}_{S_T}$ denotes its velocity with respect to the triad S_T. Expressing $\mathbf{v}_{S_T}$ by the Poisson formula and using the expression (7.4c) of $\boldsymbol{\omega}$, compute

$$\begin{aligned}
\mathbf{M} &= -2\int (P - P_o) \wedge [\omega_T \mathbf{N} \wedge \mathbf{v}_{S_T}(P)]\, d\mu(P) \\
&= -2\omega_T \int (P - P_o) \wedge \{\mathbf{N} \wedge [\boldsymbol{\omega} \wedge (P - P_o)]\}\, d\mu(P) \\
&= 2\omega_T \int (P - P_o) \wedge \{[\boldsymbol{\omega} \cdot \mathbf{N}](P - P_o) - [\mathbf{N} \cdot (P - P_o)]\boldsymbol{\omega}\}\, d\mu(P) \\
&= 2\omega_T \int [\mathbf{N} \cdot (P - P_o)]\, [\boldsymbol{\omega} \wedge (P - P_o)]\, d\mu(P).
\end{aligned}$$

Because of the gyroscopic structure of the top, the triad $\widetilde{S} = \{P_o; \mathbf{t}, \mathbf{v}, \mathbf{u}_3\}$ is principal central of inertia. Writing

$$\mathbf{N} = N_{\mathbf{t}}\mathbf{t} + N_{\mathbf{v}}\mathbf{v} + N_{\mathbf{u}_3}\mathbf{u}_3, \quad (P - P_o) = x\mathbf{t} + y\mathbf{v} + z\mathbf{u}_3,$$

the last integrand can be rewritten as

$$\begin{aligned}(N_{\mathbf{t}}x + N_{\mathbf{v}}y + &N_{\mathbf{u}_3}z)[(\dot{s}z - \omega_3 y)\mathbf{t} + \omega_3 x\mathbf{v} - \dot{s}x\mathbf{u}_3] \\ &= (\dot{s}N_{\mathbf{u}_3}z^2 - \omega_3 N_{\mathbf{v}}y^2)\mathbf{t} + \omega_3 N_{\mathbf{t}}x^2\mathbf{v} - \dot{s}N_{\mathbf{t}}x^2\mathbf{u}_3 \\ &\quad + \left\{ \begin{array}{c} \text{vectorial terms of coefficients} \\ \text{of mixed-type } xy,\ xz,\ yz \end{array} \right\}.\end{aligned}$$

The integral of these mixed terms gives the deflection moments of the top with respect to the coordinate planes of the triad $\widetilde{S}$. These are zero, since $\widetilde{S}$ is central principal of inertia. The remaining terms are computed using the gyroscopic structure of the top, and give

$$2\int x^2 d\mu(P) = 2\int y^2 d\mu(P) = \mathcal{I}_3, \qquad \int z^2 d\mu(P) = \mathcal{I}.$$

Therefore the moment generated by the Coriolis forces is

$$\frac{\mathbf{M}}{\omega_T} = [2\dot{s}N_{\mathbf{u}_3}\mathcal{I} - \omega_3 N_{\mathbf{v}}\mathcal{I}_3]\mathbf{t} + \omega_3 N_{\mathbf{t}}\mathcal{I}_3\mathbf{v} - \dot{s}N_{\mathbf{t}}\mathcal{I}_3\mathbf{u}_3.$$

Putting this in the last two equations of (7.6c) yields the differential system

$$\mathcal{I}\ddot{s} = \mathcal{I}_3\omega_3\omega_T N_{\mathbf{t}}, \qquad \mathcal{I}_3\dot{\omega}_3 = -\mathcal{I}_3\dot{s}\omega_T N_{\mathbf{t}}. \tag{7.7c}$$

7.3c Integrating the Intrinsic Equations

A first integral of (7.7c) is found by multiplying the first equation by $\dot{s}$ and the second by ω_3 and adding, which gives the energy integral in the form

$$\mathcal{I}\dot{s}^2 + \mathcal{I}_3\omega_3^2 = \text{const.}$$

More generally, (7.7c) can be integrated starting from some given initial data. Let $\theta = \widehat{\mathbf{u}_3\mathbf{e}_1}$ in the plane π. Since the second extreme of $\mathbf{u}_3$ moves over the unit circle about P_o on π, one has $s = \theta$. Moreover, $N_{\mathbf{t}} = -\sin\alpha\sin\theta$. With these remarks, the system (7.7c) takes the form

$$\begin{array}{lll} \ddot{\theta} = -a\omega_3\sin\theta, & \theta(0) = \theta_o, & a = \dfrac{\mathcal{I}_3\omega_T}{\mathcal{I}}\sin\alpha, \\ \dot{\omega}_3 = b\dot{\theta}\sin\theta, & \omega_3(0) = \omega_{o,3}, & b = \omega_T\ \sin\alpha. \end{array} \tag{7.8c}$$

The second equation can be integrated explicitly, and we obtain

$$\omega_3 = \omega_{o,3} + b(\cos\theta_o - \cos\theta).$$

Putting this in the first equation yields

$$\ddot{\theta} + a\,\omega_{o,3}\Big[1 + \frac{b}{\omega_{o,3}}(\cos\theta_o - \cos\theta)\Big]\sin\theta = 0. \tag{7.9c}$$

If $\theta_o = 0$, i.e., if initially $\mathbf{u}_3$ points to the north pole, then the only solution of (7.9c) is $\theta = 0$, that is, the figure axis continues to point toward the north pole at all times. Assume now that $\theta_o \neq 0$, and that the gyroscope is set, initially, in rapid rotation about its figure axis, with $\omega_{o,3} \gg 1$ so large that $b/\omega_{o,3}$ is negligible. Then an approximate integral of (7.9c) is

$$\ddot{\theta} + a\,\omega_{o,3} \sin\theta = 0.$$

This is the equation of a pendulum of length $\ell = g/a\omega_{o,3}$ and frequency $\nu = \sqrt{a\omega_{o,3}}$. Therefore the gyroscopic axis exhibits approximate periodic oscillations about $\mathbf{e}_1$, which itself is directed toward the north pole. In this sense this device is a *compass*.

10c Precessions of Constant Direction ($g(\theta_o) \geq 0$)

Rewrite $g(\theta)$ in the form

$$\begin{aligned} g(\theta) &= \frac{2mgd}{\mathcal{I}} \left[(b + \sin^2\theta) - a(\cos\theta_o - \cos\theta) \right], \\ a &= \frac{\mathcal{I}}{2mgd} \Big(\omega_{o,3}^2 \frac{\mathcal{I}_3^2}{\mathcal{I}^2} + \dot{\varphi}_o^2 \sin^2\theta_o \Big), \\ b &= \frac{\mathcal{I}}{2mgd} \Big(2\dot{\varphi}_o^2 \cos\theta_o - \dot{\varphi}_o \omega_{o,3} \frac{\mathcal{I}_3}{\mathcal{I}} \Big) \sin^2\theta_o. \end{aligned} \tag{10.1c}$$

The condition $g(\theta_o) > 0$ implies

$$\omega_{o,3} \frac{\mathcal{I}_3}{\mathcal{I}} < 2\dot{\varphi}_o \cos\theta_o + \frac{2mgd}{\mathcal{I}\dot{\varphi}_o}.$$

Therefore if $\omega_{o,3} \to \infty$, the condition holds if either $\dot{\varphi}_o \to \infty$ or if $\dot{\varphi}_o \to 0$. In either case $a \to \infty$.

10.1c Asymptotic Amplitude of the Nutation

Since θ_1 is the first zero of $g(\cdot)$,

$$\cos\theta_o - \cos\theta_1 = \frac{(b + \sin^2\theta_1)}{a}.$$

Therefore if $\dot{\varphi}_o \to 0$ and $\omega_{o,3} \to \infty$, then $\theta_1 \to \theta_o$, in agreement with (8.6)–(8.6)$'$. If $\dot{\varphi}_o \to \infty$ while $\omega_{o,3} = O(1)$, then

$$\cos\theta_o - \cos\theta_1 = 2\cos\theta_o, \qquad \text{i.e.,} \qquad \theta_1 = \pi - \theta_o.$$

Since $\theta_o < \theta_1 < \pi$, this is possible only if $\theta_o \in (0, \pi/2)$. Therefore the nutation angle θ exhibits periodic oscillations of amplitude $\pi - 2\theta_o$, symmetric about $\theta = \pi/2$. In the limiting case $\theta = \pi/2$ the nutation is zero, and the motion reduces to a precession of infinite speed.

10.2c Asymptotic Period of the Nutation

The expression (8.7) of the period T can be rewritten as

$$T = 2\sqrt{\frac{\mathcal{I}}{2mgd}} \int_{\theta_o}^{\theta_1} \frac{\sin\theta \, d\theta}{\sqrt{\cos\theta_o - \cos\theta}\sqrt{(b + \sin^2\theta) - a(\cos\theta_o - \cos\theta)}}.$$

Introduce the change of variables

$$a(\cos\theta_o - \cos\theta) = (b + \sin^2\theta)\sin^2 s, \qquad s \in (0, \pi/2),$$

and compute the differentials

$$\sin\theta \, d\theta = \frac{2(b + \sin^2\theta)\sin s \cos s \, ds}{a(1-\eta)}, \qquad \eta = \frac{2\cos\theta \sin^2 s}{a}.$$

For $a \gg 1$ one also has $\eta \ll 1$. Therefore proceeding as in the case of (8.7), one gets an asymptotic estimate of T as either $\omega_{o,3} \to \infty$ or $\dot{\varphi} \to \infty$ in the form

$$T = 2\pi \frac{\mathcal{I}}{\sqrt{\omega_{o,3}^2 \mathcal{I}_3^2 + \dot{\varphi}_o^2 \mathcal{I}^2 \sin^2\theta_o}} + o\left(\frac{1}{\omega_{o,3} + \dot{\varphi}_o}\right).$$

10.2.1. Compute the period of the precession starting from (8.1) and using an argument similar to the one leading to (8.7). Investigate the behavior as $a \to \infty$.

10.2.2. If $\omega_{o,3} \to 0$, the motion tends to a spherical pendulum (§7.3c of the Complements of Chapter 3). This is called a *spherical compound pendulum*. Compute the length of such a compound pendulum.

8

STABILITY AND SMALL OSCILLATIONS

1 Notion of Stability in Phase Space

Let $\{\mathcal{M}; d\mu\}$ be a mechanical system with N degrees of freedom, by the Lagrangian parameters $(q_1, \dots, q_N)$. We will assume that $\{\mathcal{M}; d\mu\}$ is subject to fixed holonomic constraints satisfying the principle of virtual work and is acted upon by conservative forces. The motion is then determined either by the Lagrange equations $(3.1)_V$ of Chapter 6 or the Hamilton equations (5.2) of the same chapter, or by some form of the cardinal equations (3.1)–(3.2) of Chapter 4, such as, for example, the system of Poinsot precessions (2.1) of Chapter 7. Since the constraints are fixed, the structural functions of these equations, such as the Lagrangian or Hamiltonian, are explicitly independent of time. These *dynamical systems* can be given a unifying mathematical formalism. Let E be an open set in $\mathbb{R}^n$, for some $n \in \mathbb{N}$, whose points are denoted by $x = (x_1, \dots, x_n)$. Given a locally Lipschitz-continuous vector-valued function

$$E \ni x \longrightarrow \mathbf{F}(x) = \big(F_1(x), \dots, F_n(x)\big),$$

the first-order system of ordinary differential equations

$$\dot{x} = \mathbf{F}(x), \qquad x(t_o) = x_o \in E, \tag{1.1}$$

admits a unique solution for t about t_o. We assume that proper conditions are placed on $\mathbf{F}$ and x_o so that (1.1) is solvable for all $t \in \mathbb{R}$. Up to a possible time shift, we will assume that $t_o = 0$ and will avoid specifying the domain of definition E of $\mathbf{F}$. The Hamiltonian system (5.2) of Chapter 6 is of this form for $n = 2N$, $x = (p, q)$, and $\mathbf{F} = (\nabla_p \mathcal{H}, -\nabla_q \mathcal{H})$. The system of Poinsot precessions (2.1) of Chapter 7 has the form (1.1) for $n = 3$ and $x = \boldsymbol{\omega}$. Referring back to $(3.1)'$ of Chapter 6, also the Lagrange equations can be recast in the form (1.1), for $n = 2N$ and

E. DiBenedetto, *Classical Mechanics: Theory and Mathematical Modeling*, Cornerstones, DOI 10.1007/978-0-8176-4648-6_8,

$$x_j = \begin{cases} q_j & \text{for } j = 1, \dots, N, \\ \dot{q}_j & \text{for } j = N+1, \dots, 2N, \end{cases}$$

$$F_j = \begin{cases} x_{j+N} & \text{for } j = 1, 2, \dots, N, \\ f_j & \text{for } j = N+1, \dots, 2N. \end{cases}$$

This last example indicates that the equation of motion can be recast in a form like (1.1) even for moving constraints and forces not necessarily conservative. If the constraints are moving, $\mathbf{F}$ bears an explicit dependence on time. Dynamical systems of the form (1.1) with $\mathbf{F}$ explicitly independent of t are called *autonomous*.

1.1 Equilibrium Configurations

A point $x^o \in E$ is an equilibrium configuration for (1.1) if for the initial datum $x_o = x^o$, the system admits the unique solution $t \to x(t) = x^o$.

An equilibrium configuration x^o is *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all initial data $\|x_o - x^o\| < \delta$, the corresponding solutions $x(\cdot; x_o)$ remain in the ball centered at x^o with radius ε, i.e., if $\|x(t; x_o) - x^o\| < \varepsilon$ for all $t \in \mathbb{R}$. An equilibrium configuration x^o is *asymptotically stable* if there exists $\varepsilon > 0$ such that for all initial data $\|x_o - x^o\| < \varepsilon$, the corresponding solutions $x(\cdot; x_o)$ tend to x^o as $t \to \infty$, i.e., if $\|x(t; x_o) - x^o\| \to 0$ as $t \to \infty$.

An equilibrium configuration x^o is *unstable* if for every pair of positive numbers $\varepsilon > 0$ and δ, there exists $\|x_o - x^o\| < \delta$ such that the corresponding solution $x(\cdot; x_o)$ does not remain in the ball centered at x^o with radius ε, i.e., if for all $t' > 0$ there exists $t > t'$ such that $\|x(t; x_o) - x^o\| > \varepsilon$.

Remark 1.1 Because of the physical significance of these notions, their definition pertains only to positive times. However, solutions of (1.1) might exist for all $t \in \mathbb{R}$, and the corresponding notions would have to be specified for $t \to \pm\infty$. For example, the scalar equation $\dot{x} = \lambda x$ for $\lambda < 0$ has solutions $x = x_o e^{\lambda t}$. For it, the origin is a configuration of stable equilibrium for $t \to \infty$ and unstable equilibrium for $t \to -\infty$.

2 Lyapunov Stability Criteria

Let x^o be an equilibrium configuration for the dynamical system (1.1). A smooth function $\mathcal{F} : E \to \mathbb{R}$, is a *Lyapunov function* for the pair $\{\mathbf{F}; x^o\}$ if

$$\begin{aligned} \mathcal{F}(x) > 0 \qquad & \text{for all } x \in E - x^o, \\ \mathcal{F}(x^o) = 0, \qquad & \\ \mathbf{F}(x) \cdot \nabla \mathcal{F}(x) \le 0 \qquad & \text{for all } x \in E. \end{aligned} \tag{2.1}$$

The first two conditions imply that x^o is an isolated minimum for $\mathcal{F}$. The last one implies that $\mathcal{F}$ is nonincreasing along solutions of (1.1). This is verified if $\mathcal{F} = \text{const}$ is a first integral of (1.1). Thus first integrals are candidates to be Lyapunov functions. For example, in the case of the Hamiltonian system (5.2) of Chapter 6, with Hamiltonian explicitly independent of t, the energy is conserved along trajectories of motion in phase space. Therefore the energy $(q,p) \to E(p,q)$ is a candidate Lyapunov function for such a system.[1]

Theorem 2.1 (Lyapunov [118]). *If $\{\mathbf{F}; x^o\}$ admits a Lyapunov function, then x^o is a configuration of stable equilibrium for (1.1).*

Proof. Denote by $B_\rho(x^o)$ the open ball in $\mathbb{R}^n$, centered at x^o with radius ρ, and for $\sigma > 0$ set

$$[\mathcal{F} < \sigma] = \{x \in E \mid \mathcal{F}(x) < \sigma\}.$$

Since x^o is an isolated minimum for $\mathcal{F}$, for every $\varepsilon > 0$ there exists $\sigma > 0$ such that $[\mathcal{F} < \sigma] \subset B_\varepsilon(x^o)$. The set $[\mathcal{F} < \sigma]$ is open, and it contains a ball

$$B_\delta(x^o) \subset [\mathcal{F} < \sigma] \subset B_\varepsilon(x^o).$$

For $x_o \in B_\delta(x^o)$ the corresponding trajectories $t \to x(t;x_o)$ remain in $[\mathcal{F} < \sigma]$, and thus in $B_\varepsilon(x^o)$, since $\mathcal{F}(x_o) < \sigma$ and $\mathcal{F}$ is nonincreasing along them. ∎

Assume now that $\{\mathbf{F}; x^o\}$ admits a Lyapunov function satisfying the more stringent condition

$$\mathbf{F}(x) \cdot \nabla\mathcal{F}(x) < 0 \quad \text{for all } x \in E - x^o. \tag{2.2}$$

This would force $\mathcal{F}$ to be strictly decreasing along any solution of (1.1).

Theorem 2.2. *If $\{\mathbf{F}; x^o\}$ admits a Lyapunov function satisfying (2.2), then x^o is a configuration of asymptotically stable equilibrium for (1.1).*

Proof. By the previous theorem, x^o is a configuration of stable equilibrium. Having fixed $\varepsilon > 0$, let $B_\delta(x^o)$ be such that all orbits $t \to x(t;x_o)$ originating from an initial datum $x_o \in B_\delta(x^o)$ remain in $B_\varepsilon(x^o)$. If x^o were not asymptotically stable, there would exist $x_o \in B_\delta(x^o)$ such that

$$\lim_{t\to\infty} \|x(t;x_o) - x^o\| > 0.$$

In particular,

$$\lim_{t\to\infty} \mathcal{F}\big(x(t;x_o)\big) = \sigma \quad \text{for some } \sigma > 0.$$

Since $\mathcal{F}$ is strictly decreasing along trajectories, $\mathcal{F}\big(x(t;x_o)\big) > \sigma$ for all $t > 0$. On the other hand, by the continuity of $\mathcal{F}$,

$$[\mathcal{F} < \sigma] \subset B_{\varepsilon'}(x^o) \quad \text{for some } 0 < \varepsilon' < \varepsilon.$$

[1] A Lyapunov function need not be differentiable. It is required only to be continuous and to admit a continuous right derivative along solutions of (1.1) [110].

Therefore $x(\cdot; x_o)$ remains confined in the closed annulus $\varepsilon' \le \|x - x^o\| \le \varepsilon$. On such a compact set there exists $\lambda > 0$ such that

$$\max_{\varepsilon' \le \|x - x^o\| \le \varepsilon} \mathbf{F}(x) \cdot \nabla \mathcal{F}(x) = -\lambda < 0.$$

Therefore $\mathcal{F}\big(x(t; x_o)\big) \to -\infty$, contradicting the continuity of $\mathcal{F}$. ■

3 Criteria of Instability

Let x^o be an equilibrium configuration and assume that there exists a smooth function $\mathcal{F} : E \to \mathbb{R}$ such that

$$\begin{aligned} &\mathcal{F}(x^o) = 0, \\ &\exists \{x_n\} \to x^o \ \text{ such that } \ \mathcal{F}(x_n) > 0, \\ &\mathbf{F}(x) \cdot \nabla \mathcal{F}(x) > 0 \quad \forall x \in E - x^o. \end{aligned} \tag{3.1}$$

The last of these implies that $\mathcal{F}$ strictly increases along solutions of (1.1).

Theorem 3.1. *If $\{\mathbf{F}; x^o\}$ admits a function $\mathcal{F}$ satisfying (3.1), then x^o is a configuration of unstable equilibrium.*

Proof. For any balls $B_\delta(x^o) \subset B_\varepsilon(x^o)$ however small, there exists $x_o \in B_\delta(x^o)$ such that $\mathcal{F}(x_o) > 0$. For such a fixed positive number, there exists $0 < \varepsilon' < \varepsilon$ such that $\mathcal{F}(x) < \mathcal{F}(x_o)$ for all $x \in B_{\varepsilon'}(x^o)$. Since $\mathcal{F}$ increases along solutions of (1.1), the orbit $t \to x(t : x_o)$ never penetrates $B_{\varepsilon'}(x^o)$. However, such an orbit must exit $B_\varepsilon(x^o)$. Indeed, if not, it would remain confined to the closed spherical annulus $\varepsilon' \le \|x - x^o\| \le \varepsilon$. On such a compact set,

$$\min_{\varepsilon' \le \|x - x^o\| \le \varepsilon} \mathbf{F}(x) \cdot \nabla \mathcal{F}(x) \ge \lambda \qquad \text{for some } \ \lambda > 0.$$

Therefore $\mathcal{F}\big(x(t; x_o)\big) \to \infty$ as $t \to \infty$. ■

Remark 3.1 The proof requires only that for any fixed $\varepsilon > 0$, the function $\mathcal{F}$ be increasing along solutions of (1.1) that remain in $B_\varepsilon(x^o)$. Therefore the last inequality of (3.1) could be replaced by

$$\dot{\mathcal{F}}\big(x(t; x_o)\big) > 0 \qquad \text{along orbits of (1.1) that remain in } B_\varepsilon(x^o). \tag{3.1$'$}$$

3.1 Some Generalizations and the Četaev Theorem

The proof of Theorem 3.1 uses only the last inequality of (3.1) for orbits originating from points x_o close to the points of $\{x_n\}$. Since $\mathcal{F}$ is continuous, for all $n \in \mathbb{N}$ there exists an open set $\mathcal{O}_n$ containing x_n and such that

$$\mathcal{F}(x) > 0 \qquad \text{for all } \ x \in \mathcal{O}(x_n).$$

Setting $\mathcal{O} = \bigcup \mathcal{O}(x_n)$, the requirement (3.1) implies

$$\begin{aligned}
&\exists \text{ an open set } \ \mathcal{O} \subset E \ \text{ such that} \\
&x^o \in \partial\mathcal{O};\ \mathcal{F} > 0 \ \text{ in } \ \mathcal{O};\ \mathcal{F} = 0 \ \text{ on } \ \partial\mathcal{O}, \\
&\mathbf{F}(x) \cdot \nabla\mathcal{F}(x) > 0 \quad \forall x \in \mathcal{O}.
\end{aligned} \tag{3.2}$$

Theorem 3.2 (Četaev [26]). *If $\{\mathbf{F}; x^o\}$ admits a function $\mathcal{F}$ satisfying (3.2), then x^o is a configuration of unstable equilibrium.*

Remark 3.2 The proof is analogous to the previous arguments. It shows that given $\varepsilon > 0$, the last inequity of (3.2) needs to be verified only along solutions of (1.1) whose orbits remain in $\mathcal{O} \cap B_\varepsilon(x^o)$.

4 Dirichlet Stability Criteria

Let $\{\mathcal{M}; d\mu\}$ be a mechanical system subject to fixed holonomic constraints, satisfying the principle of virtual work, and acted upon by conservative forces of potential V. Its dynamics are governed by the Hamilton equations for the variables (p, q). By the definition of kinetic momenta p,

$$2T(q, \dot{q}) = A_{hk}(q)\dot{q}_h\dot{q}_k \quad \Longleftrightarrow \quad 2T(q, p) = a_{hk}(q)p_h p_k,$$

where $\big(a_{hk}(q)\big)$ is the inverse matrix of $\big(A_{hk}(q)\big)$. The Hamiltonian is explicitly independent of t, and it takes the form

$$\mathcal{H}(p, q) = \tfrac{1}{2}a_{hk}(q)p_h p_k - V(q).$$

Theorem 4.1 (Dirichlet [43]). *If q^o is an isolated maximum of V, then $(0, q^o)$ is a configuration of stable equilibrium.*

Proof. Up to a change of variables, we may assume that $q^o = 0$ and $V(0) = 0$. Since the constraints are fixed, $\mathcal{H}(q, p) = E(q, p)$ is conserved along the motion. Since $\big(a_{hk}(q)\big)$ is positive definite, if q^o is an isolated maximum for V, the point $(q^o, 0)$ is an isolated minimum for E. Therefore $(q, p) \to E(q, p)$ is a Lyapunov function, relative to such a point, for the corresponding Hamiltonian system. The conclusion then follows from Theorem 2.1. ∎

Corollary 4.1 *If q^o is an isolated minimum for the potential energy, then $(0, q^o)$ is a configuration of stable equilibrium.*

Theorem 4.2. *If q^o is an isolated minimum of V, then $(0, q^o)$ is a configuration of unstable equilibrium.*

Proof. We may assume that $q^o = 0$ and $V(0) = 0$. The function $\mathcal{F}(p,q) = p\cdot q$ satisfies the first two requirements of (3.1), relative to $(0,0)$ and the Hamiltonian system. The conclusion will follow from Theorem 3.1 upon verifying that $\mathcal{F}$ satisfies (3.1)$'$. Along solutions of the Hamiltonian system (see (6.2) of Chapter 6),

$$\dot{q}_h p_h = \mathcal{H}(p,q) + \mathcal{L}(q,\dot{q}) = 2T(p,q).$$

Therefore, along such orbits,

$$\begin{aligned}\dot{\mathcal{F}}(p,q) &= \dot{q}_h p_h + q_h \dot{p}_h = 2T - \nabla_q \mathcal{H}\cdot q = 2T - \nabla_q(T-V)\cdot q \\ &= a_{hk}(q) p_h p_k - \tfrac{1}{2} a_{hk,q_\ell}(q) q_\ell p_h p_k + \nabla V\cdot q \\ &\geq a_o \|p\|^2 - \tfrac{1}{2} a_{hk,q_\ell}(q) q_\ell p_h p_k + \nabla V\cdot q\end{aligned}$$

for a positive number a_o. Since the functions $a_{hk}(\cdot)$ are regular, there exists $\varepsilon_o > 0$ such that

$$|a_{hk,q_\ell}(q) q_\ell p_h p_k| \leq a_o \|p\|^2 \quad \text{for} \quad \|q\| < \varepsilon_o.$$

Therefore, along solutions of the Hamiltonian system confined in $\|q\| < \varepsilon_o$,

$$\dot{\mathcal{F}}(p,q) \geq \tfrac{1}{2} a_o \|p\|^2 + \nabla V\cdot q.$$

If V has an isolated minimum at the origin, the number ε_o can be chosen so that $\nabla V(q)\cdot q > 0$ for all $0 < \|q\| < \varepsilon_o$. ■

5 Stability and Instability of the Poinsot Precessions

Consider the precession by inertia, or Poinsot precession of a system $\{\mathcal{M}; d\mu\}$ about a pole O. Introduce a moving triad S, clamped with $\{\mathcal{M}; d\mu\}$, with origin at O, and principal of inertia. Denoting by $\mathcal{I}_i$ the moments of inertia of $\{\mathcal{M}; d\mu\}$ with respect to the coordinate axes, the dynamics of the system are determined by (§2 of Chapter 7)

$$\begin{aligned}\mathcal{I}_1 \dot{\omega}_1 &= (\mathcal{I}_2 - \mathcal{I}_3)\omega_2\omega_3, \\ \mathcal{I}_2 \dot{\omega}_2 &= (\mathcal{I}_3 - \mathcal{I}_1)\omega_1\omega_3, \\ \mathcal{I}_3 \dot{\omega}_3 &= (\mathcal{I}_1 - \mathcal{I}_2)\omega_1\omega_2.\end{aligned} \tag{5.1}$$

The system has integrals of energy and angular momentum, i.e.,

$$\begin{aligned} T = T_o \qquad &\Longleftrightarrow \mathcal{I}_i\omega_i^2 = \mathcal{I}_i\omega_{o,i}^2, \\ \|\mathbf{K}\|^2 = \|\mathbf{K}_o\|^2 \qquad &\Longleftrightarrow \mathcal{I}_i^2\omega_i^2 = \mathcal{I}_i^2\omega_{o,i}^2,\end{aligned}$$

for a given initial configuration $\boldsymbol{\omega}_o = (\omega_{o,1}, \omega_{o,2}, \omega_{o,3})$. By the remarks of §3.4 of Chapter 7, a rotation about a principal axis of inertia is permanent. Thus in particular, the configurations $\boldsymbol{\omega}^o = (\omega_{o,1}, 0, 0)$ are of equilibrium. The stable or unstable nature of such equilibrium configurations depends uniquely on the material geometry of $\{\mathcal{M}; d\mu\}$, or more precisely, on the structure of the ellipsoid of inertia $\mathcal{E}_\lambda$. Introduce the functions

$$\mathcal{F}_1(\boldsymbol{\omega}) = \mathcal{I}_i\omega_i^2 - \mathcal{I}_1\omega_{o,1}^2,$$
$$\mathcal{F}_2(\boldsymbol{\omega}) = \mathcal{I}_i^2\omega_i^2 - \mathcal{I}_1^2\omega_{o,1}^2,$$
$$\mathcal{F}(\boldsymbol{\omega}) = \mathcal{F}_1^2(\boldsymbol{\omega}) + \mathcal{F}_2^2(\boldsymbol{\omega}).$$

The first two of these are first integrals of (5.1) and vanish for $\boldsymbol{\omega} = \boldsymbol{\omega}^o$. The last one is a nonnegative first integral of (5.1) vanishing for $\boldsymbol{\omega} = \boldsymbol{\omega}^o$.

Proposition 5.1 *The equilibrium configuration* $\boldsymbol{\omega}^o = (\omega_{o,1}, 0, 0)$ *is stable in any one of the following cases:*

$$\mathcal{I}_1 < \min\{\mathcal{I}_2; \mathcal{I}_3\}, \qquad \mathcal{I}_1 > \max\{\mathcal{I}_2; \mathcal{I}_3\}, \qquad \mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_3.$$

Proof. In the last case the ellipsoid of inertia is a sphere, and every axis through the pole of the precession is a principal axis of inertia. Therefore the assertion follows from the remarks of §3.4 of Chapter 7. To establish the assertion for the remaining two cases we will show that $\mathcal{F}$ is a Lyapunov function for the system (5.1) and the configuration $\boldsymbol{\omega}^o$. First, $\mathcal{F}$ is a non negative first integral of (5.1) vanishing for $\boldsymbol{\omega} = \boldsymbol{\omega}^o$. Thus it suffices to show that $\mathcal{F}(\boldsymbol{\omega}) > 0$ for all $\boldsymbol{\omega} \neq \boldsymbol{\omega}^o$. For one such $\boldsymbol{\omega}$, either $\mathcal{F}_1(\boldsymbol{\omega}) \neq 0$ or $\mathcal{F}_1(\boldsymbol{\omega}) = 0$. In the former case there is nothing to prove. In the latter case,

$$\mathcal{I}_1\omega_1^2 = \mathcal{I}_1\omega_{o,1}^2 - \mathcal{I}_2\omega_2^2 - \mathcal{I}_3\omega_3^2.$$

Since $\boldsymbol{\omega} \neq \boldsymbol{\omega}^o$, this implies that at least one of the components ω_2, ω_3 is not zero. Putting this in the expression of $\mathcal{F}_2$ gives

$$\mathcal{F}_2^2(\boldsymbol{\omega}) = \big[\mathcal{I}_2(\mathcal{I}_2 - \mathcal{I}_1)\omega_2^2 + \mathcal{I}_3(\mathcal{I}_3 - \mathcal{I}_1)\omega_3^2\big]^2 > 0. \qquad \blacksquare$$

Proposition 5.2 *Assume that the moments of inertia* $\mathcal{I}_i$ *are not all equal. The equilibrium configuration* $\boldsymbol{\omega}^o$ *is unstable in either of the following two cases:*

$$\mathcal{I}_2 \le \mathcal{I}_1 \le \mathcal{I}_3, \qquad \mathcal{I}_3 \le \mathcal{I}_1 \le \mathcal{I}_2.$$

Proof. Assume that the first of these holds and that $\omega_{o,1} > 0$. The function $\mathcal{F}(\boldsymbol{\omega}) = \omega_2\omega_3$ is regular and the set $\mathcal{O} = [\mathcal{F} > 0]$ is open and nonempty. Moreover, $\boldsymbol{\omega}^o \in \partial\mathcal{O}$. Thus $\mathcal{F}$ satisfies the first two conditions of (3.2). The last one will be verified in the weaker form indicated by Remark 3.2. Along solutions of (5.1),

$$\dot{\mathcal{F}}(\boldsymbol{\omega}) = \omega_1\Big(\frac{\mathcal{I}_1 - \mathcal{I}_2}{\mathcal{I}_3}\omega_2^2 + \frac{\mathcal{I}_3 - \mathcal{I}_1}{\mathcal{I}_2}\omega_3^2\Big).$$

For $\boldsymbol{\omega} \in \mathcal{O}$ the quantity in parentheses is positive. Since $\omega_{o,1} > 0$, there exists $\varepsilon > 0$ sufficiently small that $\omega_1 > 0$ for $\|\boldsymbol{\omega} - \boldsymbol{\omega}^o\| < \varepsilon$. Therefore

$$\dot{\mathcal{F}}(\boldsymbol{\omega}) > 0 \quad \text{along solutions of (5.1) in } \mathcal{O} \cap B_\varepsilon(\boldsymbol{\omega}^o).$$

The conclusion now follows from the Četaev instability theorem. ∎

Theorem 5.1. *Let $\mathcal{E}_\lambda$ be the ellipsoid of inertia of a free rotator. If $\mathcal{E}_\lambda$ is a sphere, every rotation is stable. Otherwise, rotations about the extreme axes are stable, whereas rotations about the intermediate axes are unstable.*

6 Linearized Motions

Let $\{\mathcal{M}; d\mu\}$ be subject to fixed, smooth, holonomic constraints, and acted upon by forces of potential V, and let

$$\mathcal{L}(q, \dot{q}) = \tfrac{1}{2} A_{hk}(q)\dot{q}_h \dot{q}_k + V(q)$$

be its Lagrangian, uniquely determined by stipulating that $V(0) = 0$. Assume that $q = 0$ is a stationary point of V, so that $\nabla V(0) = 0$. By Taylor's formula,

$$\mathcal{L}(q, \dot{q}) = \tfrac{1}{2} A_{hk}(0)\dot{q}_h \dot{q}_k + \tfrac{1}{2} V_{q_h q_k}(0) q_h q_k + O(|(q, \dot{q})|^3).$$

The linearized motion is that for which the terms of order higher than two are neglected, and is thereby described by the linearized Lagrangian

$$2\mathcal{L}_o = A_{hk}(0)\dot{q}_h \dot{q}_k + V_{q_h q_k}(0) q_h q_k = \dot{q}^t \big(A_{hk}(0)\big)\dot{q} + q^t \big(V_{q_h q_k}(0)\big) q.$$

The matrix $\big(A_{hk}(0)\big)$ is symmetric and positive definite. Therefore there exists a unitary matrix $\mathcal{A}$ that diagonalizes it, i.e.,

$$\mathcal{A}\big(A_{hk}(0)\big)\mathcal{A}^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_N \end{pmatrix} \stackrel{\text{def}}{=} \Lambda,$$

where λ_j are the eigenvalues of $\big(A_{hk}(0)\big)$. These eigenvalues are all positive with their multiplicity, and therefore the matrices $\Lambda^{\frac{1}{2}}$ and $\Lambda^{-\frac{1}{2}}$ are well defined. With this formalism, the approximate Lagrangian $\mathcal{L}_o$ can then be rewritten as

$$\begin{aligned} 2\mathcal{L}_o &= (\mathcal{A}\dot{q})^t \mathcal{A}\big(A_{hk}(0)\big)\mathcal{A}^{-1}(\mathcal{A}\dot{q}) + (\mathcal{A}q)^t \mathcal{A}\big(V_{q_h q_k}(0)\big)\mathcal{A}^{-1}(\mathcal{A}q) \\ &= (\mathcal{A}\dot{q})^t \Lambda(\mathcal{A}\dot{q}) + (\mathcal{A}q)^t \mathcal{A}\big(V_{q_h q_k}(0)\big)\mathcal{A}^{-1}(\mathcal{A}q) \\ &= (\Lambda^{\frac{1}{2}}\mathcal{A}\dot{q})^t \mathbb{I}(\Lambda^{\frac{1}{2}}\mathcal{A}\dot{q}) + (\Lambda^{\frac{1}{2}}\mathcal{A}q)^t (\Lambda^{-\frac{1}{2}}\mathcal{A})\big(V_{q_h q_k}(0)\big)(\Lambda^{-\frac{1}{2}}\mathcal{A})^t (\Lambda^{\frac{1}{2}}\mathcal{A}q) \\ &= \dot{Q}^t \mathbb{I}\dot{Q} + Q^t (c_{hk}) Q, \end{aligned}$$

where we have set

$$Q_h = \sqrt{\lambda_h}(\mathcal{A}q)_h \quad \text{i.e.,} \quad Q = \Lambda^{\frac{1}{2}}(\mathcal{A}q),$$
$$(c_{hk}) = (\Lambda^{-\frac{1}{2}}\mathcal{A})\big(V_{q_h q_k}(0)\big)(\Lambda^{-\frac{1}{2}}\mathcal{A})^t.$$

The matrix (c_{hk}) is symmetric, its eigenvalues μ_h are all real, and there exists a unitary matrix $\mathcal{B}$ such that

$$\mathcal{B}(c_{hk})\mathcal{B}^{-1} = \begin{pmatrix} \mu_1 & 0 & 0 & \dots\dots & 0 \\ 0 & \mu_2 & 0 & \dots\dots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & \dots \; 0 & \mu_N \end{pmatrix}.$$

Introducing new Lagrangian coordinates

$$\theta_h = (\mathcal{B}Q)_h = (\mathcal{B}\Lambda^{\frac{1}{2}}\mathcal{A}q)_h, \qquad h = 1, \dots, N, \tag{6.1}$$

the linearized Lagrangian $\mathcal{L}_o$ and the linearized potential V_o take the form

$$\mathcal{L}_o = \frac{1}{2}\sum_{h=1}^{N} |\dot{\theta}_h|^2 + V(\theta), \qquad V_o(\theta) = \frac{1}{2}\sum_{h=1}^{N} \mu_h \theta_h^2, \tag{6.2}$$

and the corresponding Lagrange equations are

$$\ddot{\theta}_h - \mu_h \theta_h = 0, \qquad h = 1, \dots, N. \tag{6.3}$$

These are the equations of motion, linearized about the origin of the phase space $\mathbb{R}^{2N}$ for the variables $(\dot{q}, q)$. They approximate the actual motion if by prescribing initial data $(\dot{\theta}_o, \theta_o)$ close to the origin, the resulting solutions $(\dot{\theta}, \theta)$ remain near the origin at all times. This occurs, for example, if the origin is a configuration of stable equilibrium for the system.

In these arguments, the origin of the configuration space $\mathbb{R}^N$ for the variables q could be replaced by any fixed configuration q^o. While q^o is a stationary point of the potential V, no restrictions are placed on the Hessian $\big(V_{q_h q_k}(q^o)\big)$. Since the eigenvalues μ_h of (c_{hk}) coincide with the eigenvalues of the Hessian $\big(V_{q_h q_k}(q^o)\big)$, information on the nature of the stationary point q^o can be given in terms of these eigenvalues.

6.1 Stability and Instability of Linearized Motions

If the eigenvalues μ_h are all negative, the linearized potential V_o has an isolated minimum at the origin. Therefore by Dirichlet's stability criterion of Theorem 4.1, the origin is a configuration of stable equilibrium for the linearized motion (6.3).

Proposition 6.1 *If $\mu_h > 0$ for some $h = 1, \dots, N$, then the origin is a configuration of unstable equilibrium for the linearized motion (6.3).*

Proof. Assume $\mu_1 > 0$. Then for the initial data $\theta_o = (\theta_{o,1}, \dots, 0)$ and $\dot{\theta}_o = 0$, the system (6.3) has the unique solution

$$\theta_1 = \theta_{o,1} \cosh \sqrt{\mu_1} t, \quad \text{and} \quad \theta_h = 0 \quad \text{for } h = 2, \dots, N.$$

Thus for every $\varepsilon > 0$ there exist initial configurations in $B_\varepsilon(0)$ whose orbits depart from the origin for $t \to \infty$. ■

If q^o is an isolated maximum for the original potential V, then $(0, q^o)$ is a configuration of stable equilibrium for the original motion. On the other hand, this occurs if and only if the eigenvalues of (c_{hk}) are all negative. Thus configurations of stable equilibrium of the original motion remain of stable equilibrium for the linearized motion.

It is natural to ask whether instability configurations for the linearized system (6.3) are also instability configurations for the original system.

Proposition 6.2 (Lyapunov) *If all the eigenvalues of* (c_{hk}) *are not zero, and if one of them is positive, then the origin is a configuration of unstable equilibrium for the original motion.*

Since the eigenvalues of (c_{hk}) coincide with the eigenvalues of the Hessian of V at the origin, the assumptions of the proposition imply that the origin is not a maximum point for V. Thus if $\det\big(V_{q_h q_k}(0)\big) \neq 0$, the mere lack of maximality of V implies that the origin is a configuration of unstable equilibrium.

Corollary 6.1 *Let* $(0, q^o)$ *be a stationary point for the potential* V *such that* $\det\big(V_{q_h q_k}(q^o)\big) \neq 0$. *Then if* $(0, q^o)$ *is not a maximum, it is a configuration of unstable equilibrium for the system.*

7 Small Oscillations

If the eigenvalues of (c_{hk}) are all negative, set $\mu_h = -\omega_h^2$ and rewrite the linearized system (6.3) in the form

$$\ddot{\theta}_h + \omega_h^2 \theta_h = 0, \qquad h = 1, \dots, N. \tag{7.1}$$

These represent N independent mathematical pendulums, or equivalently N uncoupled harmonic oscillators, each with frequency ω_h. These are called *normal modes* or *principal frequencies.* The coordinates θ_h introduced in (6.1) are the *normal coordinates.*

Proposition 7.1 *Let* $\{\mathcal{M}; d\mu\}$ *be a mechanical system subject to fixed, smooth, holonomic constraints, and acted upon by potential forces of potential* V. *For every configuration* $(0, q^o)$ *of stable equilibrium there exists a local system of normal coordinates such that the corresponding linearized motion about* $(0, q^o)$ *can be separated into* N *independent harmonic oscillators.*

The solutions of (7.1) are of the form

$$\theta_h = A_h \cos(\omega_h t + \varphi_h), \qquad h = 1, \dots, N.$$

It follows from (6.1) that the original Lagrangian coordinates q_h are a linear combination of the normal coordinates [8]. Therefore

$$q_h = \sum_{k=1}^{N} B_{hk} \cos\left(\omega_k t + \varphi_k\right), \qquad h = 1, \dots, N, \tag{7.2}$$

where B_{hk} and φ_k are real constants to be determined from the initial data. Each of the N independent solutions of (7.1) is a *fundamental vibration* and the motion. In phase space (p, q), the motion results from the composition of these fundamental vibrations. While each of the normal coordinates describes a periodic motion, the composite motion $t \to q(t)$ is not periodic in general. If a period T exists such that $q(t + T) = q(t)$ for all t, then (7.2) implies

$$\cos\big(\omega_h(t + T) + \varphi_h\big) = \cos(\omega_h t + \varphi_h) \quad \forall t \in \mathbb{R}, \quad h = 1, 2, \dots, N.$$

Since t is arbitrary, this is possible if and only if there exist positive integers n_h such that $\omega_h T = 2\pi n_h$ for all $h = 1, \dots N$. This in turn implies

$$\frac{\omega_h}{n_h} = \frac{\omega_k}{n_k}, \qquad h, k = 1, 2, \dots, N.$$

Thus the principal frequencies ω_h must be *commensurable*. Summarizing, the composite motion $t \to q(t)$ is periodic if and only if the normal modes are commensurable. A similar treatment occurs in acoustical phenomena, when sound is decomposed into its principal frequencies. In that case the sound is *clear*, and an instrument is *tuned*, if the principal frequencies are commensurable. Ordering such principal frequencies in increasing order $\omega_1 < \omega_2 < \cdots < \omega_N$, the first is the *first harmonic* and corresponds to the *bass*, whereas the following ones are the *second harmonic*, etc., up to the highest frequency corresponding to the *treble*.

8 Vibrations of Masses Subject to Elastic Forces

Two points $\{P_1; m\}$ and $\{P_2; m\}$ of equal mass m are attracted by a third point mass $\{P_3; M\}$ by elastic forces of equal constant k. Assume that the system moves in $\mathbb{R}^3$ and it is in a configuration of stable equilibrium when the three masses are aligned and P_3 is equidistant from P_1 and P_2. Choose a Cartesian system with the horizontal axis as the axis of equilibrium and such that the configuration of stable equilibrium is

$$P_{1,o} = (\ell, 0, 0), \qquad P_{3,o} = (0, 0, 0), \qquad P_{2,o} = (-\ell, 0, 0),$$

for a given positive constant ℓ. Translate the coordinates of the points about their equilibrium configuration, by setting

$$\xi = P_1 - P_{1,o}, \qquad \eta = P_2 - P_{2,o}, \qquad \zeta = P_3.$$

It is assumed that the system exhibits small oscillations about its equilibrium configuration and that in addition, the horizontal components of these oscillations are negligible with respect to those normal to the equilibrium axis, that is, $(P_i - P_{i,o}) \cdot P_{i,o} \approx 0$. This system is taken as a model for the vibrations of a 3-atom molecule, such as for example CO_2 (carbon dioxide), when the atomic potentials are approximated by elastic potentials [14, Chap. IX, §67]. With this symbolism and stipulations, the kinetic energy T and the elastic potential V are

$$\begin{aligned} 2T &= m(\dot{P}_1^2 + \dot{P}_2^2) + M\dot{P}_3^2 = m(\dot{\xi}^2 + \dot{\eta}^2) + M\dot{\zeta}^2, \\ 2V &= -k\|(P_1 - P_{1,o}) - P_3\|^2 - k\|(P_2 - P_{2,o}) - P_3\|^2 \\ &= -k(\xi^2 + \eta^2 + 2\zeta^2 - 2\xi \cdot \zeta - 2\eta \cdot \zeta), \end{aligned}$$

or in matrix form,

$$\begin{aligned} 2T &= \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix}^t \begin{pmatrix} m\mathbb{I} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & m\mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & M\mathbb{I} \end{pmatrix} \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix}, \\ -\frac{2}{k}V &= \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}^t \begin{pmatrix} \mathbb{I} & \mathbb{O} & -\mathbb{I} \\ \mathbb{O} & \mathbb{I} & -\mathbb{I} \\ -\mathbb{I} & -\mathbb{I} & 2\mathbb{I} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \end{aligned}$$

where $\mathbb{I}$ is the 3×3 identity matrix and $\mathbb{O}$ is the 3×3 zero matrix. Set $\mu^2 = m/M$ and rewrite T in the form

$$\frac{2}{m}T = \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix}^t \begin{pmatrix} \mathbb{I} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I}/\mu^2 \end{pmatrix} \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta}/\mu \end{pmatrix}^t \begin{pmatrix} \mathbb{I} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I} \end{pmatrix} \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta}/\mu \end{pmatrix}.$$

Similarly, rewrite the potential V as

$$-\frac{2}{k}V = \begin{pmatrix} \xi \\ \eta \\ \zeta/\mu \end{pmatrix}^t \begin{pmatrix} \mathbb{I} & \mathbb{O} & -\mu\mathbb{I} \\ \mathbb{O} & \mathbb{I} & -\mu\mathbb{I} \\ -\mu\mathbb{I} & -\mu\mathbb{I} & 2\mu^2\mathbb{I} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta/\mu \end{pmatrix}.$$

With this notation, the system of the Lagrange equation can be written as

$$\begin{pmatrix} \mathbb{I} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I} \end{pmatrix} \begin{pmatrix} \ddot{\xi} \\ \ddot{\eta} \\ \ddot{\zeta}/\mu \end{pmatrix} + \nu^2 \begin{pmatrix} \mathbb{I} & \mathbb{O} & -\mu\mathbb{I} \\ \mathbb{O} & \mathbb{I} & -\mu\mathbb{I} \\ -\mu\mathbb{I} & -\mu\mathbb{I} & 2\mu^2\mathbb{I} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta/\mu \end{pmatrix} = 0, \tag{8.1}$$

where we have set $\nu^2 = k/m$.

9 Normal Coordinates

The normal frequencies are found by diagonalizing the second matrix, whose eigenvalues are the roots of the algebraic equation

$$\det\begin{pmatrix}(1-\lambda)\mathbb{I} & \mathbb{O} & -\mu\mathbb{I}\\ \mathbb{O} & (1-\lambda)\mathbb{I} & -\mu\mathbb{I}\\ -\mu\mathbb{I} & -\mu\mathbb{I} & (2\mu^2-\lambda)\mathbb{I}\end{pmatrix} = 0.$$

Subtract the second row from the first and then add the first column to the second. This reduces the calculation of the determinant to

$$\det\begin{pmatrix}(1-\lambda)\mathbb{I} & \mathbb{O} & \mathbb{O}\\ \mathbb{O} & (1-\lambda)\mathbb{I} & -\mu\mathbb{I}\\ -\mu\mathbb{I} & -2\mu\mathbb{I} & (2\mu^2-\lambda)\mathbb{I}\end{pmatrix}.$$

Now multiply the second row by 2μ and add it to the last one. Then multiply the last column by 2μ and subtract it from the second. This reduces the calculation to

$$\det\begin{pmatrix}(1-\lambda)\mathbb{I} & \mathbb{O} & \mathbb{O}\\ \mathbb{O} & (1+2\mu^2-\lambda)\mathbb{I} & -\mu\mathbb{I}\\ -\mathbb{I} & \mathbb{O} & -\lambda\mathbb{I}\end{pmatrix} = -\lambda^3(1-\lambda)^3(1+2\mu^2-\lambda)^3.$$

Therefore the eigenvalues of the second matrix in (8.1) are $\lambda = 0$, $\lambda = 1$, and $\lambda = (1+2\mu^2)$, each with multiplicity 3. The corresponding eigenvectors are

$$\text{for } \lambda = 0 \qquad \begin{cases} \mathbf{u}_1 = \dfrac{(\mu,0,0,\ \mu,0,0,\ 1,0,0)}{\sqrt{1+2\mu^2}}, \\ \mathbf{u}_2 = \dfrac{(0,\mu,0,\ 0,\mu,0,\ 0,1,0)}{\sqrt{1+2\mu^2}}, \\ \mathbf{u}_3 = \dfrac{(0,0,\mu,\ 0,0,\mu,\ 0,0,1)}{\sqrt{1+2\mu^2}}, \end{cases}$$

$$\text{for } \lambda = 1 \qquad \begin{cases} \mathbf{v}_1 = \dfrac{(1,0,0,\ -1,0,0,\ 0,0,0)}{\sqrt{2}}, \\ \mathbf{v}_2 = \dfrac{(0,1,0,\ 0,-1,0,\ 0,0,0)}{\sqrt{2}}, \\ \mathbf{v}_3 = \dfrac{(0,0,1,\ 0,0,-1,\ 0,0,0)}{\sqrt{2}}, \end{cases}$$

$$\text{for } \lambda = 1+2\mu^2 \qquad \begin{cases} \mathbf{w}_1 = \dfrac{(1,0,0,\ 1,0,0,\ -2\mu,0,0)}{\sqrt{2(1+2\mu^2)}}, \\ \mathbf{w}_2 = \dfrac{(0,1,0,\ 0,1,0,\ 0,-2\mu,0)}{\sqrt{2(1+2\mu^2)}}, \\ \mathbf{w}_3 = \dfrac{(0,0,1,\ 0,0,1,\ 0,0,-2\mu)}{\sqrt{2(1+2\mu^2)}}. \end{cases}$$

Therefore the unitary matrix that diagonalizes the second matrix in (8.1) is given by

$$\mathcal{B} = \begin{pmatrix} \dfrac{\mu\,\mathbb{I}}{\sqrt{1+2\mu^2}} & \dfrac{\mu\,\mathbb{I}}{\sqrt{1+2\mu^2}} & \dfrac{\mathbb{I}}{\sqrt{1+2\mu^2}} \\ \dfrac{\mathbb{I}}{\sqrt{2}} & \dfrac{-\mathbb{I}}{\sqrt{2}} & \mathbb{O} \\ \dfrac{\mathbb{I}}{\sqrt{2(1+2\mu^2)}} & \dfrac{\mathbb{I}}{\sqrt{2(1+2\mu^2)}} & \dfrac{-2\mu\mathbb{I}}{\sqrt{2(1+2\mu^2)}} \end{pmatrix}.$$

By means of $\mathcal{B}$ the Lagrange equations (8.1) can be rewritten as

$$\mathcal{B}\begin{pmatrix} \ddot{\xi} \\ \ddot{\eta} \\ \ddot{\zeta}/\mu \end{pmatrix} + \nu^2 \begin{pmatrix} \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & (1+2\mu^2)\mathbb{I} \end{pmatrix} \mathcal{B}\begin{pmatrix} \xi \\ \eta \\ \zeta/\mu \end{pmatrix} = 0. \tag{9.1}$$

10 Degenerate Vibrations

The first three components of the normal coordinates written in vector form are

$$\theta_1 = \frac{\mu}{\sqrt{1+2\mu^2}}\Big(\xi + \eta + \frac{1}{\mu^2}\zeta\Big).$$

Recalling the definition of ξ, η, ζ and that $\mu^2 = m/M$, compute

$$\begin{aligned} \theta_1 &= \frac{\mu}{\sqrt{1+2\mu^2}}\frac{1}{m}[m(P_1+P_2)+MP_3] \\ &= \frac{\mu}{\sqrt{1+2\mu^2}}\frac{2m+M}{m}\frac{m(P_1+P_2)+MP_3}{2m+M} \\ &= \frac{\sqrt{1+2\mu^2}}{\mu}P_o, \end{aligned}$$

where P_o is the center of mass of the system. Therefore the first three equations of (9.1), written in vector form, are

$$\ddot{P}_o = 0, \qquad \text{or} \qquad \dot{P}_o = \text{const.} \tag{10.1}$$

This integral was already known expressing the conservation of momentum. Thus the effect of the first three normal coordinates is to permit vibrations to occur in a reference system that translates rectilinear uniform motion with respect to the center of mass. The Lagrange equations (9.1) are in intrinsic vector form and independent of a reference frame. Having assumed that a configuration of the system is in stable equilibrium does not preclude such a configuration being realized in some other inertial system. This is referred to as *degenerate* fundamental vibration.

Fix now an inertial Cartesian system Σ centered at P_o and in (10.1) choose $\dot{P}_o = 0$. This implies $\theta_1 = 0$ identically and

$$P_o = O \quad \text{and} \quad P_1 + P_2 = -\frac{P_3}{\mu^2}. \tag{10.2}$$

11 Fundamental Vibrations

The remaining six normal coordinates are, in vector form,

$$\theta_2 = \frac{(P_1 - P_{1,o}) - (P_2 - P_{2,o})}{\sqrt{2}},$$
$$\theta_3 = \frac{(P_1 - P_3) + (P_2 - P_3)}{\sqrt{2(1+2\mu^2)}}.$$

The corresponding Lagrange equations are, still in vector form,

$$\begin{aligned} \ddot{\theta}_2 + \omega_2^2 \theta_2 = 0, \quad \omega_2 = \sqrt{\frac{k}{m}}, \\ \ddot{\theta}_3 + \omega_3^2 \theta_3 = 0, \quad \omega_3 = \sqrt{\frac{k(1+2\mu^2)}{m}}. \end{aligned} \tag{11.1}$$

Since these are *independent* oscillators, the physical significance of each of them is drawn by setting the remaining ones equal to zero. Setting $\theta_3 = 0$ and taking into account the last of (10.2) implies that P_3 is fixed at the origin and consequently $P_1 = -P_2$. Then the first equation of (11.1) can be written equivalently in terms of $P_1 - P_{1,o}$ or $P_2 - P_{2,o}$ as

$$(P_i - P_{i,o})'' + \omega_2^2 (P_i - P_{i,o}) = 0, \quad i = 1, 2.$$

These have solutions

$$P_i - P_{i,o} = \mathbf{A}_i \cos(\omega_2 t + \varphi_i), \qquad i = 1, 2,$$

where $\mathbf{A}_i$ are constant vectors and φ_i are real constants. Thus the fundamental vibration θ_2 describes the oscillation of P_i, $i = 1, 2$, about their equilibrium configurations $P_{i,o}$. Since $P_1 + P_2 = 0$, the two oscillations have the same amplitudes and are in opposition of phase, i.e., $\varphi_1 = \varphi_2 + \pi$.

For the fundamental vibration corresponding to ω_3 set $\theta_2 = 0$. Then taking into account (10.2),

$$(P_1 - P_{1,o}) - (P_2 - P_{2,o}) = 0, \qquad \ddot{P}_3 + \omega_3^2 P_3 = 0.$$

Taking into account that $P_{1,o} + P_{2,o} = 0$, compute

$$\begin{aligned} -P_3 &= \mu^2 (P_1 + P_2) \\ &= \mu^2 [(P_1 - P_{1,o}) + (P_2 - P_{2,o})] \\ &= \mathbf{B} \cos(\omega_3 t + \varphi), \end{aligned}$$

where $\mathbf{B}$ is a constant vector and φ is a real constant. From these, we obtain

$$P_i - P_{i,o} = \frac{\mathbf{B}}{2\mu^2}\cos(\omega_3 t + \varphi), \qquad P_3 = \mathbf{B}\cos(\omega_3 t + \varphi + \pi).$$

Therefore the points P_i exhibit oscillations of frequency ω_3 about their equilibrium configurations $P_{i,o}$, of equal amplitude and in concurrence of phase. The point P_3 exhibits oscillations about the origin of amplitude $\|\mathbf{B}\|$, frequency ω_3, and in opposition of phase with respect the two "exterior" points P_1 and P_2.

Problems and Complements

2c Lyapunov Stability Criteria

2.1c Harmonic Oscillator and Exponential Decay

For fixed $\omega \in \mathbb{R}$ consider the two dynamical systems

$$\ddot{q} \pm \omega^2 q = 0, \qquad \text{or equivalently} \qquad \begin{cases} \dot{q} = p, \\ \dot{p} = \mp\omega^2 q. \end{cases} \tag{2.1c}$$

Verify that $\mathcal{F}(q,p) = \frac{1}{2}(\omega^2 q^2 + p^2)$ is a Lyapunov function for $(2.1c)_+$ and the origin, and conclude that such a point is a configuration of stable equilibrium. Verify that $\mathcal{F}(q,p) = qp$ is a function relative to $(2.1c)_-$ that satisfies (3.2) and identify the configurations of unstable equilibrium.

2.2c Damped Oscillator

Consider the damped oscillator (3.3c) of Chapter 3,

$$\ddot{q} + 2\varepsilon\dot{q} + \omega^2 q = 0, \qquad \text{or equivalently} \qquad \begin{cases} \dot{q} = p, \\ \dot{p} = -(2\varepsilon p + \omega^2 q). \end{cases} \tag{2.2c}$$

A Lyapunov function relative to the origin is

$$2\mathcal{F}(q,p) = \left(\omega^2 + 2\varepsilon^2\right) q^2 + p^2 + 2\varepsilon qp.$$

This is positive in a neighborhood of the origin of $\mathbb{R}^2$, except at the origin. Prove that along solutions of (2.2c),

$$\dot{\mathcal{F}}(q,p) = -\varepsilon\left(\omega^2 q^2 + p^2\right),$$

and conclude that the origin is a configuration of asymptotically stable equilibrium. Observe that the energy is not conserved and that asymptotic stability is possible only because of dissipations, i.e., only if $\varepsilon > 0$.

4c Dirichlet Stability Criteria

4.1c Point Mass Constrained on a Curve

Let a curve on the vertical plane of (x, y) be represented as the graph of a smooth function f defined in $\mathbb{R}$ and such that $f(0) = f'(0) = 0$. A point mass $\{P; m\}$ slides on it, with no friction, and subject to its weight. The system has one degree of freedom, and taking $q = x$ as Lagrangian coordinate, one computes

$$V = -mgf(q), \qquad T = \tfrac{1}{2}m\dot{q}^2\left(1 + f'^2(q)\right),$$
$$E(q, \dot{q}) = \tfrac{1}{2}m\dot{q}^2\left(1 + f'^2(x)\right) + mgf(q).$$

Since the energy is conserved, $E = E_o$ and one computes

$$\dot{q}^2 = \frac{2}{m}\frac{E_o - mgf(q)}{1 + f'^2(q)}, \qquad \text{provided} \qquad f(q) \le \frac{E_o}{mg}. \tag{4.1c}$$

4.1.1c Isolated Maxima of the Potential *V*

If the origin is an isolated minimum for f, then $E_o > 0$, and the equation $f(q) = E_o/g$ has two distinct roots

$$q_1 < 0 < q_2 \quad \text{such that} \quad f(q) < \frac{E_o}{g} \quad \text{in the interval} \quad (q_1, q_2).$$

Therefore any initial position $q_o \in (q_1, q_2)$ and initial velocity $\dot{q}_o$ for which $E(q_o, \dot{q}_o) = E_o$ generate a motion whose Lagrangian trajectories remain in the interval (q_1, q_2). Quantify the configuration of stable equilibrium of the origin in terms of the two variables $(q, \dot{q})$.

The stable nature of the origin as an equilibrium configuration can be established indirectly by tracing the trajectories $(q, \dot{q})$ in phase space. Using the first equation of (4.1c), prove that such trajectories are symmetric with respect to the q-axis are closed curves surrounding the origin and represent periodic motions.

4.1.2c Isolated Minima of the Potential *V*

If the origin is an isolated maximum for f, then E_o is of variable sign. If $E_o > 0$ the first equation of (4.1c) implies that $|\dot{q}| > 0$. Therefore the abscissa q starting from q_o is always increasing or decreasing, and $\{P; m\}$ departs indefinitely from its equilibrium configuration. Assuming $E_o > 0$, trace the curves $(q, \dot{q})$ in phase space and show that while these might temporarily approach the origin, they will eventually depart from it indefinitely.

If $E_o < 0$, the motion is possible only outside the interval (q_1, q_2). Prove that there exist trajectories that depart indefinitely from such an interval. Prove that a trajectory originating from a point q_o outside such an interval never approaches either q_1 or q_2. Discuss the case $E_o = 0$.

4.1.3c Nonisolated Minima of the Potential V

Assume that there is an interval $(-\delta, \delta)$ where $f = 0$. Prove that the origin is a configuration of unstable equilibrium. Examine the cases in which the points of $(-\delta, \delta)$ are maxima or minima or neither for f.

4.2c On the Dirichlet Instability Criterion

For systems subject only to their weight, the Dirichlet criterion can be given the following form.

Proposition 4.1c *Let $\{\mathcal{M}; d\mu\}$ be a mechanical system subject only to its weight and constrained by smooth, fixed, holonomic constraints. Then the configurations of stable equilibrium are the isolated minima of the quote of the center of mass. Stationary points of the quote of the center of mass that are not minima are configurations of unstable equilibrium.*

4.3c Rigid Rod with Extremities on a Parabola

The extremities A and B of a material homogeneous rod are constrained to slide on the workless parabola $2hy = x^2$, for some given $h \in \mathbb{R}$.

The system has one degree of freedom and as Lagrangian parameter choose the abscissa q of the center of mass P_o of the rod. The coordinates (x_A, y_A) and (x_B, y_B) of the extremities of the rod satisfy the equation of the parabola, and the coordinates of the center of mass satisfy

$$2(x_{P_o}, y_{P_o}) = [(x_A + x_B), (y_A + y_B)] = 2(q, y(q)).$$

From these, we have

$$2hy(q) = q^2 + \frac{h^2\ell^2}{4(h^2 + q^2)}.$$

Assuming first that $h > 0$, prove that:

(i) If $h < \ell$, there are three equilibrium configurations. Of these, two are stable, whereas the one corresponding to $q = 0$ is unstable.
(ii) If $h \geq \ell$, then $q = 0$ is the only equilibrium configuration, and it is stable.

Assuming now that $h < 0$ prove that:

(i) If $-h < \ell$, there are three equilibrium configurations, of which only $q = 0$ is stable.
(ii) If $-h \geq \ell$, then $q = 0$ is the only equilibrium configuration, and it is unstable.

4.4c Miscellaneous Problems

Complete part **(c)** of the problem of §3.4c of the Complements of Chapter 5, by determining the nature of the equilibrium configurations.

Complete the problems of §3.4.4c and §3.4.5c of the Complements of Chapter 5, by determining the nature of the equilibrium configurations.

Complete the problem of §7c of the Complements of Chapter 6, by determining the nature of the equilibrium configurations.

Assume that the double pendulum of §4.5c of the Complements of Chapter 5 is subject only to its weight. Find all the equilibrium configurations, and prove that $\varphi = \theta = 0$ is the only configuration of stable equilibrium.

4.5c Material Rod and Point Mass $\{P; m\}$ Connected by a Spring

A material homogeneous rod of extremities O and B, mass M, and length 2ℓ moves with its extremity O constrained by a fixed cylindrical hinge. A point mass $\{P; m\}$ slides on the horizontal workless guide taken as the axis of abscissas. The second extremity B of the rod is connected to P by a spring of elasticity constant k. Determine the equilibrium configurations and their nature.

The system has two degrees of freedom, and as Lagrangian coordinates, take the abscissa q of P and the angle $\varphi = \widehat{BOP}$ formed by the rod with the positively oriented horizontal axis. The potential of the external forces is

$$V = \ell M g \sin\varphi - \tfrac{1}{2}k\left(q^2 - 4\ell q\cos\varphi\right) + \text{const},$$

and the equilibrium configurations are determined by setting

$$\nabla V = -kq + 2\ell k\cos\varphi, \quad \ell M g\cos\varphi - 2k\ell q\sin\varphi = 0.$$

Therefore the equilibrium configurations are the solutions of the system

$$q = 2\ell\cos\varphi, \qquad \cos\varphi(Mg - 4k\ell q\sin\varphi) = 0.$$

Apart from the trivial solution $(0, \pm\pi/2)$, the systems admits the configurations

$$\begin{gathered}\left(\frac{1}{2k}\sqrt{(4k\ell)^2 - (Mg)^2}, \arcsin\frac{Mg}{4k\ell}\right),\\ \left(\frac{-1}{2k}\sqrt{(4k\ell)^2 - (Mg)^2}, \pi - \arcsin\frac{Mg}{4k\ell}\right),\end{gathered} \tag{4.2c}$$

provided $(Mg/4k\ell) \le 1$. The Hessian of V is

$$\begin{aligned}\text{Hess}(V) &= \det\begin{pmatrix} -k & -2k\ell\sin\varphi \\ -2k\ell\sin\varphi & -\ell M g\sin\varphi - 2k\ell q\cos\varphi\end{pmatrix}\\ &= k\ell\left(Mg\sin\varphi + 2kq\cos\varphi\right) - 4k^2\ell^2\sin^2\varphi.\end{aligned}$$

From this one computes

$$\text{Hess}(V)\big|_{0,\pm\pi/2} = \pm k\ell\,(Mg \mp 4k\ell)\,, \quad V_{qq} = -k < 0.$$

Therefore $(0, -\pi/2)$ is a configuration of unstable equilibrium, whereas $(0, \pi/2)$ is stable if $Mg > 4k\ell$ and unstable if $Mg < 4k\ell$. In this latter case also the two expressions of (4.2c) are equilibrium configurations. Computing the Hessian of V in each of them gives the same value

$$4k^2\ell^2 - \tfrac{1}{4}M^2g^2 > 0.$$

Thus if $Mg < 4k\ell$, the configurations (4.2c) are of stable equilibrium. Finally, is $Mg = 4k\ell$, one recovers the equilibrium configurations $\varphi = \pm\pi/2$.

5c Stability and Instability of Poinsot Precessions

5.1c Instability of Rotations about the Intermediate Axis

Let the moments of inertia $\mathcal{I}_i$ and the semiaxes a_i of the ellipsoid of inertia be ordered as

$$\mathcal{I}_1 < \mathcal{I}_2 < \mathcal{I}_3, \qquad a_1 > a_2 > a_3, \qquad a_i = \frac{\lambda}{\sqrt{\mathcal{I}_i}}.$$

Consider a Poinsot precession with initial datum $\boldsymbol{\omega}_o$ close to the intermediate axis, say, for example,

$$\omega_{o,1}^2 + (a_2 - \omega_{o,2})^2 + \omega_{o,3}^3 \le \varepsilon^2, \qquad 0 < \varepsilon \ll a_2.$$

Compute $\mathbf{I}$ from (3.3) and (4.1) of Chapter 7,

$$\mathbf{I} = \left(\frac{\lambda}{h}\right)^2 = \frac{\mathcal{I}_i^2\omega_i^2}{\mathcal{I}_i\omega_i^2} = \frac{\mathcal{I}_i^2\omega_{o,i}^2}{\mathcal{I}_i\omega_{o,i}^2},$$

and observe that $\mathbf{I} < \mathcal{I}_3$. There are three possible cases:

$$\mathcal{I}_1 < \mathbf{I} < \mathcal{I}_2 < \mathcal{I}_3, \qquad \mathcal{I}_1 < \mathcal{I}_2 < \mathbf{I} < \mathcal{I}_3, \qquad \mathcal{I}_1 < \mathcal{I}_2 = \mathbf{I} < \mathcal{I}_3.$$

In the first case, the polhode, i.e., the trace of $\boldsymbol{\omega}$ on $\mathcal{E}_\lambda$, is a curve surrounding the major semiaxis a_1. In the second case, the polhode is a closed curve traced on $\mathcal{E}_\lambda$ and surrounding the minor semiaxis. This follows from the classification of the polhodes in §3.2.1 of Chapter 7. In either case $\boldsymbol{\omega}$ departs from its initial configuration $\boldsymbol{\omega}_o$, along its polhode, thereby generating instability. Using the integration procedure of §4 of Chapter 7, prove that $\boldsymbol{\omega}$ covers the entire polhode, with a period that can be computed only in terms of $\mathcal{I}_i$ and $\mathbf{I}$. In the third case, the polhode is the intersection of $\mathcal{E}_\lambda$ with the degenerate Poinsot cones. These intersections consist of two ellipses each through the points $(0, \pm a_2, 0)$. This follows from §3.2.3 of Chapter 7. The vector $\boldsymbol{\omega}$ departs from its initial configuration $\boldsymbol{\omega}_o$ along one of these ellipses.

Using the integration procedure of §4 of Chapter 7, prove that $\boldsymbol{\omega}$ covers the entire polhode in an infinite time. Thus $\boldsymbol{\omega}$ moves very slowly along its polhode. For this reason the instability of such a motion is essentially undetectable, and for small times the motion appears to be stable.

5.2c Ellipsoids of Rotation

Let $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}$ and take an initial datum close to any one of the equatorial axes, say, for example,

$$(\omega_{o,1} - a\cos\theta)^2 + (\omega_{o,2} - a\sin\theta)^2 + \omega_{o,3}^2 \le \varepsilon^2, \qquad a = \frac{\lambda}{\sqrt{\mathcal{I}}}.$$

Computing $\mathbf{I}$, one has $\mathcal{I}_1 = \mathcal{I}_2 < \mathbf{I} < \mathcal{I}_3$. By Proposition 3.1 of Chapter 7, the polhodes are circles traced on $\mathcal{E}_\lambda$, parallel to the equatorial circle and surrounding the gyroscopic axis. Thus the configuration of $\boldsymbol{\omega}$ on an equatorial axis is unstable. Using the integration procedure of §4 of Chapter 7, prove that $\boldsymbol{\omega}$ covers the entire polhode, with a period that is inversely proportional to the number ε. Thus $\boldsymbol{\omega}$ moves very slowly along its polhode, and the motion appears to be stable. For $\varepsilon = 0$, the initial datum $\boldsymbol{\omega}_o$ lies on an equatorial axis; such an axis, remains fixed, and the resulting rotation is stable. Thus, roughly speaking, as $\varepsilon \to 0$, the *slow* instability tends to stability.

Finally, if $\mathcal{E}_\lambda$ is a sphere, all rotations are permanent, and every configuration is of stable equilibrium.

6c Small Oscillations

6.1c Small Oscillations of the System in §4.5c

Assume $Mg > 4k\ell$ and study the small oscillations about the equilibrium configuration $(0, \pi/2)$. One computes

$$\begin{pmatrix} V_{qq} & V_{q\varphi} \\ V_{q\varphi} & V_{\varphi\varphi} \end{pmatrix} \Bigg|_{(q,\varphi)=(0,\pi/2)} = -\begin{pmatrix} k & 2k\ell \\ 2k\ell & \ell Mg \end{pmatrix}.$$

Therefore for (q, φ) near $(0, \pi/2)$,

$$V(q,\varphi) = -(q,\varphi)\begin{pmatrix} k & 2k\ell \\ 2k\ell & \ell Mg \end{pmatrix}\begin{pmatrix} q \\ \varphi \end{pmatrix} + O\left(|(q,\varphi),(\dot q,\dot\varphi)|^3\right).$$

One also computes

$$2T = m\dot q^2 + \tfrac{4}{3}Mg\ell^2\dot\varphi^2 = (\dot q, \dot\varphi)\begin{pmatrix} m & 0 \\ 0 & \tfrac{4}{3}Mg\ell^2 \end{pmatrix}\begin{pmatrix} \dot q \\ \dot\varphi \end{pmatrix}.$$

Determine the normal coordinates and the normal modes about the configuration of stable equilibrium $(0, \pi/2)$.

6.2c Computing the Normal Frequencies

The following method provides an efficient procedure to compute the normal modes ω_h independently of the normal coordinates. Linearize the Lagrangian and write down the linearized equations

$$A_{hk}\ddot{q}_k + B_{hk}q_k = 0, \qquad A_{kh} = A_{hk}(0), \quad B_{hk} = V_{q_h q_k}(0).$$

Motivated by (7.2), we look for solutions of the type

$$q_h = C_h \cos(\omega t + \varphi_h), \qquad h = 1, 2, \ldots, N,$$

where C_h and φ_h are constants to be determined. Putting this in the linearized Lagrange equations gives

$$A_{hk}C_k\omega^2 - B_{hk}C_k = 0, \qquad h = 1, 2, \ldots, N.$$

This is a homogeneous linear algebraic system in the unknowns C_k, which has a nontrivial solution $\mathbf{C} = (C_1, C_2, \ldots, C_N)^t$ if the determinant of the coefficient is zero, i.e., if

$$\det\left(A_{hk}\omega^2 - B_{hk}\right) = 0.$$

The latter in an algebraic equation of degree N that admits N solutions, real or complex, counted with their multiplicities.

Prove that if the origin is a configuration of stable equilibrium, these roots are precisely the normal modes of the system.

6.3c Double Pendulum in §4.5c of the Complements of Chapter 5

Kinetic energy T and potential V are computed as

$$\begin{aligned} 2T &= \ell^2(M + \tfrac{1}{3}m)\dot{\varphi}^2 + \tfrac{1}{3}ML^2\dot{\theta}^2 + M\ell L\dot{\varphi}\dot{\theta}\cos(\theta - \varphi), \\ V &= \tfrac{1}{2}mg\ell\cos\varphi + (\ell\cos\varphi + \tfrac{1}{2}L\cos\theta)Mg + \text{const.} \end{aligned}$$

Linearizing these about the configuration of stable equilibrium $\varphi = \theta = 0$ gives

$$2T_o = \left(\dot{\varphi}\ \dot{\theta}\right) \begin{pmatrix} \ell^2\left(M + \frac{1}{3}m\right) & \frac{1}{2}M\ell L \\ \frac{1}{2}M\ell L & \frac{1}{3}ML^2 \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \end{pmatrix},$$

$$2V_o = -g\left(\varphi\ \theta\right) \begin{pmatrix} \ell(M + \frac{1}{2}m) & 0 \\ 0 & \frac{1}{2}LM \end{pmatrix} \begin{pmatrix} \varphi \\ \theta \end{pmatrix}.$$

Therefore the linearized Lagrange equations are

$$\begin{pmatrix} \ell^2(M+\frac{1}{3}m) & \frac{1}{2}M\ell L \\ \frac{1}{2}M\ell L & \frac{1}{3}ML^2 \end{pmatrix} \begin{pmatrix} \ddot{\varphi} \\ \ddot{\theta} \end{pmatrix} + g \begin{pmatrix} \ell\left(M+\frac{1}{2}m\right) & 0 \\ 0 & \frac{1}{2}LM \end{pmatrix} \begin{pmatrix} \varphi \\ \theta \end{pmatrix} = 0.$$

Find the normal coordinates and the normal modes in the case that $L = \ell$ and hence $M = m$.

7c Degenerate Vibrations

We have assumed that the configuration $\{m; P_{1,o}\}$, $\{m; P_{2,o}\}$, $\{M; 0\}$ is of stable equilibrium. We will derive a condition on the masses m and M that would guarantee such a stability independently of the elastic constant k. Eliminating the variable ζ from (10.2), one computes the kinetic energy T and the potential V only in terms of ξ and η:

$$\begin{aligned} \frac{2}{m}T &= (1+\mu^2)(\dot{\xi}^2+\dot{\eta}^2) + 2\mu^2\dot{\xi}\cdot\dot{\eta}, \\ \frac{2}{k}V &= -(1+2\mu^2)(\xi^2+\eta^2) - 4\mu^2(1+\mu^2)\xi\cdot\eta. \end{aligned}$$

In matrix form,

$$\frac{2}{m}T = \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix}^t (\mathcal{A}_{hk}) \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix}, \qquad -\frac{2}{k}V = \begin{pmatrix} \xi \\ \eta \end{pmatrix}^t (\mathcal{B}_{hk}) \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

where the matrices $(\mathcal{A}_{hk})$ and $(\mathcal{B}_{hk})$ are given by

$$\begin{aligned} (\mathcal{A}_{hk}) &= \begin{pmatrix} (1+\mu^2)\mathbb{I} & \mu^2\mathbb{I} \\ \mu^2\mathbb{I} & (1+\mu^2)\mathbb{I} \end{pmatrix}, \\ (\mathcal{B}_{hk}) &= \begin{pmatrix} (1+2\mu^4)\mathbb{I} & 2\mu^2(1+\mu^2)\mathbb{I} \\ 2\mu^2(1+\mu^2)\mathbb{I} & (1+2\mu^4)\mathbb{I} \end{pmatrix}. \end{aligned}$$

The indicated equilibrium configuration is stable if $(\mathcal{B}_{hk})$ is positive definite, i.e., if for all $(\xi, \eta)^t \in \mathbb{R}^6 - \{0\}$,

$$\begin{aligned} \begin{pmatrix} \xi \\ \eta \end{pmatrix}^t (\mathcal{B}_{hk}) \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= (1+2\mu^4)(\xi^2+\eta^2) + 4\mu^2(1+\mu^2)\xi\cdot\eta \\ &= 2\mu^2(1+\mu^2)(\xi+\eta)^2 + (1-2\mu^2)(\xi^2+\eta^2) > 0. \end{aligned}$$

The condition is then $2\mu^2 < 1$, i.e., $M > 2m$.

9

VARIATIONAL PRINCIPLES

1 Maxima and Minima of Functionals

Given two points q_o and q_1 in $\mathbb{R}^N$ and an interval $[t_o, t_1] \subset \mathbb{R}$, consider the convex set $\mathcal{K}$ of all smooth curves parameterized with $t \in [t_o, t_1]$, and of extremities q_o and q_1, i.e.,

$$\mathcal{K} = \left\{ q \in C^1[t_o, t_1] \;\middle|\; q(t_o) = q_o,\; q(t_1) = q_1 \right\}.$$

Given $q \in \mathcal{K}$ and a vector-valued function $\varphi \in C_o^\infty(t_o, t_1)$, the curve $\{q + \lambda\varphi\}$ is still in $\mathcal{K}$ for all $\lambda \in \mathbb{R}$. More generally, any modified path $q + \delta q$ of a curve $q \in \mathcal{K}$ remains in $\mathcal{K}$ if δq is smooth and $\delta q(t_o) = \delta q(t_1) = 0$. In such a case δq is called a *synchronous variation* of the curve $q \in \mathcal{K}$.

If $q \in \mathcal{K}$ is the orbit of a mechanical system in configuration space, a synchronous variation δq might be regarded as a virtual variation from the Lagrangian path q into the virtual Lagrangian path $q + \delta q$ that keeps its extremities fixed. Given a function $(\xi, \eta; t) \to F(\xi, \eta; t) \in C^2\left(\mathbb{R}^{2N+1}\right)$, consider the *functional*

$$\mathcal{K} \ni q \longrightarrow J(q) = \int_{t_o}^{t_1} F\big[q(t), \dot{q}(t); t\big]\, dt.$$

For each $q \in \mathcal{K}$, the functional returns a real number. We ask whether $J(q)$ has a greatest lower bound as q ranges over $\mathcal{K}$, i.e., if

$$\inf_{q \in \mathcal{K}} \int_{t_o}^{t_1} F\big[q(t), \dot{q}(t); t\big]\, dt > -\infty.$$

The infimum might not exist, or if it does, it might not be achieved by any $q \in \mathcal{K}$ (see Problems 1.1c and 1.2c of the Complements). If the infimum exists and it is achieved by some $q \in \mathcal{K}$, such an orbit is called a *minimum point* for J. Similarly one might ask whether J has a least upper bound. If such a supremum exists and it is achieved for some $q \in \mathcal{K}$, the orbit q is a *maximum point* for the functional J. If q is a minimum point for J, for any $\varphi \in C_o^\infty(t_o, t_1)$

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the function of one real variable $\lambda \to J(q+\lambda\varphi)$ has a minimum for $\lambda = 0$. Therefore

$$\frac{d}{d\lambda} J(q+\lambda\varphi)\Big|_{\lambda=0} = 0 \qquad \forall \varphi \in C_o^\infty(t_o, t_1). \tag{1.1}$$

The same holds if q is a maximum point for J. However, if (1.1) holds for some $q \in \mathcal{K}$, such an orbit need not be a maximum or a minimum point (see Problem 1.1c of the Complements).

1.1 Stationary Points of Functionals

If $q \in \mathcal{K}$ undergoes a synchronous variation δq, its derivative $\dot{q}$ undergoes the variation $\delta\dot{q}$. If δq and $\delta\dot{q}$ are both infinitesimal, we compute the corresponding variation of $F(q, \dot{q}; t)$ as

$$\begin{aligned}\delta F\big(q, \dot{q}; t\big) &= F\big(q + \delta q, \dot{q} + \delta\dot{q}; t\big) - F\big(q, \dot{q}; t\big)\\ &= \frac{\partial F(q, \dot{q}; t)}{\partial q_h}\delta q_h + \frac{\partial F(q, \dot{q}; t)}{\partial \dot{q}_h}\delta\dot{q}_h\\ &= -\left(\frac{d}{dt}\frac{\partial F}{\partial \dot{q}_h} - \frac{\partial F}{\partial q_h}\right)\delta q_h + \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_h}\delta q_h\right).\end{aligned}$$

The corresponding variation of the functional J is

$$\begin{aligned}\delta J(q) &= \int_{t_o}^{t_1} \big[F\big(q + \delta q, \dot{q} + \delta\dot{q}; t\big) - F\big(q, \dot{q}; t\big]\big)\, dt\\ &= \int_{t_o}^{t_1} \delta F\big(q, \dot{q}; t\big) dt\\ &= \int_{t_o}^{t_1} \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_h}\delta q_h\right) dt - \int_{t_o}^{t_1}\left(\frac{d}{dt}\frac{\partial F}{\partial \dot{q}_h} - \frac{\partial F}{\partial q_h}\right)\delta q_h dt\\ &= \left(\frac{\partial F}{\partial \dot{q}_h}\delta q_h\right)\Big|_{t_o}^{t_1} - \int_{t_o}^{t_1}\left(\frac{d}{dt}\frac{\partial F}{\partial \dot{q}_h} - \frac{\partial F}{\partial q_h}\right)\delta q_h dt.\end{aligned} \tag{1.2}$$

Since $\delta q(t_o) = \delta q(t_1) = 0$, we finally compute

$$\delta J(q) = -\int_{t_o}^{t_1}\left(\frac{d}{dt}\frac{\partial F}{\partial \dot{q}_h} - \frac{\partial F}{\partial q_h}\right)\delta q_h dt. \tag{1.3}$$

Whenever the synchronous variation δq is of the type $\delta q = \varphi d\lambda$, for some $\varphi \in C_o^\infty(t_o, t_1)$, the variation of J in (1.3) takes the form

$$\delta J(q) = -d\lambda \int_{t_o}^{t_1}\left(\frac{d}{dt}\frac{\partial F}{\partial \dot{q}_h} - \frac{\partial F}{\partial q_h}\right)\varphi_h dt. \tag{1.3$'$}$$

Remark 1.1 The differential in (1.3) is, in general, topologically different from the differential in (1.3)′. The latter is called a *Gâteaux differential.* Having fixed $q \in \mathcal{K}$ and $\varphi \in C_o^\infty(t_o, t_1)$, such a differential is computed along the orbits $\lambda \to q + \lambda\varphi$. These can be regarded as “lines” in $\mathcal{K}$ through q and with fixed “slope” φ. The differential appearing in (1.3) is the *Fréchet differential.* For $q \in \mathcal{K}$ fixed, such a differential is computed along any “path” $q + \delta q \in \mathcal{K}$. In what follows we will not distinguish them further, since for smooth $F(\xi, \eta; t)$, these two notions of differential coincide [4].

Having (1.1)–(1.3)′ as a guideline, we say that $q \in \mathcal{K}$ is a *stationary point* for the functional J if $\delta J(q) = 0$. Extremum points, if any, are stationary points. Indeed, if $q \in \mathcal{K}$ is a minimum point, (1.2) implies that $\delta J(q) \geq 0$ for all $\delta\hat{q}$. Writing (1.2) for δq and $-\delta q$ gives $\pm\delta J(q) \geq 0$, and thus $\delta J(q) = 0$.

The next proposition provides a characteristic condition for $q \in \mathcal{K}$ to be a stationary point for J.

Proposition 1.1 *A curve $q \in \mathcal{K}$ is a stationary point for the functional J if and only if it is a solution of the system of differential equations*

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{q}_h} - \frac{\partial F}{\partial q_h} = 0, \qquad h = 1, 2, \ldots, N. \tag{1.4}$$

Proof. If $q \in \mathcal{K}$ solves (1.4), it follows from (1.3)–(1.3)′ that it is a stationary point for J. Conversely, if $q \in \mathcal{K}$ is a stationary point for J, then (1.3)′ gives

$$\int_{t_o}^{t_1} \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{q}_h} - \frac{\partial F}{\partial q_h} \right) \varphi_h dt = 0 \qquad \forall \varphi \in C_o^\infty(t_o, t_1). \qquad \blacksquare$$

By virtue of (1.4), stationary points for J are only those that satisfy a system of differential equations similar to the Lagrange equations. The characteristic condition (1.4) does not contain sufficient information to determine the nature of a stationary point. Conditions that would identify a stationary point as an extremum involve the second variation of J and thus the structure of the function F [144], [12, Chapter II, 37–64]. Likewise, (1.4) does not contain sufficient information for one to infer uniqueness of stationary points.

2 The Least Action Principle

The motion $t \to q(t)$ of a mechanical system subject to smooth, holonomic constraints satisfying the principle of virtual work and acted upon only by conservative forces is determined by the Lagrange equations through the Lagrangian $\mathcal{L}$, starting from some given initial conditions. Fix a time interval (t_o, t_1) along the time evolution of the motion and set $q(t_o) = q_o$ and $q(t_1) = q_1$. Then for $t \in (t_o, t_1)$, the orbit $t \to q(t)$ might also be regarded as a solution of the two-point boundary value problem (see §2.1c of the Complements)

$$\begin{aligned} &\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_h} - \frac{\partial \mathcal{L}}{\partial q_h} = 0, \quad h = 1, \dots, N, \\ &q(t_o) = q_o, \quad q(t_1) = q_1. \end{aligned} \tag{2.1}$$

Having determined the Lagrangian $\mathcal{L}$ of a mechanical system, the functional

$$\mathcal{K} \ni q \to S(q) = \int_{t_o}^{t_1} \mathcal{L}(q, \dot{q}; t) dt \tag{2.2}$$

is the *action* or the *Hamiltonian action* of the system. The variational principle of Proposition 1.1 in this context takes the following form.

Theorem 2.1 (Hamilton Principle of Stationary Action). *The motion of a mechanical system subject to smooth, holonomic constraints satisfying the principle of virtual work and acted upon only by conservative forces evolves along a Lagrangian trajectory $q \in \mathcal{K}$ in configuration space, which is a stationary point for the Hamiltonian action.*

Remark 2.1 It can be shown that stationary points of the Hamiltonian action cannot be maxima (§2.2c of the Complements). Moreover, while solutions of (2.1) need not be a minimum for the action $q \to S(q)$, they are always minima in some "local sense"(§2.3c of the Complements). For this reason the Hamilton principle of *stationary action* is commonly referred to as the *principle of least action.*[1]

2.1 Reduced Action and Maupertuis Principle

If the constraints are fixed, the Lagrangian, and hence the Hamiltonian, are independent of time, and $\mathcal{H}(p, q) = \gamma$, for some constant γ, is an integral of motion. Rewrite the action as

$$\begin{aligned} S(q) &= \int_{t_o}^{t_1} \big[p_h \dot{q}_h - \mathcal{H}(q, p)\big] d\tau \\ &= \int_{q_o}^{q_1} p_h dq_h - \int_{t_o}^{t_1} \mathcal{H}(q, p) d\tau. \end{aligned}$$

[1] A first form of such a principle was introduced by Hamilton in [69]. While motivated by the principle of *path of least time* of geometrical optics (see §4), Hamilton perceived its general breadth. From his introduction: *... A certain quantity which in one physical theory is the "action" and in another "time," expended by light in going from any first to any second point, is found to be less than if the light had gone in any other than its actual path ... The mathematical novelty of my method consists in considering this quantity as a function... and in reducing all researches respecting optical systems of rays to the study of this single function: a reduction which presents mathematical optics under an entirely novel view, and one analogous (as it appears to me) to the aspect under which Descartes presented the application of Algebra to Geometry....* Applications to mechanics are in [72, 73].

In these integrals the parameter t need not be time, and it could be replaced by any other parameter $s \in [s_o, s_1]$, provided $q(s_o) = q_o$ and $q(s_1) = q_1$. Extrema of $S(q)$ could then be found for curves $s \to q(s)$ in configuration space, expressed in terms of such a new parameter. The energy integral $\mathcal{H} = \gamma$ suggests looking for the stationary points of the action among those curves in phase space, parameterized to enforce the energy integral. Set

$$\mathcal{K}_\gamma = \left\{ \begin{array}{c} [s_o, s_1] \ni s \to \big(q(s), p(s)\big) \in C^1[s_o, s_1] \text{ such that} \\ q(s_o) = q_o,\ q(s_1) = q_1,\ \text{and } \mathcal{H}(q,p) = \gamma \end{array} \right\}.$$

For such curves in phase space,

$$S(q) = \int_{q_o}^{q_1} p_h dq_h - \gamma(t_1 - t_o).$$

The integral on the right-hand side is the *reduced action* and is denoted by S_o. Since stationary points of $S(q)$ yield a curve in $\mathcal{K}_\gamma$, they can be found as the stationary points of the reduced action in $\mathcal{K}_\gamma$. Thus, for fixed constraints, the motion can be determined by

$$\delta_\gamma S_o = \delta_\gamma \int_{q_o}^{q_1} p_h dq_h = 0, \tag{2.3}$$

where δ_γ denotes the variation effected in $\mathcal{K}_\gamma$. This is a special case of the Hamilton principle of stationary action, and is referred to as the *Maupertuis principle*. It was visualized by Maupertuis in 1744 [119, Vol. II, n° 328] and mathematically formalized by Euler [53] and Lagrange [98].

The variational principle in (2.3) is of purely geometrical nature, since it involves curves in $\mathcal{K}_\gamma$ irrespective of their parameterization, provided $\mathcal{H} = \text{const}$. As such it determines the geometrical form of the trajectories irrespective of the time-law on them, as illustrated by the following example.

2.1.1 Motion along Geodesics

A point moves on a fixed smooth surface $\mathcal{S}$ and is unsolicited and unconstrained otherwise. Then denoting by ds its elemental arc length, we obtain[2]

$$\frac{1}{2}\left(\frac{ds}{dt}\right)^2 = T = \mathcal{L} = \mathcal{H} \quad \text{and} \quad p_h \dot{q}_h = \frac{\partial T}{\partial \dot{q}_h}\dot{q}_h = 2T.$$

The reduced action takes the form

$$S_o = \int_{t_o}^{t_1} p_h \dot{q}_h dt = \int_{t_o}^{t_1} \sqrt{2T}\left(\frac{ds}{dt}\right) dt = \sqrt{2\gamma}\int_{q_o}^{q_1} ds,$$

whose stationary points are the geodesics on $\mathcal{S}$.

[2] Combine the remarks of §1.4c with those in §§3.2 and 4 of Chapter 6.

3 The Hamilton–Jacobi Equation

If in (2.2) the second limit of integration $(q_1; t_1)$ is regarded as variable $(q; t)$, the Hamiltonian action can be regarded as a function of $(q; t)$:

$$V(q;t) = \int_{t_o}^{t} \mathcal{L}(q, \dot{q}; \tau) d\tau = \int_{t_o}^{t} \left[p_h \dot{q}_h - \mathcal{H}(q, p; \tau) \right] d\tau.$$

For a fixed t consider a variation $q + \delta q$ from the path q that keeps the Lagrangian configuration $q(t_o)$ and velocity $\dot{q}(t_o)$ at time t_o and is otherwise unrestricted. Thus in particular, $\delta q(t_o) = \delta \dot{q}(t_o) = 0$, and no further restrictions are placed at time t. Computing the corresponding variation of $V(q; t)$ gives

$$\begin{aligned} \delta V(q;t) &= \delta \int_{t_o}^{t} \mathcal{L}(q, \dot{q}; \tau) d\tau \\ &= \int_{t_o}^{t} \left(\frac{\partial \mathcal{L}(q, \dot{q}; \tau)}{\partial q_h} \delta q_h + \frac{\partial \mathcal{L}(q, \dot{q}; \tau)}{\partial \dot{q}_h} \delta \dot{q}_h \right) d\tau \\ &= \frac{\partial \mathcal{L}(q, \dot{q}; \tau)}{\partial \dot{q}_h} \delta q_h \Bigg|_{t_o}^{t} - \int_{t_o}^{t} \left(\frac{d}{dt} \frac{\partial \mathcal{L}(q, \dot{q}; \tau)}{\partial \dot{q}_h} - \frac{\partial \mathcal{L}(q, \dot{q}; \tau)}{\partial q_h} \right) \delta q_h d\tau. \end{aligned}$$

Therefore if q is a solution of (2.1),

$$\delta V(q;t) = p_h \delta q_h.$$

Since the virtual variation δq is arbitrary,

$$\frac{\partial V(q;t)}{\partial q_h} = p_h, \qquad h = 1, \dots, N. \tag{3.1}$$

Starting from (3.1), we now compute the total derivative of $t \to V\big(q(t); t\big)$. This gives

$$\frac{d}{dt} V(q;t) = \frac{\partial V(q;t)}{\partial t} + \frac{\partial V(q;t)}{\partial q_h} \dot{q}_h = \mathcal{L}(q, \dot{q}; t).$$

From this and (3.1),

$$\frac{\partial V(q;t)}{\partial t} = \mathcal{L}(q, \dot{q}; t) - p_h \dot{q}_h = -\mathcal{H}(q, p; t).$$

Therefore the action $V(q; t)$ is a solution of the *Hamilton–Jacobi equation* [73, 90]

$$\frac{\partial V}{\partial t} + \mathcal{H}\left(q, \nabla_q V; t\right) = 0. \tag{3.2}$$

4 The Functional of Geometrical Optics

The Fermat principle states that a light ray travels between any two points q_o and q_1 of a medium along the path of least time [56, Vol. I, 170–172]. Parameterizing the trajectory q of the ray with either time t or arc length s, the instantaneous unit tangent along the direction of propagation is[3]

$$\boldsymbol{\tau}(s) = \mathbf{t}(t) = \frac{\dot{q}}{\|\dot{q}\|} = \frac{d}{ds}\, q(s).$$

The speed $\|\mathbf{v}\|(q, \mathbf{t})$ of propagation of a light ray is, in general, a function of the position q of the ray and its direction $\mathbf{t}$. For example, in crystals the speed of a light ray depends on its direction of propagation. The *refraction index* $\nu(q, t)$ is defined as

$$\nu(q, \mathbf{t}) \stackrel{\text{def}}{=} \frac{1}{\|\mathbf{v}\|(q, \mathbf{t})}.$$

A medium is *optically isotropic* if its refraction index is independent of $\mathbf{t}$; it is *optically homogeneous* if the refraction index is constant. The refraction index is a property of the medium; it gives the speed of a light ray when it goes instantaneously through the position q with direction $\mathbf{t}$.

With this symbolism, the infinitesimal time dt it takes the light ray to cover the infinitesimal distance ds is given by

$$dt = \nu(q, \boldsymbol{\tau}) ds = \nu(q, \mathbf{t}) \|\dot{q}\| dt.$$

If the trajectory $t \to q(t)$ of a light ray from q_o to q_1 is known, the time of travel is

$$T = \int_{t_o}^{t_1} \nu(q, \mathbf{t}) \|\dot{q}\| dt = \int_{q_o}^{q_1} \nu(q, \boldsymbol{\tau}) ds.$$

In the latter of these equalities the time dependence is eliminated and the integral depends only on the geometrical properties of the trajectory, through the refraction index. This suggests that we seek the trajectory of a light ray by expressing Fermat's principle of least time in the form

$$T = \inf_{q \in \mathcal{K}} \int_{t_o}^{t_1} \nu(q, \mathbf{t}) \|\dot{q}\| dt = \inf_{q \in \mathcal{K}(q_o, q_1)} \int_{q_o}^{q_1} \nu(q, \boldsymbol{\tau}) ds, \tag{4.1}$$

where $\mathcal{K}(q_o, q_1)$ is the set of all Smooth curves of endpoints q_o and q_1 parameterized with their intrinsic arc length. For optically homogeneous media the latter equality implies that the curve of least time is the line segment joining q_o to q_1. Since q_o and q_1 are arbitrary, this implies that in media with constant refraction index, light rays travel along straight lines. By Proposition 1.1 the

[3]With perhaps improper symbolism we have used the same symbol to denote $q(t)$ and $q(s)$, the instantaneous position of the ray, with different parametric representations.

trajectory of least time must be a solution of the system of the Euler–Lagrange equations[4]

$$\frac{d}{dt}\frac{\partial \nu(q,\mathbf{t})\|\dot{q}\|}{\partial \dot{q}_h} - \frac{\partial \nu(q,\mathbf{t})\|\dot{q}\|}{\partial q_h} = 0, \qquad h = 1, 2, \dots, N. \tag{4.2}$$

Moreover, recalling that the refraction index is the reciprocal of the speed of propagation of a light ray,

$$\nu(q,\mathbf{t})\|\dot{q}\| = 1 \quad \text{along trajectories solutions (4.2)}. \tag{4.3}$$

Equation (4.2)–(4.3) are $(N+1)$ relations to be satisfied by the N components of the trajectory. However, of these only N are independent, since

$$\dot{q} \to \nu(q,\mathbf{t})\|\dot{q}\| \quad \text{is homogeneous of degree 1 in the variable } \dot{q}. \tag{4.4}$$

In the next sections we investigate the structure of these systems, and we will put them in their canonical form.

5 Huygens Systems

Let $(\xi,\eta) \to F(\xi,\eta) \in C^2(\mathbb{R}^{2N})$ be homogeneous of degree 1 in the variables η, for all fixed ξ. By Euler's theorem on homogeneous functions (§1.1c of the Complements of Chapter 6),

$$F(\xi,\eta) = \frac{\partial F(\xi,\eta)}{\partial \eta_h}\eta_h.$$

Taking the derivative with respect to the generic variable η_k gives

$$\frac{\partial^2 F(\xi,\eta)}{\partial \eta_h \partial \eta_k}\eta_h = 0, \qquad k = 1, \dots, N.$$

This can be regarded as an algebraic homogeneous linear system in the unknowns η_h. Since η is arbitrary, such a system admits nontrivial solutions. Therefore

$$\det\left(\frac{\partial^2 F(\xi,\eta)}{\partial \eta_h \partial \eta_k}\right) = 0.$$

In particular, the system (4.2) is not normal. Assume momentarily that the refraction index is independent of $\mathbf{t}$. Then one verifies that

$$\operatorname{rank}\left(\frac{\partial^2\, \nu(q)\,\|\dot{q}\|}{\partial \dot{q}_h \partial \dot{q}_k}\right) = N - 1.$$

[4]Although optical trajectories are curves in $\mathbb{R}^3$, we will regard them as in $\mathbb{R}^N$, to stress that the principles of geometrical optics are dimension-independent.

Therefore for isotropic media, (4.3) is the extra equation that makes the system normal. One also verifies that $\dot{q} \to [\nu(q)\|\dot{q}\|]^2$ is homogeneous of degree 2 and

$$\operatorname{rank}\left(\frac{\partial^2 \left(\nu(q)\|\dot{q}\|\right)^2}{\partial \dot{q}_h \partial \dot{q}_k}\right) = N.$$

In analogy with light propagation in isotropic media, consider systems of the type

$$\begin{aligned} &\frac{d}{dt}\frac{\partial F(q,\dot{q})}{\partial \dot{q}_h} - \frac{\partial F(q,\dot{q})}{\partial \dot{q}_h} = 0, \qquad h = 1,\dots,N, \\ &F\left(q(t),\dot{q}(t)\right) = 1, \end{aligned} \tag{5.1}$$

where $F(\xi,\eta)$ is homogeneous of degree 1 in η and

$$\operatorname{rank}\left(\frac{\partial^2 F(q,\dot{q})}{\partial \dot{q}_h \partial \dot{q}_k}\right) = N-1, \qquad \operatorname{rank}\left(\frac{\partial^2 F^2(q,\dot{q})}{\partial \dot{q}_h \partial \dot{q}_k}\right) = N. \tag{5.2}$$

Systems of the type (5.1) in which F satisfies (5.2) are called *Huygens systems* or systems of geometrical optics.[5]

Proposition 5.1 *A smooth trajectory $t \to q(t)$ is a solution of the Huygens system (5.1)–(5.2) if and only if is a solution of the normal system*

$$\frac{d}{dt}\frac{\partial \frac{1}{2}F^2(q,\dot{q})}{\partial \dot{q}_h} - \frac{\partial \frac{1}{2}F^2(q,\dot{q})}{\partial q_h} = 0, \qquad h = 1,\dots,N. \tag{5.3}$$

Proof. Let q be a solution of (5.1). Multiplying the first equation by $F(q,\dot{q})$ gives

$$\frac{d}{dt}\frac{\partial \frac{1}{2}F^2(q,\dot{q})}{\partial \dot{q}_h} - \frac{\partial \frac{1}{2}F^2(q,\dot{q})}{\partial q_h} = \frac{\partial F(q,\dot{q})}{\partial \dot{q}_h}\frac{d}{dt}F(q,\dot{q}).$$

Conversely, if q solves (5.3), then

$$F(q,\dot{q})\left(\frac{d}{dt}\frac{\partial F(q,\dot{q})}{\partial \dot{q}_h} - \frac{\partial F(q,\dot{q})}{\partial \dot{q}_h}\right) = \frac{\partial F(q,\dot{q})}{\partial \dot{q}_h}\frac{d}{dt}F(q,\dot{q}).$$

Therefore the converse will follow if $F(q,\dot{q}) = 1$ is an integral of (5.3). This, in turn, is a consequence of the canonical form of (5.3). Such a system is normal and has the structure of the Lagrange equations for the "Lagrangian"

$$\widetilde{\mathcal{L}}(q,\dot{q}) = \tfrac{1}{2}F^2(q,\dot{q})$$

[5]The property of $\eta \to F(\eta)$ being homogeneous of degree 1 does not imply any restriction on the rank of the Hessian of F. Give examples of homogeneous functions of degree 1 whose Hessian has rank $1, 2, \dots, (N-1)$.

whose Hessian with respect to the variables $\dot{q}$ has maximum rank. Then, introducing the "kinetic momenta"

$$p_h = \frac{\partial \frac{1}{2} F^2(q,\dot{q})}{\partial \dot{q}_h}, \qquad h = 1, \dots, N,$$

the pair of variables (q,p) and the Hamiltonian

$$\widetilde{\mathcal{H}}(q,p) = p_h \dot{q}_h - \tfrac{1}{2} F^2(q,\dot{q}) \tag{5.4}$$

satisfy the canonical system

$$\dot{q}_h = \frac{\partial \widetilde{\mathcal{H}}(q,p)}{\partial p_h}, \qquad \dot{p}_h = -\frac{\partial \widetilde{\mathcal{H}}(q,p)}{\partial q_h}, \qquad h = 1, \dots, N. \tag{5.5}$$

Since $\dot{q} \to F^2(q,\dot{q})$ is homogeneous of degree 2, by Euler's theorem we have

$$p_h \dot{q}_h = \frac{\partial \frac{1}{2} F^2(q,\dot{q})}{\partial \dot{q}_h}\, \dot{q}_h = F^2(q,\dot{q}).$$

From this and (5.4),

$$\widetilde{\mathcal{H}}(q,p) = \tfrac{1}{2} F^2(q,\dot{q}) = \tfrac{1}{2} p_h \dot{q}_h. \tag*{(5.4)$'$}$$

Therefore the Hamiltonian $\widetilde{\mathcal{H}}(q,p)$ coincides with the Lagrangian $\frac{1}{2}F^2(q,\dot{q})$, whenever in the latter $\dot{q}$ is expressed in terms of the new independent variables (q,p). Now, systems of the type (5.5), with $\widetilde{\mathcal{H}}$ independent of time, have the "energy integral" (§5.2 of Chapter 6)

$$\widetilde{\mathcal{H}}(q,p) = \tfrac{1}{2} F^2(q,\dot{q}) = \text{const.}$$

■

6 Canonical Form of Huygens Systems

Even though the system (5.1) is not normal, the previous remarks permit one to put it in canonical form. From (5.4) and (5.4)$'$,

$$2\widetilde{\mathcal{H}}(q,p) = \frac{\partial \widetilde{\mathcal{H}}(q,p)}{\partial p_h}\, p_h.$$

Therefore by Euler's theorem and its converse, $\widetilde{\mathcal{H}}(q,p)$ is, for all fixed $q \in \mathbb{R}^N$, homogeneous of degree 2 in the variables p. Write now the canonical Hamiltonian system (5.5) and the Hamiltonian (5.4), computed along the curve $t \to q(t)$, a solution of the Lagrangian system (5.1). Setting

$$\mathcal{H}(q,p) = \sqrt{2\widetilde{\mathcal{H}}(q,p)} = F(q,\dot{q})$$

and taking into account (5.4)′ gives

$$\mathcal{H}(q,p) = \sqrt{2\widetilde{\mathcal{H}}(q,p)} = 1 \tag{6.1}$$

and

$$\dot{q}_h = \frac{\partial \mathcal{H}(q,p)}{\partial p_h}, \qquad \dot{p}_h = -\frac{\partial \mathcal{H}(q,p)}{\partial q_h}, \qquad h = 1,\dots,N. \tag{6.2}$$

These is the canonical form of (5.1) with Hamiltonian $\mathcal{H}(q,p) = F(q,\dot{q})$, where $\dot{q}_h$ is regarded as expressed in terms of (q,p). Since the Hamiltonian $\widetilde{\mathcal{H}}$ is homogeneous of degree 2 in p, the new Hamiltonian $\mathcal{H}$ is homogeneous of degree 1 in p, and by Euler's theorem,

$$\mathcal{H}(q,p) = \frac{\partial \mathcal{H}(q,p)}{\partial p_h} p_h, \qquad \forall q \in \mathbb{R}^N. \tag{6.3}$$

Computing this along solutions of (6.1)–(6.2), we obtain

$$\mathcal{H}(q,p) = p_h \dot{q}_h \equiv 1. \tag{6.4}$$

Proposition 6.1 *Light rays travel along trajectories (q,p) in phase space, such that $p_h dq_h = dt$.*

6.1 Contact Virtual Differential Forms

The differential form $p_h \delta q_h$, where δ denotes a virtual variation, is called *of contact*. A remarkable property of these forms is that they are invariant along solutions of the canonical system (6.1)–(6.2).

Proposition 6.2 (Lie [114]) *Let (q,p) be a solution of (6.1)–(6.2). Then, as a function of time, $p\delta q = const.$*

Proof. Since the time derivative $\frac{d}{dt}$ and the virtual variation δ commute,

$$\begin{aligned}
\frac{d}{dt}(p_h \delta q_h) &= \dot{p}_h \delta q_h + p_h \delta \dot{q}_h \\
&= -\frac{\partial \mathcal{H}(q,p)}{\partial q_h}\delta q_h + p_h \delta \frac{\partial \mathcal{H}(q,p)}{\partial p_h} \\
&= -\frac{\partial \mathcal{H}}{\partial q_h}\delta q_h - \frac{\partial \mathcal{H}}{\partial p_h}\delta p_h + \delta\left(\frac{\partial \mathcal{H}(q,p)}{\partial p_h} p_h\right) \\
&= -\delta \mathcal{H} + \delta\left(\frac{\partial \mathcal{H}(q,p)}{\partial p_h} p_h\right).
\end{aligned}$$

The assertion follows, since $\mathcal{H}(q,p)$ is homogeneous of the degree 1 in p and (6.3) holds. ∎

7 Wave Fronts

For each pair (q, p) consider the elemental portion of the plane through q and normal p, i.e.,

$$d\pi(q,p) = \left\{\xi \in \mathbb{R}^N \;\middle|\; p\cdot(\xi - q) = 0;\ \|\xi - q\| < \|\delta q\|\right\},$$

where δq is an infinitesimal variation of q. The portion $d\pi(q,p)$ is an infinitesimal front normal to p. Consider now the canonical system (6.1)–(6.2) with initial conditions

$$q(t_o) = q_o \in \mathbb{R}^N \ \text{ fixed}, \qquad p(t_o) = p_o \in \mathbb{R}^N \ \text{ arbitrary}. \tag{7.1}$$

Thus the initial origin q_o of the light ray is specified, but its initial direction of propagation p_o is not restricted and it may vary in $\mathbb{R}^N$. Following the previous geometric interpretation of a pair $(p, q) \in \mathbb{R}^{2N}$, these initial data represent a bundle of $\infty^{(N-1)}$ elemental fronts, all through q_o. Since light propagates in all directions (in general with variable speed), studying the canonical system (6.1)–(6.2) with the initial conditions (7.1) amounts to investigating light propagation from a point source. Denote by $t \to \left[q(t; p_o), p(t; p_o)\right]$ the solution of (6.1)–(6.2) originating from the initial datum (q_o, p_o). The first quantity is the trajectory of the ray, and the second describes the evolution of an elemental front, normal to $p(t; p_o)$ at time t, originating from an elemental front, normal to p_o at time t_o.

7.1 First Definition of Wave Front

The wave front at time t of light originating from a point source in q_o is the geometric set

$$\Phi(q_o; t) = \bigcup_{p_o \in \mathbb{R}^N} q(t; p_o). \tag{7.2}$$

Proposition 7.1 *The wave front $\Phi(q_o; t)$ is a smooth $(N-1)$-dimensional surface, at least for t sufficiently close to t_o.*

Proof. Since $\mathcal{H}(q, p)$ is homogeneous of degree 1 in p, the first equation of (6.2) can be rewritten in the form (Proposition 1.1c of the Complements of Chapter 6)

$$\dot{q}_h = \frac{\partial \mathcal{H}(q, p/\|p\|)}{\partial p_h/\|p\|}, \qquad h = 1, \ldots, N. \tag{6.2$'$}$$

Set

$$\mathbf{n}(t) = \frac{p(t; p_o)}{\|p(t; p_o)\|}, \qquad \mathbf{n}_o = \frac{p_o}{\|p_o\|}.$$

As p_o ranges over $\mathbb{R}^N$, the unit vectors $\mathbf{n}_o$ range over the whole unit sphere $\mathbb{S}^{N-1}$, and $\mathbf{n}(t)$ ranges over a subset of $\mathbb{S}^{N-1}$. For t close to t_o, from (6.2)′ we have

$$q_h(t;\mathbf{n}) = q_{o,h} + \frac{\partial \mathcal{H}(q_o,\mathbf{n})}{\partial n_h}(t-t_o) + o_h(|t-t_o|;\mathbf{n}).$$

This is a smooth map from $\mathbb{S}^{N-1}$ into $\mathbb{R}^N$, and it represents a smooth manifold whose dimension is the rank of the matrix

$$\left(\frac{\partial q_h}{\partial n_k}\right) = \left(\frac{\partial^2 \mathcal{H}(q_o,\mathbf{n})}{\partial n_h \partial n_k}(t-t_o) + \frac{\partial o_h(|t-t_o|;\mathbf{n})}{\partial n_k}\right).$$

Since $\mathcal{H}(q,p) = F(q,\dot{q})$ and the Hessian of F with respect to $\dot{q}$ has rank $N-1$, also the Hessian of $\mathcal{H}$ with respect to $\mathbf{n}$ has rank $N-1$. Therefore, for t sufficiently close to t_o,

$$\operatorname{rank}\left(\frac{\partial q_h}{\partial n_k}\right) = \operatorname{rank}\left(\frac{\partial^2 \mathcal{H}(q_o,\mathbf{n})}{\partial n_h \partial n_k}\right) = N-1. \qquad \blacksquare$$

7.2 Second Definition of Wave Front

Each of the ∞^{N-1} elemental portions of planes $d\pi(q_o,p_o)$ is transformed, at time $t \neq t_o$, into the infinitesimal portion of plane $d\pi\big(q(t;p_o),p(t;p_o)\big)$. The envelope of these infinitesimal wave fronts, as p_o ranges over the unit sphere of $\mathbb{R}^N$, is the *wave front* of light generated at time t by a point source in q_o.

Such a front can be visualized as the “assembled motion” of the infinitesimal portions of planes of the bundle centered at q_o. They move from their initial position at time t_o and become reassembled, at each further time t, as infinitesimal tiles of a surface.

7.3 Equivalence of the Two Definitions

The surface $\Phi(q_o;t)$ might be regarded as a moving constraint for its own points. Therefore, the virtual displacements of a point $q \in \Phi(q_o;t)$ are those tangent to $\Phi(q_o;t)$. Since $\Phi(q_o;t_o) = q_o$, the only virtual displacements of q_o are zero. It follows from Proposition 6.2 that

$$p(t;p_o)\,\delta q(t;p_o) = p_o\,\delta q_o = 0$$

for every virtual displacement $\delta q(t;p_o)$. Thus, for all p_o, the vector $p(t;p_o)$ is normal to the tangent plane to the front $\Phi(q_o;t)$ at $q(t,p_o)$.

7.4 Velocity of the Light Ray and of the Wave Front

Let the function $F(q, \dot{q})$ appearing in the Huygens system (5.1)–(5.2), be given by $\nu(q; \mathbf{t})\|\dot{q}\|$. From this one computes the "kinetic momenta"

$$p_h = \frac{\partial \frac{1}{2} F^2(q, \dot{q})}{\partial \dot{q}_h} = \dot{q}_h \nu^2 + \|\dot{q}\| \nu \frac{\partial \nu}{\partial t_k} \left(\delta_{hk} - \frac{\dot{q}_h \dot{q}_k}{\|\dot{q}\|} \right). \tag{7.3}$$

In isotropic media, ν depends only on the position q, and in such a case, (7.3) becomes

$$\nu^2(q)\, \dot{q}_h = p_h, \qquad h = 1, \dots, N. \tag{7.3$'$}$$

Therefore in isotropic media, the Lagrangian velocity $\dot{q}$ of a light ray, along its trajectory q, is normal to the wave front to which q belongs.

In an isotropic media this is no longer the case, and the direction of propagation of the front $\Phi(q_o; t)$, differs from the direction of a light ray that penetrates it.

In isotropic and optically homogeneous media, the refraction index is a constant ν_o. In such a case the second equation of (6.2) implies that all components of the initial unit sphere $\|p_o\| = 1$ remain constant along the motion. Therefore by (7.3), every light ray travels along half-lines originating at q_o at constant speed ν_o^{-1}. Thus the wave front $\Phi(q_o; t)$ is a sphere centered at q_o and radius t/ν_o.

7.5 Normal Velocity and Normal "Slowness" of the Wave Front

The *normal velocity* of a wave front $\Phi(q_o; t)$ at one of its points q is the projection of the velocity of q along the normal to the front, i.e., $\dot{q}_h p_h / \|p\|$. Since this is computed along optical trajectories,

$$\dot{q}_h \frac{p_h}{\|p\|} = \frac{1}{\|p\|} \frac{\partial \mathcal{H}(q, p)}{\partial p_h} p_h = \frac{\mathcal{H}(q, p)}{\|p\|} = \frac{1}{\|p\|}. \tag{7.4}$$

Here we have taken into account that $\mathcal{H}(q, p) \equiv 1$ along optical trajectories. The same expression could be computed starting from (7.3), or alternatively from Proposition 6.2. It follows from (7.4) that the larger the value of $\|p\|$, the slower the front moves. For this reason, Hamilton called p the vector of *normal slowness* [71].

8 The Huygens Principle

Let $\Phi(q_o; t)$ be the wave front at time t, originating from a point source q_o. Every $q \in \Phi(q_o; t)$ might be considered itself as a point source, which generates its own front $\Phi(q; \tau)$ at time $\tau > 0$. The Huygens principle connects the front $\Phi(q_o; t + \tau)$ originating at q_o with the fronts $\Phi(q; \tau)$ as q ranges over $\Phi(q_o; t)$.

Theorem 8.1 (Huygens [83, 84]). *The front $\Phi(q_o; t+\tau)$ is the envelope of the fronts $\Phi(q;\tau)$, as q ranges over $\Phi(q_o; t)$.*

Proof. It will suffice to establish that for all $\xi \in \Phi(q_o; t+\tau)$, there exists some $q \in \Phi(q_o; t)$ such that the front $\Phi(q;\tau)$ is tangent to $\Phi(q_o; t+\tau)$ at ξ. Fix $\xi \in \Phi(q_o; t+\tau)$, and let q be the intersection of its trajectory of least time, with $\Phi(q_o; t)$. Such a trajectory, when restricted to its portion between q and ξ, is the trajectory of least time connecting these two points. This implies that $\xi \in \Phi(q;\tau)$. Moreover, the same trajectory defines in ξ the same "kinetic momentum" p. Therefore the two surfaces $\Phi(q_o; t+\tau)$ and $\Phi(q;\tau)$ have the same tangent plane at ξ. ∎

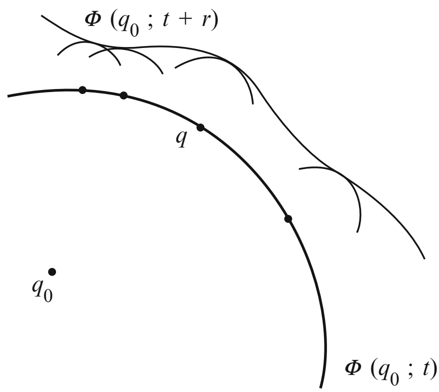

Fig. 8.1.

8.1 Optical Length

A similar argument, still based on the local nature of the Fermat principle, permits one to characterize further the wave fronts. For $q \in \mathbb{R}^N$ let $T(q_o; q)$ be the time it takes a light ray originating at q_o to reach q. The function $q \to T(q_o; q)$ is called *optical length* of the path from q_o to q. Now, all points $q \in \Phi(q_o; t)$ reach the wave front, starting from q_o, at the same time t. Therefore the wave front $\Phi(q_o; t)$ is a level set for the optical length, i.e.,

$$\Phi(q_o; t) = \left\{ q \in \mathbb{R}^N \;\middle|\; T(q_o; q) = t \right\}.$$

This implies that the gradient of $q \to T(q_o; q)$ is parallel to the "kinetic momentum" p. The next proposition asserts that indeed these two vectors coincide.

Proposition 8.1 $\nabla_q T(q_o; q) = p$.

Proof. Having fixed $q \in \Phi(q_o; t)$, let Δq be a variation of q and denote by Δt the corresponding variation of t, linked by

$$q + \Delta q \in \Phi(q_o; t + \Delta t), \qquad \Delta t = O(|\Delta q|).$$

By the Fermat principle, there exists a ray originating at $q \in \Phi(q_o; t)$ that reaches $\Phi(q_o; t + \Delta t)$ in the least time Δt.[6] By the local nature of such a principle, this ray must be the extension to the time interval $(t, t + \Delta t)$ of the ray that in the time interval (t_o, t) goes from q_o to q. For such a ray, by (6.4) the trajectory of such a ray, there must hold, by

$$\nu(q, \mathbf{t})\|\dot{q}\| = F(q, \dot{q}) = \mathcal{H}(q, p) = p_h \dot{q}_h, \qquad \mathbf{t} = \frac{\dot{q}}{\|\dot{q}\|}.$$

Therefore

$$\begin{aligned} T(q_o; q + \Delta q) - T(q_o; q) &= \int_t^{t+\Delta t} \nu(q, \mathbf{t})\|\dot{q}\| d\tau \\ &= \int_t^{t+\Delta t} p_h \dot{q}_h d\tau \\ &= p_h \Delta q_h + o(|\Delta q|). \end{aligned}$$

■

It follows from (7.4) that those surfaces defined intrinsically by

$$\|\nabla_q T(q_o; q)\| = \text{const} \tag{8.1}$$

are those wave fronts of constant normal velocity. Equation (8.1), introduced by Hamilton in [71], is called the *eikonal* equation of geometrical optics. Special solutions are functions whose level sets are spheres or hyperplanes.

Problems and Complements

1c Maxima and Minima of Functionals

1.1c A Functional with Stationary Points and No Extrema

Let $N = 1$ and consider the functional

$$J(q) = \int_0^1 q^3(t) dt, \qquad \mathcal{K} = \{q \in C^1[0,1] \mid q(0) = q(1) = 0\}.$$

Verify that $q \equiv 0$ is a stationary point for J and prove that J admits no maxima or minima in $\mathcal{K}$.

[6] It is not asserted that the front $\Phi(q_o; t + \Delta t)$ is reached at $q + \Delta q$, only that such a front is reached in the least time Δt.

1.2c A Functional with a Minimum That Is Not Achieved

Let $N = 1$ and consider the functional

$$J(q) = \int_0^1 \frac{\dot{q}^2(t)}{1+\dot{q}^2(t)} dt, \qquad \mathcal{K} = \{q \in C^1[0,1] \mid q(0) = 1,\, q(1) = 0\}.$$

Verify that $0 < J(q) \le 1$ for all $q \in \mathcal{K}$ and $\inf_{q \in \mathcal{K}} J(q) = 0$. However, such an infimum is not achieved in $\mathcal{K}$. One such q would have to satisfy

$$\frac{d}{dt} \frac{\dot{q}}{1+\dot{q}^2} = 0.$$

This implies $\dot{q} = \text{const}$, and therefore $q(t) = 1 - t$. However, $J(1-t) = \frac{1}{2}$.

Consider now the sequence of curves

$$t \to q_n(t) = \begin{cases} 1 - nt & t \in \left[0, \frac{1}{n}\right], \\ 0 & t \in \left[\frac{1}{n}, 1\right]. \end{cases}$$

Verify that $J(q_n) \to 0$ as $n \to \infty$. The functions q_n are not in $C^1[0,1]$. Modify their construction to get the same conclusions for functions $q_n \in \mathcal{K}$.

1.3c The Brachistochrone

In a vertical plane fix two points $P_o = (x_o, y_o)$ and $P_1 = (x_1, y_1)$ such that $x_o < x_1$ and $y_o > y_1$. Denote by $\mathcal{K}$ the set of all curves joining P_o and P_1 that can be parameterized with $x \in (x_o, x_1)$, i.e.,

$$\mathcal{K} = \left\{ y \in C^1[x_o, x_1], \,\middle|\, y(x_o) = y_o,\; y(x_1) = y_1 \right\}.$$

A point mass "falls" by gravity from P_o to P_1 along one of these curves. Denoting by s the arc length along one such curve,

$$ds = \dot{s}\, dt = \sqrt{\dot{x}^2 + \dot{y}^2} dt = \sqrt{1 + y'^2(x)}\, dx,$$

and the speed of the point is $v = \dot{s}$. Therefore the time it takes from the "falling" point to reach P_1 is

$$T = \int_{x_o}^{x_1} \frac{\sqrt{1 + y'^2(x)}}{v} dx.$$

If v_o is the initial speed, by the energy integral we have

$$v^2 - v_o^2 = -2g(y - y_o),$$

which implies

$$v = \sqrt{2g(y_* - y)}, \qquad 2gy_* = v_o^2 + 2gy_o.$$

Therefore,

$$T = \frac{1}{\sqrt{2g}} \int_{x_o}^{x_1} \frac{\sqrt{1+y'^2(x)}}{\sqrt{y_* - y(x)}}\, dx. \tag{1.1c}$$

The problem of the brachistrochrone is to find the curve in $\mathcal{K}$ for which the time T is least. If such a curve exists, by Proposition 1.1 it must be a solution of

$$\begin{aligned} &\frac{d}{dx}\frac{\partial}{\partial y'} \frac{\sqrt{1+y'^2(x)}}{\sqrt{y_* - y(x)}} - \frac{\partial}{\partial y} \frac{\sqrt{1+y'^2(x)}}{\sqrt{y_* - y(x)}} = 0, \\ &y(x_o) = y_o, \quad v(x_o) = v_o. \end{aligned} \tag{1.2c}$$

The problem of the brachistochrone, introduced formally by J. Bernoulli in 1696, is regarded as the beginning of the calculus of variations.

1.3.1c Finding the Brachistochrone (Abel [1, 2])

Assume first $v_o = 0$ and consequently $y_* = y_o$. Prove that (1.2c) implies

$$y'(x) = -\frac{\sqrt{2R - (y_* - y)}}{\sqrt{y_* - y}}$$

for a given positive constant R. From this,

$$x(y) = -\int \frac{\sqrt{y_o - y}}{\sqrt{2R - (y_o - y)}} dy.$$

By the substitution $(y_o - y) = 2R\sin^2(\varphi/2)$, this integral gives

$$x = x_o + R(\varphi - \sin\varphi), \qquad y = y_o - 2R(1 - \cos\varphi).$$

This is a cycloid through (x_o, y_o) for $\varphi = 0$. Choose R so that it also goes through (x_1, y_1) for $\varphi = \pi/2$. Investigate the case that the point has positive initial kinetic energy and find the corresponding cycloid.

1.3.2c On the Minimality of the Brachistochrone

The previous calculations show only that cycloids are stationary points of the functional in (1.1c). Having determined the curve $y = y(x)$, let $\varphi \in C_o^\infty(x_o, x_1)$ and set

$$\mathcal{T}(\varphi; \lambda) = \frac{\sqrt{1 + (y+\lambda\varphi)'^2}}{\sqrt{y_* - (y + \lambda\varphi)}}, \qquad \mathcal{B}(\varphi;\lambda) = \int_{x_o}^{x_1} \mathcal{T}(\varphi;\lambda) dx,$$

where λ is a real parameter. The function $\lambda \to \mathcal{B}(\varphi; \lambda)$ has a critical point for $\lambda = 0$, which is a minimum if

$$\frac{\partial^2}{\partial \lambda^2} \mathcal{B}(\varphi; \lambda) \Big|_{\lambda=0} > 0.$$

To verify such a condition, compute

$$\begin{aligned} \frac{d}{d\lambda} \mathcal{T}(\varphi; \lambda) &= \frac{(y + \lambda\varphi)' \varphi'}{\sqrt{y_* - (y + \lambda\varphi)} \sqrt{1 + (y + \lambda\varphi)'^2}} \\ &+ \frac{\varphi \sqrt{1 + (y + \lambda\varphi)'^2}}{2 \left[y_* - (y + \lambda\varphi)\right]^{3/2}}, \\ \frac{d^2}{d\lambda^2} \mathcal{T}(\varphi; \lambda) \Big|_{\lambda=0} &= \frac{\varphi'^2}{\sqrt{y_* - y}\,(1 + y'^2)^{3/2}} + \frac{3}{4} \frac{\varphi^2 \sqrt{1 + y'^2}}{(y_* - y)^{5/2}} \\ &+ \frac{y' \varphi' \varphi}{(y_* - y)^{3/2} \sqrt{1 + y'^2}}. \end{aligned}$$

From this, by Cauchy's inequality, for all $\varepsilon > 0$,

$$\frac{d^2}{d\lambda^2} \mathcal{T}(\varphi; \lambda) \Big|_{\lambda=0} \geq \frac{(1 - \varepsilon)\varphi'^2}{\sqrt{y_* - y}\,(1 + y'^2)^{3/2}} + \left(\frac{3}{4} - \frac{1}{4\varepsilon}\right) \frac{\varphi^2 \sqrt{1 + y'^2}}{(y_* - y)^{5/2}}.$$

Choose $\varepsilon = \frac{1}{3}$. ■

1.4c Motion along Geodesics

Let $\mathcal{S}$ be a smooth surface in $\mathbb{R}^3$ of parametric equations $P = P(u, v)$ as in §3 of Chapter 2. Let also γ be a curve lying on $\mathcal{S}$ parameterized with t and parametric equations

$$\mathcal{S} \ni P(t) = P\big(u(t), v(t)\big).$$

The infinitesimal arc length on γ is

$$ds = \sqrt{\Delta}\, dt, \qquad \text{where} \qquad \Delta = A\dot{u}^2 + 2B\dot{u}\dot{v} + C\dot{v}^2,$$

where A, B, C are the elements of the first fundamental form of $\mathcal{S}$. The length of the portion of γ between any two of its points $P_o = P(t_o)$ and $P_1(t_1)$ is

$$L(\gamma) = \int_{t_o}^{t_1} \sqrt{\Delta}\, dt.$$

The curve γ is a *geodesic* on $\mathcal{S}$ if for any two of its points P_o and P_1 the length $L(\gamma)$ is the least among all curves lying on $\mathcal{S}$ with extremities P_o and P_1.

By Proposition 1.1 the parametric representation $t \to P(t)$ of one such curve, if it does exist, must satisfy the system (4.1) of Chapter 2. From the remarks of §3.2 of Chapter 6 it follows that the motion of a point mass, constrained on a surface $\mathcal{S}$ and unsolicited otherwise, takes place along geodesics on $\mathcal{S}$. Equivalently, a point constrained on a surface $\mathcal{S}$ to which no other forces are applied moves spontaneously along trajectories of least length on $\mathcal{S}$.

1.5c An Isoperimetric Problem

Among all smooth closed curves in a plane of fixed length ℓ, one seeks the one that encloses the largest area. Denote by (x, y) the coordinates of the plane and regard these curves as parameterized by

$$[0, 2\pi] \ni t \to P(t) = x(t)\,\mathbf{e}_1 + y(t)\,\mathbf{e}_2, \qquad P(0) = P(2\pi).$$

The area enclosed by such a curve and its length are

$$A = \frac{1}{2}\int_0^{2\pi} (x\dot{y} - \dot{x}\,y)\,dt, \qquad \ell = \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt. \tag{1.3c}$$

The problem reduces to maximizing the first integral subject to the constraint represented by the second. Denoting by λ a Lagrange multiplier, introduce the functional

$$J = \int_0^{2\pi} \Big[(x\dot{y} - \dot{x}y) + 2\lambda\sqrt{\dot{x}^2 + \dot{y}^2}\Big]\,dt \stackrel{\text{def}}{=} \int_0^{2\pi} F(x, y; \dot{x}, \dot{y})dt.$$

Compute

$$\frac{\partial F}{\partial \dot{x}} = -y + \frac{2\lambda}{\sqrt{\dot{x}^2 + \dot{y}^2}}\dot{x}, \qquad \frac{\partial F}{\partial x} = \dot{y},$$

$$\frac{\partial F}{\partial \dot{y}} = \ x + \frac{2\lambda}{\sqrt{\dot{x}^2 + \dot{y}^2}}\dot{y}, \qquad \frac{\partial F}{\partial y} = -\dot{x}.$$

Therefore, by Proposition 1.1, the curve that encloses the largest area, for a given fixed perimeter ℓ, satisfies the system

$$\frac{d}{dt}\left(y - \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0, \qquad \frac{d}{dt}\left(x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0.$$

By integration,

$$x - x_o = -\frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \qquad y - y_o = \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}},$$

where x_o and y_o are generic integration constants. Squaring and adding these gives

$$\|P - P_o\| = |\lambda|, \qquad P_o \equiv (x_o, y_o).$$

Thus these curves are circles of radius $|\lambda|$ centered anywhere in the plane. The parameter λ is computed by imposing the constraint $2\pi|\lambda| = \ell$. Choosing $P_o = O$ and $\lambda > 0$, the parametric equations of such a curve are

$$[0, 2\pi) \ni t \to P(t) = \lambda\big(\cos t\mathbf{e}_1 + \sin t\mathbf{e}_2\big).$$

This circle is a stationary point for the functional J. To prove that it actually is a maximum, we will show that there exists no closed curve $P + \delta P$, the perturbation of such a circle, of the same circumference and of larger area. Such a perturbation is represented as $\delta P = (\delta x, \delta y)$, where $t \to \delta x(t), \delta y(t)$ are smooth functions in $[0, 2\pi]$ and $\delta P(0) = \delta P(2\pi)$. The perturbation is small in the sense $\|(\delta P, \delta\dot{P})\| < \varepsilon$, for some $0 < \varepsilon \ll 1$. The length of such a perturbed curve must be the same, i.e.,

$$\int_0^{2\pi} \sqrt{(\dot{x} + \delta\dot{x})^2 + (\dot{y} + \delta\dot{y})^2}\,dt = \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt, \tag{1.4c}$$

and its area must be larger, i.e.,

$$\int_0^{2\pi} \big[(x + \delta x)(\dot{y} + \delta\dot{y}) - (y + \delta y)(\dot{x} + \delta\dot{x})\big]\,dt > \int_0^{2\pi} (x\dot{y} - y\dot{x})\,dt. \tag{1.5c}$$

By a Taylor expansion of the integrand on the left-hand side of (1.4c),

$$\begin{aligned} &\int_0^{2\pi} \big(\delta x \cos t + \delta y \sin t\big)\,dt \\ &\qquad + \frac{1}{4\lambda}\int_0^{2\pi} \big(\delta\dot{x}\cos t + \delta\dot{y}\sin t\big)^2 dt + O\left(\varepsilon^3\right) = 0. \end{aligned} \tag{1.4c$'$}$$

A similar Taylor expansion in (1.5c) gives

$$\int_0^{2\pi} \big(\delta x\ \cos t + \delta y\ \sin t\big)\,dt + \int_0^{2\pi} \delta x\,\delta\dot{y}\,dt\ >\ 0. \tag{1.5c$'$}$$

Without loss of generality, the perturbation δP might be taken of the form

$$\delta P = \varepsilon\varphi(t)\left\{\cos t\mathbf{e}_1 + \sin t\mathbf{e}_2\right\},$$

where ε is positive and φ is a smooth periodic function of t with period 2π. Putting this expression of δP in (1.4c)$'$–(1.5c)$'$ yields

$$\begin{aligned} \int_0^{2\pi} \varphi\,dt + \frac{\varepsilon}{4\lambda}\int_0^{2\pi} \dot{\varphi}^2\,dt &= O(\varepsilon^2), \\ \int_0^{2\pi} \varphi\,dt + \frac{\varepsilon}{2}\int_0^{2\pi} \varphi^2\,dt &> 0. \end{aligned} \tag{1.6c}$$

From these we obtain

$$-\frac{1}{2}\int_0^{2\pi} \varphi^2\,dt < \frac{1}{\varepsilon}\int_0^{2\pi} \varphi\,dt = -\frac{1}{4\lambda}\int_0^{2\pi} \dot{\varphi}^2\,dt + O(\varepsilon).$$

Assume first that φ does not have zero average in $[0,2\pi)$. Then this is impossible for ε sufficiently small. If, on the other hand, φ has zero average in $[0,2\pi)$, then the first equation of 1.6c implies

$$\int_0^{2\pi} \dot\varphi^2 dt = O(\varepsilon).$$

This, in turn, is impossible for ε sufficiently small unless $\dot\varphi \equiv 0$. This would imply $\varphi = \varphi_o$ and the perturbed curve $P+\delta P$ would be

$$[0,2\pi) \ni t \to (P+\delta P)(t) = (\lambda + \varepsilon\varphi_o)\big(\cos t\mathbf{e}_1 + \sin t\mathbf{e}_2\big).$$

However, this is a circle whose length is not $\ell = 2\pi\lambda$ unless $\varepsilon\varphi_o = 0$.

A similar property holds in $\mathbb{R}^N$, i.e., among all "smooth, oriented, closed" hypersurfaces of equal given area, the sphere is the one that encloses the largest volume [38].

2c The Least Action Principle

2.1c On the Two-Point Boundary Value Problem (2.1)

The existence of a unique solution to (2.1) is implied by the a priori knowledge of the motion $t \to q(t)$. Indeed, the extreme points q_o and q_1 have been chosen accordingly. However, if one were to prescribe arbitrary points q_o and q_1, the system (2.1) in general does not have a solution, nor is such a solution, if any, unique. As an example, consider

$$\ddot q + q = 0, \quad t\in[0,\tau], \quad q(0)=0,\ q(\tau)=\delta.$$

Prove that the parameters δ and τ can be chosen so that such a problem has no solution. If (2.1) is solvable, the solution in general is not unique, as shown by the following problem:

$$\ddot q + q = 0, \qquad q(0) = q(\pi) = 0.$$

This has the solutions $q\equiv 0$ and $q(t) = \sin t$.

2.2c Stationary Points of the Action Cannot Be Maxima

Assume first that the constraints are workless. By Proposition 1.1 and Corollary 1.2 of Chapter 6, the kinetic energy T is a positive definite quadratic form of $\dot q$. Therefore

$$\sup_{q\in\mathcal{K}} \int_{t_o}^{t_1} \mathcal{L}(q,\dot q;t)dt \ge \sup_{q\in\mathcal{K};\|q\|\le 1} c_o \int_{t_o}^{t_1} |\dot q|^2 dt - \text{const}.$$

Choose a sequence of curves $q_n \in \mathcal{K}$ such that

$$\|q_n\| \le 1 \quad \forall n\in\mathbb{N} \quad \text{and} \quad \sup_{q_n} \int_{t_o}^{t_1} |\dot q_n|^2 dt = \infty.$$

Modify the argument for moving constraints.

2.3c Critical Points of the Action Are Local Minima

An orbit $q \in \mathcal{K}$, a solution of (2.1), is a *local minimum* for the Hamiltonian action $\mathcal{S}(\cdot)$ if there exists $\varepsilon \in (0,1)$ such that

$$\mathcal{S}(q) < \mathcal{S}(q+\delta q) \quad \forall \text{ synchronous } \delta q \text{ such that } \sup_{[t_o,t_1]} \|\delta q\| < \varepsilon.$$

Proposition 2.1c *Let $q \in \mathcal{K}$ be a solution of (2.1). There exist two numbers $\varepsilon, \sigma \in (0,1)$, which can be determined a priori only in terms of $\mathcal{L}$ and q, such that if $|t_1 - t_o| \le \sigma$, then q is a local minimum for the Hamiltonian action.*

Proof. Assume first that the constraints are workless and compute the variation $\delta\mathcal{L}$ of the Lagrangian, corresponding to a synchronous small variation δq of the orbit q:

$$\begin{aligned}
\delta\mathcal{L} &= \mathcal{L}(q+\delta q, \dot{q}+\delta\dot{q}) - \mathcal{L}(q,\dot{q}) \\
&= \tfrac{1}{2} A_{hk}(q+\delta q)[(\dot{q}_h + \delta\dot{q}_h)(\dot{q}_k + \delta\dot{q}_k) - \dot{q}_h\dot{q}_k] \\
&\quad + \tfrac{1}{2}[A_{hk}(q+\delta q) - A_{hk}(q)]\dot{q}_h\dot{q}_k + U(q+\delta q) - U(q) \\
&= \tfrac{1}{2} A_{hk}(q+\delta q)\delta\dot{q}_h\delta\dot{q}_k + A_{hk}(q+\delta q)\dot{q}_h\delta\dot{q}_k \\
&\quad + \tfrac{1}{2}[A_{hk}(q+\delta q) - A_{hk}(q)]\dot{q}_h\dot{q}_k + U(q+\delta q) - U(q) \\
&= \tfrac{1}{2} A_{hk}(q+\delta q)\delta\dot{q}_h\delta\dot{q}_k + [A_{hk}(q+\delta q) - A_{hk}(q)]\dot{q}_h\delta\dot{q}_k \\
&\quad + \tfrac{1}{2}\Big[A_{hk}(q+\delta q) - A_{hk}(q) - \frac{\partial}{\partial q_k}A_{hk}(q)\delta q_k\Big]\dot{q}_h\dot{q}_k \\
&\quad + U(q+\delta q) - U(q) - \frac{\partial}{\partial q_k}\mathcal{U}(q)\delta q_k \\
&\quad + \frac{\partial}{\partial \dot{q}_k}\mathcal{L}(q,\dot{q})\delta\dot{q}_k + \frac{\partial}{\partial q_k}\mathcal{L}(q,\dot{q})\delta q_k .
\end{aligned} \tag{2.1c}$$

Set also

$$\mathcal{A}(q) = \sum_{h,k,i,j=1}^{N} \Big(\Big|\frac{\partial A_{hk}(q)}{\partial q_i}\Big| + \Big|\frac{\partial^2 A_{hk}(q)}{\partial q_i \partial q_j}\Big| + \Big|\frac{\partial^2 \mathcal{U}(q)}{\partial q_i \partial q_j}\Big| \Big).$$

Since $A_{hk}(\cdot)$ and $\mathcal{U}(\cdot)$ are smooth, one may choose ε sufficiently small, possibly dependent on $\mathcal{L}$ and q, such that

$$\sup_{[t_o,t_1]} \mathcal{A}\big(q(t)\big) = C \quad \text{and} \quad \sup_{[t_o,t_1]} \mathcal{A}\big(q(t)+\delta q(t)\big) \le 2C.$$

Since $\xi \to A_{hk}\xi_h\xi_k$ is a positive definite quadratic form, there exists a positive constant c_o, independent of δq such that

$$\tfrac{1}{2} A_{hk}(q+\delta q)\,\delta\dot{q}_h\delta\dot{q}_k \ge c_o\|\delta\dot{q}\|^2.$$

Moreover,

$$\begin{aligned}|A_{hk}(q+\delta q) - A_{hk}(q)||\dot{q}_h||\delta\dot{q}_k| &\le 2C\|\dot{q}\|\|\delta q\|\|\delta\dot{q}\| \\ &\le \tfrac{1}{2}c_o\|\delta\dot{q}\|^2 + \frac{2C^2}{c_o}\|\dot{q}\|^2\|\delta q\|^2 \\ &\le \tfrac{1}{2}c_o\|\delta\dot{q}\|^2 + C_1\|\delta q\|^2,\end{aligned}$$

where we have used Cauchy's inequality and have set

$$C_1 = \frac{2C^2}{c_o}\sup_{[t_o,t_1]}\|\dot{q}\|^2.$$

Since $q \in \mathcal{K}$ is known as a solution of (2.1), the quantity C_1 can be regarded as a known constant. The remaining terms in (2.1c), except the last, are estimated as

$$\begin{aligned}&\frac{1}{2}\Big|A_{hk}(q+\delta q) - A_{hk}(q) - \frac{\partial A_{hk}(q)}{\partial q_k}\,\delta q_k\Big||\dot{q}_h||\dot{q}_k| \\ &\quad + \Big|U(q+\delta q) - U(q) - \frac{\partial}{\partial q_k}U(q)\,\delta q_k\Big| \\ &\quad\le 2C(1+\|\dot{q}(t)\|^2)\|\delta q\|^2 \le C_2\|\delta q\|^2.\end{aligned}$$

The last is transformed by using that $t \to q(t)$ is a solution of (2.1),

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\delta\dot{q}_k + \frac{\partial\mathcal{L}}{\partial q_k}\delta q_k &= -\left(\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_k} - \frac{\partial\mathcal{L}}{\partial q_k}\right)\delta q_k + \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\delta q_k\right) \\ &= \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\delta q_k\right).\end{aligned}$$

Combining these estimates in (2.1c) gives

$$\delta\mathcal{L} \ge \frac{1}{2}c_o\|\delta\dot{q}\|^2 - C_3\|\delta q\|^2 + \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\,\delta q_k\right),$$

where the constant $C_3 = (2C + C_1 + C_2)$ can be regarded as known a priori. Integrate this dt over (t_o, t_1) and recall that $\delta q(t_o) = \delta q(t_1) = 0$ to get

$$\mathcal{S}(q+\delta q) - \mathcal{S}(q) \ge \frac{1}{2}c_o\int_{t_o}^{t_1}\|\delta\dot{q}\|^2 dt - C_3(t_1 - t_o)\sup_{[t_o,t_1]}\|\delta q\|^2.$$

Using that the virtual variations δq vanish at t_o and t_1, estimate

$$\sup_{[t_o,t_1]}\|\delta q\|^2 \le (t_1 - t_o)\int_{t_o}^{t_1}\|\delta\dot{q}\|^2 dt.$$

Combining these estimates gives

$$\mathcal{S}(q+\delta q) - \mathcal{S}(q) \geq \left[\tfrac{1}{2}c_o - C_3(t_1-t_o)^2\right] \int_{t_o}^{t_1} \|\delta\dot{q}\|^2 dt.$$

It follows that if the time interval (t_o, t_1) is small enough to satisfy

$$\tfrac{1}{2}c_o - C_3(t_1-t_o)^2 \geq \tfrac{1}{4}c_o > 0,$$

then $\mathcal{S}(q) < \mathcal{S}(q+\delta q)$, and the stationary orbit $q \in \mathcal{K}$, a solution of (2.1), is a *local* minimum for the Hamiltonian action. ■

Modify the proof to include the case of moving constraints. The proposition was first proved, by a different entirely geometrical argument, by Darboux [37, Tome II, n° 246, 545, 568].

4c The Functional of Geometrical Optics

Assume that the entire space $\mathbb{R}^3$ is divided into two connected optically homogeneous media Ω_o and Ω_1, separated by a smooth surface Σ, and with refraction indices ν_o and ν_1 respectively. In each of them, light rays propagate along straight lines, with speeds ν_o^{-1} and ν_1^{-1} respectively. Therefore the minimum problem (4.1) reduces to finding the point $q \in \Sigma$ for which the time

$$t = \nu_o\|q-q_o\| + \nu_1\|q_1-q\|$$

is the least. Computing its variation, we have

$$\delta t = \left(\nu_o \frac{q-q_o}{\|q-q_o\|} - \nu_1 \frac{q_1-q}{\|q_1-q\|}\right) \cdot \delta q = 0.$$

Since q varies on Σ, there exists a parameter $\lambda \in \mathbb{R}$ such that

$$\nu_o \frac{q-q_o}{\|q-q_o\|} - \nu_1 \frac{q_1-q}{\|q_1-q\|} = \lambda \mathbf{n}(q), \tag{4.1c}$$

where $\mathbf{n}(q)$ is the unit normal to Σ in q. This implies that the three vectors

$$\frac{q-q_o}{\|q-q_o\|}, \qquad \frac{q_1-q}{\|q_1-q\|}, \qquad \mathbf{n}(q),$$

are *coplanar*. Let then π be the plane that contains them and let $\mathbf{t}$ be the tangent to Σ in q lying in π. Taking the scalar product of (4.1c) by $\mathbf{t}$ gives

$$\nu_o \sin\widehat{i} - \nu_1 \sin\widehat{r} = 0, \tag{4.2c}$$

where

$$\widehat{i} \equiv \text{ angle between } \frac{q-q_o}{\|q-q_o\|} \text{ and } \mathbf{n}(q) \quad \text{is the angle of incidence,}$$

$$\widehat{r} \equiv \text{ angle between } \frac{q_1-q}{\|q_1-q\|} \text{ and } \mathbf{n}(q) \quad \text{is the angle of refraction.}$$

Consider the case of an optical medium consisting (at least locally) of $(n+1)$ layers of optically homogeneous media Ω_h with refraction indices ν_h, separated by n portions of smooth surfaces Σ_h.

Formulas (4.1c)–(4.2c) are Snell's laws (1621) reported by Huygens in [84] and Descartes in [39].

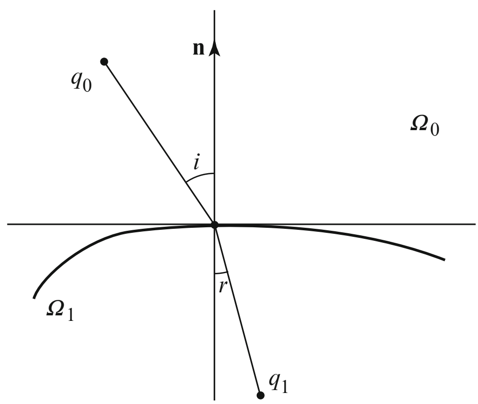

Fig. 4.1c.

10

CANONICAL TRANSFORMATIONS

1 Changing the Variables of Motion

Let $\mathcal{H}(p, q; t)$ be the Hamiltonian of a mechanical system with N degrees of freedom. By the least action principle, an orbit $t \to (p(t), q(t))$ in phase space $\mathbb{R}^{2N}$ is a solution of the Hamilton canonical equations if and only if it is a stationary point of the action, i.e.,

$$\delta \int_{t_o}^{t_1} [p\dot{q} - \mathcal{H}(p, q; t)]dt = 0 \quad \Longleftrightarrow \quad \begin{aligned} \dot{p} &= -\nabla_q \mathcal{H}, \\ \dot{q} &= \quad \nabla_p \mathcal{H}, \end{aligned} \tag{1.1}$$

where (t_o, t_1) is any subinterval of the time of motion, and the virtual variation δ is synchronous in such an interval. Consider now smooth, invertible transformations

$$\begin{aligned} P &= P(p, q; t), \qquad & p &= p(P, Q; t), \\ Q &= Q(p, q; t), \qquad & q &= q(P, Q; t), \end{aligned} \qquad J = \begin{pmatrix} \left(\dfrac{\partial P}{\partial p}\right) & \left(\dfrac{\partial P}{\partial q}\right) \\ \left(\dfrac{\partial Q}{\partial p}\right) & \left(\dfrac{\partial Q}{\partial q}\right) \end{pmatrix}, \tag{1.2}$$

with $\det J \neq 0$. Among these look for those that preserve the variational structure of (1.1). Precisely those for which, for any given Hamiltonian $\mathcal{H}(p, q; t)$, there exists a new one $\mathcal{K}(P, Q; t)$ such that the orbits satisfying (1.1), when transformed by (1.2), are those and only those for which

$$\delta \int_{t_o}^{t_1} [P\dot{Q} - \mathcal{K}(P, Q; t)]dt = 0 \quad \Longleftrightarrow \quad \begin{aligned} \dot{P} &= -\nabla_Q \mathcal{K}, \\ \dot{Q} &= \quad \nabla_P \mathcal{K}, \end{aligned} \tag{1.3}$$

where the virtual variation δ is synchronous in (t_o, t_1). We call these *variational* transformations, since they mutually keep the form of the Hamilton equations and their corresponding least action principle. These transformations, whenever they exist, are local, i.e., they are well defined, in general within a sufficiently small neighborhood of a given point $(p_o, q_o; t_o)$, taken as

E. DiBenedetto, *Classical Mechanics: Theory and Mathematical Modeling*, Cornerstones, DOI 10.1007/978-0-8176-4648-6_10,

initial data in (1.1). In what follows we will avoid specifying the local nature of (1.2) and their respective domains of definition.

Remark 1.1 It is not required that the new Hamiltonian $\mathcal{K}$ be obtained from $\mathcal{H}$ by substituting (p, q) in terms of $(P, Q; t)$, nor that Q and P have the physical significance of Lagrangian configurations and kinetic momenta. It is required only that the variational structures of (1.1) and (1.3) be mutually preserved, for *any* Hamiltonian $\mathcal{H}$, and a corresponding new one $\mathcal{K}$. For this reason, the variational transformations are independent of $\mathcal{H}$, as further clarified by the next example.

1.1 Pointwise Transformations

The choice of the Lagrangian coordinates is not unique, and from a given choice $q = (q_1, \dots, q_N)$, one can construct another one, by prescribing a smooth, locally invertible transformation $Q = Q(q; t)$. These are called *pointwise* transformations of the Lagrangian coordinates. Describing the motion in terms of these, one constructs a new Lagrangian $\mathcal{L}'(Q, \dot{Q}; t)$, and defines new kinetic momenta $P = \nabla_{\dot{Q}} \mathcal{L}'$. Then, along the motion, the new Hamiltonian

$$\mathcal{K}(P, Q; t) = P_h \dot{Q}_h - \mathcal{L}'(P, Q; t)$$

satisfies the Hamilton equations and its corresponding least action principle, as indicated in (1.3). Thus pointwise transformations are variational transformations, and because of their construction, they are independent of the original Hamiltonian $\mathcal{H}(p, q; t)$. The new variables (Q, P) keep their physical meaning of Lagrangian configurations and kinetic momenta. However, neither the new Lagrangian $\mathcal{L}'$ nor the new Hamiltonian $\mathcal{H}'$ is obtained from $\mathcal{L}$ or $\mathcal{H}$ by substitution of variables.

2 Variational and Canonical Transformations

The requirement that the transformation (1.2) be variational amounts to imposing conditions on the integrals in (1.1) and (1.3), so that the vanishing of anyone of them, for all the proper virtual synchronous variations, implies the vanishing of the other for all corresponding virtual synchronous variations. One such condition is the existence of a smooth function

$$(p, q, P, Q; t) \to F(p, q, P, Q; t) \tag{2.1}$$

such that

$$pdq - \mathcal{H}dt = PdQ - \mathcal{K}dt + dF. \tag{2.2}$$

Indeed, when computed along any curve $t \to (p(t), q(t))$ in phase space and the corresponding transformed curve $t \to (P(t), Q(t))$, the integrands in (1.1) and

(1.3) differ by the virtual synchronous variation of the total time derivative of F. The differentials dq and dQ and dF are meant with respect to the set of variables $\{p, q, P, Q; t\}$. Given an invertible transformation as in (1.2), only $2N$ of $\{p, q, P, Q\}$ are independent. To avoid specifying the set of independent variables, we consider them as a full set of variables, possibly dependent or independent. Introduce the differential $\bar{d}$ at frozen time, acting on the variables $\{p, q, P, Q\}$ only [111]. Thus for a smooth function f depending on the full set of variables $\{p, q, P, Q; t\}$,

$$\bar{d}f = df - f_t dt = \nabla_p f \bar{d}p + \nabla_q f \bar{d}q + \nabla_P f \bar{d}P + \nabla_Q f \bar{d}Q, \tag{2.3}$$

where $\bar{d}p$, $\bar{d}q$, $\bar{d}P$, $\bar{d}Q$ are expressed in terms of the $2N$ variables chosen as independent in (1.2). Notice, however, that if ξ is an independent variable, then $\xi_t = 0$ and $\bar{d}\xi = d\xi$. Thus if, for example, Q is chosen as independent, then $Q_t = 0$ and $\bar{d}Q = dQ$. With this notation rewrite (2.2) in the equivalent form

$$p\bar{d}q - P\bar{d}Q - \bar{d}F = [(\mathcal{H} - pq_t) - (\mathcal{K} - PQ_t) + F_t]dt. \tag{2.4}$$

This relation, roughly speaking, separates the differentials with respect to the space variables $\{p, q, P, Q\}$ from the differential with respect to time. The latter is satisfied if

$$\bar{d}\omega \stackrel{\text{def}}{=} p\bar{d}q - P\bar{d}Q = \bar{d}F \tag{2.5}$$

and

$$\mathcal{K} \stackrel{\text{def}}{=} \mathcal{H} - (pq_t - PQ_t) + F_t. \tag{2.6}$$

Given an invertible transformation as in (1.2), if a function F exists satisfying (2.5), then the differential form $\bar{d}\omega$ in the space variables $\{p, q, P, Q\}$ only is exact. Then with such a function at hand, (2.6) gives the form of the new Hamiltonian $\mathcal{K}$ in terms of the old one $\mathcal{H}$. Conversely, if $\bar{d}\omega$ is exact, there exists a function F as in (2.1) such that, defining $\mathcal{K}$ by (2.6), the sufficient condition (2.2) is satisfied. This, in turn, implies that the invertible transformation (1.2) preserves the variational structure of (1.1) and (1.3).

2.1 Canonical Transformations

A locally invertible transformation as in (1.2) is *canonical* if the differential form $\bar{d}\omega$ in the variables $\{p, q, P, Q\}$ is exact, equivalently if there exists a smooth function F as in (2.1) satisfying (2.5). The function F is the primitive of $\bar{d}\omega$. The requirement that $\bar{d}\omega$ be exact is a condition on the transformation (1.2) alone and not on the Hamiltonian $\mathcal{H}$. Thus, once a transformation of the type of (1.2) has been identified for which $\bar{d}\omega$ is exact, any Hamiltonian system as in (1.1), for any given Hamiltonian $\mathcal{H}$, is transformed into a new Hamiltonian system of the form (1.3) for a new Hamiltonian $\mathcal{K}$ defined by (2.6). A canonical transformation is *completely* canonical if it is explicitly independent of time. In such a case $\bar{d} = d$ and $d\omega = dF$ and $\mathcal{K} = \mathcal{H}$. Canonical transformations form a strict subclass of the variational transformations. The transformation

$$Q = q, \quad P = \lambda p, \quad \lambda \neq 1, \quad N = 1, \quad \mathcal{K}(\cdot, \cdot; t) = \mathcal{H}(\lambda \cdot, \cdot; t), \tag{2.7}$$

preserves the variational form (1.1) for the new Hamiltonian $\mathcal{K}$; however, there is no F such that $d\omega = dF$. We postpone to §7 the characterization of all variational transformations, and focus next on canonical transformations, in view of their remarkable invariance properties (§§7–8).

3 Constructing Classes of Canonical Transformations

The function F could in principle be dependent on all four N-tuples of variables p, q, P, Q. However, by the mutual invertibility requirement in (1.2), only $2N$ of these are independent. Assume momentarily that the transformation in (1.2) is canonical and that the variables (q, Q) can be chosen as independent, so that the remaining variables can be expressed in terms of these. Then, having identified the function F, a primitive of $\bar{d}\omega$, set

$$(q, Q; t) \to F_1(q, Q; t) = F\big(p(q, Q; t), q, P(q, Q; t), Q; t\big). \tag{3.1}$$

Then the condition of $\bar{d}\omega$ being exact implies

$$p = \nabla_q F_1(q, Q; t), \qquad P = -\nabla_Q F_1(q, Q; t). \tag{3.2}$$

This suggests constructing canonical transformations for which (q, Q) are independent as follows [120]. Fix a smooth function $F_1(q, Q; t)$ satisfying

$$\det \Big(\frac{\partial^2 F_1(q, Q; t)}{\partial q_h \partial Q_k} \Big) \neq 0 \tag{3.3}$$

in its domain of definition. Then define p and P from (3.2). The first of these permits expressing Q in terms of $(p, q; t)$. Putting this in the second expresses P in terms of $(p, q; t)$. This identifies the first two equations of (1.2). The form of the second two is determined by inversion, which is guaranteed by (3.3). The form of the new Hamiltonian is

$$\mathcal{K}(P, Q; t) = \mathcal{H}(p, q; t) + F_{1,t}(q, Q; t). \tag{3.4}$$

The function $F_1(q, Q; t)$ is called the *generator* of the transformation. The construction procedure is independent of the Hamiltonian $\mathcal{H}$ and identifies the subclass of canonical transformations for which (q, Q) can be taken as independent variables. Such a class would not include the pointwise transformations, since $q = q(Q; t)$ and in general the kinetic momenta (p, P) cannot be expressed in terms of $(q, Q; t)$ alone. A trivial example of this is the identity transformation $q = Q$ and $p = P$.

4 Constructing Canonical Transformations by Other Pairs of Independent Variables

The requirement that the differential form $d\omega$ be exact can be written in any of the equivalent forms

$$p\bar{d}q - P\bar{d}Q = \bar{d}F, \tag{4.1}$$

$$p\bar{d}q + Q\bar{d}P = \bar{d}F_2, \qquad \text{where} \quad F_2 = F + PQ, \tag{4.2}$$

$$q\bar{d}p + P\bar{d}Q = \bar{d}F_3, \qquad \text{where} \quad F_3 = F - pq, \tag{4.3}$$

$$q\bar{d}p + Q\bar{d}P = \bar{d}F_4, \qquad \text{where} \quad F_4 = F + PQ - pq. \tag{4.4}$$

The form of (4.1) suggests constructing the class of canonical transformations for which (q, P) are independent by a similar procedure. First, if (q, P) are independent, rewrite (2.4) in the form

$$p\bar{d}q + Q\bar{d}P - \bar{d}F_2 = (\mathcal{H} - \mathcal{K} + F_{2,t})dt. \tag{4.5}$$

Next, fix a function $F_2 = F_2(q, P; t)$ satisfying

$$\det\left(\frac{\partial^2 F_2(q, P; t)}{\partial q_h \partial P_k}\right) \neq 0 \tag{4.6}$$

in its domain of definition, and set

$$p = \nabla_q F_2(q, P; t), \qquad Q = \nabla_P F_2(q, P; t). \tag{4.7}$$

Similarly, the class of canonical transformations for which (p, Q) can be taken as independent is constructed by taking $F_3 = F_3(p, Q; t)$ satisfying the analogue of (3.3) and (4.6) and the analogue of the differential equalities (2.4) and (4.5). Then one sets

$$q = \nabla_p F_3(p, Q; t), \qquad P = -\nabla_Q F_3(p, Q; t). \tag{4.8}$$

Finally, the class of canonical transformations for which (p, P) can be taken as independent is constructed by taking $F_4 = F_4(p, P; t)$ satisfying the analogue of (3.3) and (4.7) and the analogue of the differential equalities (2.4) and (4.5), and setting

$$q = -\nabla_p F_4(p, P; t), \qquad Q = \nabla_P F_4(p, P; t). \tag{4.9}$$

In all cases, the new Hamiltonian is given by

$$\mathcal{K}(P, Q; t) = \mathcal{H}(p, q; t) + F_{j,t}, \qquad j = 1, 2, 3, 4. \tag{4.10}$$

5 Examples of Canonical Transformations

The function $F_2(q, P) = qP$ generates through (4.7) the identity transformation. The function $F_1(q, Q) = qQ$ generates through (3.2) the transformation $Q = p$ and $P = -q$. The original kinetic momenta p become the Lagrangian coordinates in the transformed system, whereas the original Lagrangian coordinates q become the opposite of the "new" kinetic momenta P. This example indicates that in general, canonical transformations do not preserve the physical meaning of the original variables. The pointwise transformations of §1.1 are generated by $F_2(q, P; t) = Q(q; t)P$. Given a smooth positive function $t \to f(t)$, the transformation

$$P = \frac{1}{f(t)} p, \quad Q = f(t)q \ \text{ is canonical with primitive } \ F = PQ. \tag{5.1}$$

5.1 The Flow Map Is a Canonical Transformation

The flow map transforms (P, Q) into variables

$$\begin{array}{lclcl} p = p(P,Q;t), & \text{solutions of} & \dot{p} = -\nabla_q \mathcal{H}, & \text{with} & p(0) = P, \\ q = q(P,Q;t), & & \dot{q} = \nabla_p \mathcal{H}, & & q(0) = Q. \end{array} \tag{5.2}$$

For $|t|$ sufficiently small, the transformation is locally smooth and invertible. It is also canonical with primitive F given by

$$F(P, Q; t) = \int_0^t [pq_t - \mathcal{H}(p, q; s)]\, ds, \tag{5.3}$$

where in the integrand p and q are expressed in terms of $\{P, Q; t\}$, as indicated in (5.2). By direct calculation,

$$\begin{aligned} \nabla_Q F &= \int_0^t (q_t \nabla_Q p + p \nabla_Q q_t - \nabla_p \mathcal{H} \nabla_Q p - \nabla_q \mathcal{H} \nabla_Q q)\, ds \\ &= \int_0^t (q_t \nabla_Q p + p \nabla_Q q_t - q_t \nabla_Q p + p_t \nabla_Q q)\, ds \\ &= \int_0^t (p \nabla_Q q)_t ds = p \nabla_Q q - P. \end{aligned}$$

Similarly, one computes $\nabla_P F = p \nabla_P q$. Therefore

$$\bar{d}F = p\nabla_Q q dQ + p \nabla_P q dP - P dQ = \bar{d}\omega.$$

Finally, since $F_t = pq_t - \mathcal{H}$, it follows from (2.6) that $\mathcal{K} = 0$, since Q is an independent variable.

5.2 On the Primitives of $\bar{d}\omega$

Given a canonical transformation (1.2), the form of the primitive F depends on a choice of $2N$ independent variables. In (5.1) and in the flow map, the independent variables have been taken as (P, Q). However, in these examples, the pair (p, Q) or (q, P) could also be taken as independent, and the primitive could be rewritten in terms of these.

The $2N$ independent variables are selected N out of (p, q) and N out of the (P, Q). Thus in principle, there could be $2(2n!)/(n!)$ forms of a primitive F, each corresponding to a pair of N-tuples of independent variables. If any one of the pairs of N variables (q, Q), (q, P), (p, Q), (p, P) can be taken as independent, the generators F_j can be found as the primitives of the respective differential forms in (4.1)–(4.4). If none of these pairs can be taken as independent variables, then the primitive F differs from the F_j. However, it can be written as one of these, modulo the interchange of some of the kinetic momenta p, or Lagrangian coordinates q, into the Lagrangian coordinates Q and the kinetic momenta P respectively. Indeed, these transformations are canonical. In this sense the F_j are essentially all the primitives of a canonical transformation.

5.3 Jacobi Integration of Hamilton Equations [87]

For a given Hamiltonian $\mathcal{H}(\cdot, \cdot; t)$, choose a smooth function $F_1(q, Q; t)$, a solution of the Hamilton–Jacobi equation (§3 of Chapter 9)

$$\frac{\partial F_1(q, Q; t)}{\partial t} + \mathcal{H}\left(\nabla_q F_1(q, Q; t), q; t\right) = 0. \tag{5.4}$$

The variables Q are regarded as parameters, and $F_1(\cdot, Q; t)$ is regarded as a family of solutions of (5.4) parameterized with Q. Assuming that (5.4) admits a family of smooth solutions satisfying (3.3), one might use such $F_1(q, Q; t)$ as the generator of a canonical transformation as in (3.2). These transformations resolve the motion in the following sense. First, by (3.4) the transformed Hamiltonian $\mathcal{K}(P, Q; t)$ is identically zero. Therefore, if $t \to (p(t), q(t))$ is a solution of the canonical Hamiltonian system (1.1), the transformed system is $\dot{P} = 0$ and $\dot{Q} = 0$ identically. Thus P and Q are constants, and their value can be computed from the resulting transformations (1.2) at some fixed time t_o, that is,

$$P = Q(p_o, q_o; t_o), \qquad Q = P(p_o, q_o; t_o),$$

by using the initial data (q_o, p_o) associated to the original canonical system. Putting these constants into the second equation of (1.2) determines the motion in phase space

$$t \to p(t) = p(P, Q; t), \qquad t \to q(t) = q(P, Q; t). \tag{5.5}$$

While the method is simple and elegant, it rests upon solving the Hamilton–Jacobi equation (5.4) depending on N scalar parameters Q. Solution techniques of (5.4) will be given in the next chapter. One could start from any one of the generators of the form F_j and arrive at the same solution methods. The choice of F_j in this method is not unique, and it depends on the possibility of solving the associated Hamilton–Jacobi equation.

6 Symplectic Product in Phase Space and Symplectic Matrices

The variables p and q in the Hamilton equations play an antisymmetric role in the sense

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \mathbb{E} \begin{pmatrix} \nabla_p \mathcal{H} \\ \nabla_q \mathcal{H} \end{pmatrix}, \qquad \text{where} \qquad \mathbb{E} = \begin{pmatrix} \mathbb{O} & -\mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}.$$

The matrix $\mathbb{E}$ is the antisymmetric or *symplectic* identity, and it satisfies

$$\mathbb{E}^t = \mathbb{E}^{-1} = -\mathbb{E} \qquad \text{and} \qquad \mathbb{E}^2 = -\mathbb{I}.$$

Given any two vectors $\mathbf{u} = (p, q)^t$ and $\mathbf{v} = (P, Q)^t$ in $\mathbb{R}^{2N}$, the matrix $\mathbb{E}$ defines a symplectic scalar product in $\mathbb{R}^{2N}$, by the formula

$$[\mathbf{u}, \mathbf{v}] = \mathbb{E}\mathbf{u} \cdot \mathbf{v} = -\mathbb{E}\mathbf{v} \cdot \mathbf{u},$$

where "$\cdot$" is the usual operation of scalar product in $\mathbb{R}^{2N}$. From the definition we have

$$[\mathbf{u}, \mathbf{v}] = -[\mathbf{v}, \mathbf{u}], \quad [\mathbf{u}, \mathbf{u}] = 0, \quad [\mathbf{u}, \mathbf{v}] = p_h Q_h - q_h P_h.$$

If $N = 1$, then $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^2$, and if they are nontrivial, the number

$$[\mathbf{u}, \mathbf{v}] = p_1 Q_1 - q_1 P_1$$

is the only nontrivial component of the exterior product $\mathbf{v} \wedge \mathbf{u}$. Moreover, $p_1 Q_1 - q_1 P_1$ is the measure of the "oriented area" of the parallelogram of $\mathbf{u}$ and $\mathbf{v}$. If $N > 1$, the vectors $\mathbf{u}$ and $\mathbf{v}$ form a parallelogram in $\mathbb{R}^{2N}$, whose projections on the planes of the coordinate axes (q_h, p_h) are parallelograms of oriented area $p_h Q_h - q_h P_h$. Therefore $[\mathbf{u}, \mathbf{v}]$ may be interpreted as the sum of the oriented areas of such projections.

6.1 Symplectic Linear Transformations

Let $\mathcal{T}$ be a linear transformation of $\mathbb{R}^{2N}$ into itself, which we identify with its representative matrix. The transformation $\mathcal{T}$ is *symplectic* if it preserves the symplectic product, that is, if

$$[\mathcal{T}\mathbf{u}, \mathcal{T}\mathbf{v}] = [\mathbf{u}, \mathbf{v}] \quad \text{for all } \mathbf{u} \text{ and } \mathbf{v} \text{ in } \mathbb{R}^{2N}.$$

The antisymmetric identity $\mathbb{E}$ preserves the symplectic product, since

$$[\mathbb{E}\mathbf{u}, \mathbb{E}\mathbf{v}] = \mathbb{E}^2\mathbf{u} \cdot \mathbb{E}\mathbf{v} = -\mathbf{u} \cdot \mathbb{E}\mathbf{v} = \mathbb{E}\mathbf{u} \cdot \mathbf{v} = [\mathbf{u}, \mathbf{v}].$$

If $\mathcal{T}$ is symplectic, then

$$\mathbb{E}\mathbf{u} \cdot \mathbf{v} = [\mathbf{u}, \mathbf{v}] = [\mathcal{T}\mathbf{u}, \mathcal{T}\mathbf{v}] = \mathbb{E}\mathcal{T}\mathbf{u} \cdot \mathcal{T}\mathbf{v} = (\mathcal{T}^t\mathbb{E}\mathcal{T})\mathbf{u} \cdot \mathbf{v}.$$

Since this has to hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^{2N}$, a linear transformations $\mathcal{T} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is symplectic if and only if either one of the following equivalent conditions holds:

$$\mathcal{T}^t\mathbb{E}\mathcal{T} = \mathbb{E}, \qquad \mathcal{T}^{-1} = -\mathbb{E}\mathcal{T}^t\mathbb{E}, \qquad (\mathcal{T}^{-1})^t\mathbb{E}(\mathcal{T}^{-1}) = \mathbb{E}. \tag{6.1}$$

Therefore the transpose and inverse of a symplectic matrix are symplectic matrices. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are linear symplectic transformations of $\mathbb{R}^{2N}$ into itself, one verifies that

$$(\mathcal{T}_1\mathcal{T}_2)^t\mathbb{E}(\mathcal{T}_1\mathcal{T}_2) = \mathbb{E}.$$

Therefore the product of two symplectic matrices is symplectic. Thus the collection of all linear symplectic transformations from $\mathbb{R}^{2N}$ into itself form a group under composition, called the *symplectic group* of order N. By (6.1), $\det(\mathcal{T}) = \pm 1$, and it can be shown that indeed $\det(\mathcal{T}) = 1$.

Proposition 6.1 *A linear, symplectic transformation $\mathcal{T} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$, preserves orientations and volumes.*

7 Characterizing Canonical Transformations by Symplectic Jacobians

Proposition 7.1 *The transformation (1.2) is canonical if and only if the Jacobian matrix J is symplectic.*

Proof. For a smooth curve $s \to [q(s), p(s)]$ and its transform by (1.2), denote by δ a synchronous virtual variation of $\{p, q, P, Q\}$. These variations are effected at time t frozen, and are synchronous with respect to the parameter s. Let $\bar{d}\omega$ be the differential form defined in (2.5) and compute

$$\delta\bar{d}\omega = \bar{d}(p\delta q - P\delta Q) + \delta_*\bar{d}\omega,$$

where we have set

$$\delta_*\bar{d}\omega = (\delta p\bar{d}q - \delta q\bar{d}p) - (\delta P\bar{d}Q - \delta Q\bar{d}P).$$

By the Pfaff–Jacobi theorem [86, 126], the differential form $\bar{d}\omega$ is exact if and only if $\delta_*\bar{d}\omega = 0$ for all virtual synchronous variations of curves in (p, q)-space and the corresponding variations in (P, Q)-space. Compute

$$\begin{aligned} -\delta_*\bar{d}\omega &= \big[(\delta p, \delta q)\mathbb{E} - (\delta P, \delta Q)\mathbb{E}J\big](\bar{d}p, \bar{d}q)^t \\ &= (\delta p, \delta q)\big[\mathbb{E} - J^t\mathbb{E}J\big](\bar{d}p, \bar{d}q)^t. \end{aligned}$$

■

If (1.2) is canonical, then J is symplectic and $\det J = 1$. Therefore we have the following corollary.

Corollary 7.1 *Canonical transformations are volume-preserving. In particular, the flow map is volume-preserving.*

The last statement is known as Liouville's theorem.

8 Poisson Brackets [131]

For any two scalar-valued smooth functions $(p, q; t) \to F(p, q; t)$, $G(p, q; t)$, the Poisson brackets of F and G are defined by

$$\begin{aligned} \{F, G\} &= \frac{\partial F}{\partial p_h}\frac{\partial G}{\partial q_h} - \frac{\partial F}{\partial q_h}\frac{\partial G}{\partial p_h} \\ &= \mathbb{E}\nabla_{p,q}F \cdot \nabla_{p,q}G = [\nabla_{p,q}F, \nabla_{p,q}G]. \end{aligned} \tag{8.1}$$

One verifies the calculus properties

$$\begin{aligned} \{F, G\} &= -\{G, F\}, \quad \{F, C\} = 0 \text{ for all constants } C, \\ \{F_1 + F_2, G\} &= \{F_1, G\} + \{F_2, G\}, \\ \{F_1 F_2, G\} &= F_1\{F_2, G\} + F_2\{F_1, G\}, \\ \{q_h, q_k\} &= \{p_h, p_k\} = 0, \quad \{p_h, q_k\} = \delta_{hk}, \\ \{F, q_h\} &= F_{p_h}, \quad \{F, p_h\} = -F_{q_h}. \end{aligned} \tag{8.2}$$

Moreover, if x is any scalar variable taken out of $(p, q; t)$, then

$$\{F, G\}_x = \{F_x, G\} + \{F, G_x\}. \tag{8.3}$$

By these properties, the Hamiltonian system (1.1) can be written in the form

$$\dot{p}_h = \{\mathcal{H}, p_h\}, \qquad \dot{q}_h = \{\mathcal{H}, q_h\}. \tag{8.4}$$

8.1 Invariance of the Poisson Brackets by Canonical Transformations

The last term in (8.1) is the symplectic product of the two vectors $\nabla_{q,p}F$ and $\nabla_{q,p}G$. Performing on F and G the change of variables (1.2) yields

$$\{F, G\}_{p,q} = [\nabla_{p,q}F, \nabla_{p,q}G] = [J\nabla_{P,Q}F, J\nabla_{P,Q}G].$$

Therefore if J preserves the symplectic product, then $\{F, G\}_{p,q} = \{F, G\}_{P,Q}$, that is, J preserves the Poisson brackets. In particular, if $F = Q_h$ and $G = P_k$, then

$$\begin{aligned} &\{P_h, P_k\}_{q,p} = \{P_h, P_k\}_{P,Q} = \{Q_h, Q_k\}_{p,q} = \{Q_h, Q_k\}_{P,Q} = 0, \\ &\{P_h, Q_k\}_{p,q} = \{P_h, Q_k\}_{P,Q} = \delta_{hk}. \end{aligned} \tag{8.5}$$

For any transformation as in (1.2) with Jacobian J, symplectic or not, one computes

$$J\mathbb{E}J^t = \begin{pmatrix} \{P_h, P_k\} & \{P_h, Q_k\} \\ \{Q_h, P_k\} & \{Q_h, Q_k\} \end{pmatrix}.$$

Therefore J is symplectic if and only if equations (8.5) hold. In particular, (8.5) characterizes canonical transformations.

Proposition 8.1 *The transformation (1.2) is canonical if and only if*

$$\begin{aligned} \{p_h, p_k\}_{P,Q} &= \{q_h, q_k\}_{P,Q} = 0, \qquad & \{p_h, q_k\}_{P,Q} &= \delta_{hk}, \\ \{P_h, P_k\}_{p,q} &= \{Q_h, Q_k\}_{p,q} = 0, \qquad & \{P_h, Q_k\}_{p,q} &= -\delta_{hk}. \end{aligned} \tag{8.6}$$

Equivalently, the transformation (1.2) is canonical if and only if it preserves the Poisson brackets.

9 The Jacobi Identity [89]

Proposition 9.1 *For every triple F, G, H of smooth functions of* $(p, q; t)$,

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0. \tag{9.1}$$

Proof. Regard (9.1) as a homogeneous linear form of the second derivatives of F. These occur only in the last two terms

$$\{G, \{H, F\}\} - \{H, \{G, F\}\}. \tag{9.2}$$

We show first that such an expression does not contain second derivatives of F. The action of G through the Poisson brackets is formally given by

$$D_G = \frac{\partial G}{\partial p_h}\frac{\partial}{\partial q_h} - \frac{\partial G}{\partial q_h}\frac{\partial}{\partial p_h}.$$

Introducing the notation

$$y = (y_1, y_2, \dots, y_N, y_{N+1}, \dots, y_{2N}) = (q, p),$$

the operation D_G can be expressed as $D_G = a_h \partial/\partial y_h$, where the coefficients a_h depend only on the first derivatives of G. Similarly, the action of H through the Poisson brackets is given by $D_H = b_k \partial/\partial y_k$. With this notation,

$$\begin{aligned} \{G, \{H, F\}\} - \{H, \{G, F\}\} &= D_G D_H F - D_H D_G F \\ &= a_h \frac{\partial}{\partial y_h}\Big(b_k \frac{\partial F}{\partial y_k}\Big) - b_h \frac{\partial}{\partial y_h}\Big(a_k \frac{\partial F}{\partial y_k}\Big). \end{aligned}$$

In this expression the terms containing the second derivatives of F are

$$a_h b_k \frac{\partial^2 F}{\partial y_h \partial y_k} - a_k b_h \frac{\partial^2 F}{\partial y_h \partial y_k} = 0.$$

It follows that the expression in (9.2) involves only first derivatives of F, and therefore it takes the form

$$\{G, \{H, F\}\} - \{H, \{G, F\}\} = A_h \frac{\partial F}{\partial p_h} + B_h \frac{\partial F}{\partial q_h}, \tag{9.3}$$

where the coefficients A_h and B_h depend on the first and second derivatives of G and H, and are independent of F. Such independence permits one to compute them by choosing $F = p_h$ and $F = q_h$. This gives

$$\begin{aligned}\{G, \{H, p_h\}\} - \{H, \{G, p_h\}\} &= A_h,\\ \{G, \{H, q_h\}\} - \{H, \{G, q_h\}\} &= B_h.\end{aligned}$$

Therefore

$$\begin{aligned}A_h &= \{H_{q_h}, G\} + \{H, G_{q_h}\} = -\{G, H\}_{q_h},\\ B_h &= \{G_{p_h}, H\} + \{G, H_{p_h}\} = \{G, H\}_{p_h}.\end{aligned}$$

These in (9.3) give

$$\begin{aligned}\{G, \{H, F\}\} + \{H, \{F, G\}\} &= \{G, H\}_{p_h} F_{q_h} - \{G, H\}_{q_h} F_{p_h}\\ &= -\{F, \{G, H\}\}.\end{aligned}$$

■

10 Generating First Integrals of Motion by the Poisson Brackets

The total time derivative of a smooth function F of $(q, p; t)$ along a solution of the canonical system (1.1) is computed as

$$\begin{aligned}\frac{d}{dt} F &= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_h} \dot{q}_h + \frac{\partial F}{\partial p_h} \dot{p}_h\\ &= \frac{\partial F}{\partial t} + \frac{\partial \mathcal{H}}{\partial p_h} \frac{\partial F}{\partial q_h} - \frac{\partial \mathcal{H}}{\partial q_h} \frac{\partial F}{\partial p_h} = \frac{\partial F}{\partial t} + \{\mathcal{H}, F\}.\end{aligned}$$

Proposition 10.1 *A smooth function $(q, p; t) \to F(q, p; t)$ is an integral of motion if and only if $F_t + \{\mathcal{H}, F\} = 0$. If F is explicitly independent of time, it is an integral of motion if and only if $\{\mathcal{H}, F\} = 0$.*

Theorem 10.1 (Poisson [131]). *If F and G are two integrals of motion, then $\{F, G\}$ is also an integral of motion.*

Proof. Compute the derivative of $\{F, G\}$ along any orbit solution of (1.1). Using the Jacobi identity (9.1) and the calculus properties (8.2) of the Poisson brackets, we obtain

$$\begin{aligned}\frac{d}{dt}\{F,G\} &= \{F,G\}_t + \{F,G\}_{p_h}\dot{p}_h + \{F,G\}_{q_h}\dot{q}_h \\ &= \{F,G\}_t - \{F,G\}_{p_h}\mathcal{H}_{q_h} + \{F,G\}_{q_h}\mathcal{H}_{p_h} \\ &= \{F,G\}_t + \{\mathcal{H},\{F,G\}\} \\ &= \{F,G\}_t - \{F,\{G,\mathcal{H}\}\} - \{G,\{\mathcal{H},F\}\} \\ &= \{F_t,G\} + \{F,G_t\} + \{F,\{\mathcal{H},G\}\} + \{\{\mathcal{H},F\},G\} \\ &= \{F_t + \{\mathcal{H},F\},G\} + \{F,G_t + \{\mathcal{H},G\}\} = 0.\end{aligned}$$

■

Remark 10.1 While the theorem guarantees that given any two integrals of motion F and G, one can generate another one by $\{F, G\}$, it does not assert that the latter is independent of F and G. If this were the case, by iteration one could generate infinitely many independent integrals of motion, contradicting that the mechanical system at hand has finitely many degrees of freedom.

11 Infinitesimal Canonical Transformations

A family of canonical transformations parameterized by a small parameter ε is close to the identity if it is generated by (§4)

$$F_2(q,P;t,\varepsilon) = qP + \varepsilon G(q,P;t,\varepsilon), \qquad |\varepsilon| \ll 1, \tag{11.1}$$

for a smooth function G of its arguments. The transformation is close to the identity, since it appears, at least formally, as a small perturbation of qP, which is the generator of the identity (§5). From (11.1) one computes the new variables

$$\begin{aligned}\nabla_q F_2 = p = P + \varepsilon\nabla_q G(q,P;t,\varepsilon), \\ \nabla_P F_2 = Q = q + \varepsilon\nabla_P G(q,P;t,\varepsilon),\end{aligned}$$

and the increments

$$\begin{aligned}\delta p = P - p = -\varepsilon\nabla_q G(q,P;\varepsilon), \\ \delta q = Q - q = \varepsilon\nabla_P G(q,P;\varepsilon).\end{aligned}$$

These transformations are canonical for all ε, so small as to guarantee that (4.6) is satisfied. Therefore, for all such ε, the variables (p, q) and (P, Q) satisfy the characteristic condition (8.6) for the transformation to be canonical. Although the generator is F_2, we say that G is the generator of the infinitesimal canonical transformation, close to the identity.

Even though G depends on $(q, P; \varepsilon)$, it can be regarded as a function of $(p, q; \varepsilon)$, up to an infinitesimal of higher order in ε. Indeed, by Taylor's formula, $G(q, P; t, \varepsilon) = G(q, p; t, \varepsilon) + O(\varepsilon)$. With this notation,

$$\begin{aligned} \delta p &= -\varepsilon \nabla_q G(p, q; t, \varepsilon) + o(\varepsilon), \\ \delta q &= \varepsilon \nabla_p G(p, q; t, \varepsilon) + o(\varepsilon). \end{aligned} \tag{11.2}$$

Remark 11.1 Transformations of the type of (11.2), with generic terms $o(\varepsilon)$, are close to the identity, but not necessarily canonical. They are canonical if they are generated by a family of functions $G(q, P; t, \varepsilon)$ as in (11.1).

As an example consider the flow map from (P, Q) into (p, q) as defined in (5.2). For an infinitesimal dt, such a map is close to the identity, since

$$\begin{aligned} p &= P + dt\dot{p} + o(dt) & \qquad q &= Q + dt\dot{q} + o(dt) \\ &= p - dt\nabla_q \mathcal{H} + o(dt), & &= q + dt\nabla_p \mathcal{H} + o(dt). \end{aligned}$$

It is also canonical since the flow map is a canonical transformation. Therefore $\varepsilon G = (dt)\mathcal{H}$, where $\mathcal{H}$ is computed in terms of (q, P), and where dt plays the role of the small parameter ε appearing in (11.1). The motion can be interpreted as a continuous progression of infinitesimal canonical transformations.

Given a smooth function G of the arguments $(p, q; t, \varepsilon)$, for a small parameter ε, consider transformations $(p, q) \to (P, Q)$ defined by (11.2) without higher-order corrections, that is,

$$\begin{aligned} P - p = \delta p &= -\varepsilon \nabla_q G(p, q; t, \varepsilon), \\ Q - q = \delta q &= \quad \varepsilon \nabla_p G(p, q; t, \varepsilon). \end{aligned} \tag{11.3}$$

These in general are not canonical, since the characteristic conditions (8.6) are satisfied only up to terms $o(\varepsilon)$ of higher order in ε. While not necessarily canonical, they are called infinitesimal canonical transformations close to the identity.

11.1 Variations for Infinitesimal Canonical Transformations

The variation of a smooth function $F(p, q; t)$ induced by (11.3) is computed as

$$\delta F = \nabla_p F \delta p + \nabla_q F \delta q = -\varepsilon \frac{\partial F}{\partial p_h} \frac{\partial G}{\partial q_h} + \varepsilon \frac{\partial F}{\partial q_h} \frac{\partial G}{\partial p_h} = \varepsilon \{G, F\}.$$

If $F = \mathcal{H}$, then

$$\delta \mathcal{H}(p, q; t) = \varepsilon \{G, \mathcal{H}\}.$$

If $G(p, q)$ is explicitly independent of t and ε, and is an integral of motion, then $\{G, \mathcal{H}\} = 0$ by Proposition 10.1, and consequently $\delta \mathcal{H} = 0$. Thus the corresponding infinitesimal canonical transformation generated by (11.3) for such a G preserves the Hamiltonian. Conversely, if (11.3) preserves the Hamiltonian, and G is explicitly independent of t, then G is an integral of motion.

Proposition 11.1 *The integrals of motion explicitly independent on t are those and only those G that generate, by (11.3), an infinitesimal canonical transformation that preserves the Hamiltonian $\mathcal{H}$.*

Remark 11.2 The proposition permits one to seek integrals of motion among those transformations that preserve the Hamiltonian. For example, if $\mathcal{H}$ has a symmetry with respect to some of its variables, those rotations that keep the symmetry of $\mathcal{H}$ are first integrals of motion.

If $\mathcal{H}$ has a cyclic variable, say for example q_h, then $\mathcal{H}$ remains unchanged for transformations of the type $\delta q_k = \varepsilon\delta_{hk}$ and $\delta p_h = 0$. The function G that generates such a transformation through (11.3) is $G = p_h + \text{const}$. Such a G remains constant along the motion, according to Proposition 11.1 and the remarks of §6.1 of Chapter 6.

12 Variational and Canonical Transformations

Let (1.2) be variational, so the it preserves the variational structure (1.1). To every smooth Hamiltonian $(p, q; t) \to \mathcal{H}(p, q; t)$ there corresponds a new smooth Hamiltonian $(P, Q; t) \to \mathcal{K}_{\mathcal{H}}(P, Q; t)$, dependent upon $\mathcal{H}$. Denote by $\mathcal{K}_o$ the function $\mathcal{K}_{\mathcal{H}}$ corresponding to the choice $\mathcal{H} = 0$. Moreover, for every Hamiltonian $\mathcal{H}$, denote by H its pointwise transform through (1.2), that is,

$$(P, Q; t) \to H(P, Q; t) = \mathcal{H}(q(P, Q; t), p(P, Q; t); t).$$

Theorem 12.1 (Lie [113]). *Let (1.2) be variational. There exists $t \to f(t)$ such that for every Hamiltonian $\mathcal{H}$,*

$$(P, Q; t) \to \mathcal{K}_{\mathcal{H}}(P, Q; t) = f(t)H(P, Q; t) + \mathcal{K}_o(P, Q; t).$$

Moreover, the transformation (1.2) is canonical if and only if $f(t) = 1$.

Proof. If the transformation preserves the variational structure of the Hamiltonian system, then

$$\begin{pmatrix} \dot{P} \\ \dot{Q} \end{pmatrix} = \mathbb{E} \begin{pmatrix} \nabla_P \mathcal{K}_{\mathcal{H}} \\ \nabla_Q \mathcal{K}_{\mathcal{H}} \end{pmatrix}$$

and also

$$\begin{pmatrix} \dot{P} \\ \dot{Q} \end{pmatrix} = J \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} + \frac{\partial}{\partial t} \begin{pmatrix} P \\ Q \end{pmatrix} = J\mathbb{E}J^t \begin{pmatrix} \nabla_P H \\ \nabla_Q H \end{pmatrix} + \frac{\partial}{\partial t} \begin{pmatrix} P \\ Q \end{pmatrix}.$$

Therefore

$$J\mathbb{E}J^t \begin{pmatrix} \nabla_P H \\ \nabla_Q H \end{pmatrix} + \frac{\partial}{\partial t} \begin{pmatrix} P \\ Q \end{pmatrix} = \mathbb{E} \begin{pmatrix} \nabla_P \mathcal{K}_{\mathcal{H}} \\ \nabla_Q \mathcal{K}_{\mathcal{H}} \end{pmatrix},$$

for all smooth functions $(P, Q; t) \to H(P, Q; t)$. Taking $H = 0$, we obtain

$$\frac{\partial}{\partial t}\begin{pmatrix} P \\ Q \end{pmatrix} = \mathbb{E}\begin{pmatrix} \nabla_P \mathcal{K}_o \\ \nabla_Q \mathcal{K}_o \end{pmatrix}.$$

Putting this in the previous expression gives the identity

$$J\mathbb{E}J^t \begin{pmatrix} \nabla_P H \\ \nabla_Q H \end{pmatrix} = \mathbb{E}\begin{pmatrix} \nabla_P(\mathcal{K}_{\mathcal{H}} - \mathcal{K}_o) \\ \nabla_Q(\mathcal{K}_{\mathcal{H}} - \mathcal{K}_o) \end{pmatrix}. \tag{12.1}$$

Lemma 12.1 *Let (1.2) be variational, so that (12.1) holds. There exists a smooth function $t \to f(t)$ determined only in terms of (1.2) such that $J\mathbb{E}J^t = f(t)\mathbb{E}$.*

Assuming the lemma for the moment, (12.1) provides the form of the new Hamiltonian, as claimed by Proposition 12.1. The transformation (1.2) is canonical if and only if J is symplectic. By the structure (6.1) of the symplectic matrices, (1.2) is canonical if and only if $f(\cdot) = 1$. ■

12.1 Proof of Lemma 12.1

Write the matrix $J\mathbb{E}J^t$ in the form

$$J\mathbb{E}J^t = \begin{pmatrix} F_{1,1} & F_{1,2} \\ F_{2,1} & F_{2,2} \end{pmatrix}, \qquad \text{where } F_{i,j} \text{ are } N \times N \text{ matrices.}$$

With this symbolism rewrite (12.1) as

$$\begin{aligned} F_{1,1}\nabla_P H + F_{1,2}\nabla_Q H &= -\nabla_Q(\mathcal{K}_{\mathcal{H}} - \mathcal{K}_o), \\ F_{2,1}\nabla_P H + F_{2,2}\nabla_Q H &= \nabla_P(\mathcal{K}_{\mathcal{H}} - \mathcal{K}_o). \end{aligned} \tag{12.2}$$

Take the divergence of the first equation with respect to P and of the second with respect to Q. Adding the resulting expressions gives

$$\mathrm{div}_P(F_{1,1}\nabla_P H + F_{1,2}\nabla_Q H) + \mathrm{div}_Q(F_{2,1}\nabla_P H + F_{2,2}\nabla_Q H) = 0, \tag{12.3}$$

valid for all smooth functions $(Q, P; t) \to H(Q, P; t)$. Fix two indices k and ℓ, and in (12.3) choose first $H = P_k$ and then $H = P_\ell P_k$. This gives the $2N$ relations

$$\begin{aligned} \frac{\partial}{\partial P_h} F_{1,1;hk} + \frac{\partial}{\partial Q_h} F_{2,1;hk} &= 0, \\ \frac{\partial}{\partial P_h}(F_{1,1;hk}P_\ell) + \frac{\partial}{\partial Q_h}(F_{2,1;hk}P_\ell) &= 0. \end{aligned}$$

These imply $F_{1,1} = \mathbb{O}$. Similarly, choosing first $H = Q_k$ and then $H = Q_\ell Q_k$ gives $F_{2,2} = \mathbb{O}$. This permits one to rewrite (12.3) as

$$\mathrm{div}_P(F_{1,2}\nabla_Q H) + \mathrm{div}_Q(F_{2,1}\nabla_P H) = 0.$$

Choosing $H = Q_k$ or $H = P_k$ implies that $F_{1,2;hk}$ are independent of P_h, and $F_{2,1;hk}$ are independent of Q_h. Choosing $H = Q_\ell P_k$ gives

$$F_{1,2;hk}\delta_{\ell k} + F_{2,1;hk}\delta_{h\ell} = 0,$$

from which

$$F_{1,2;hk} = -F_{2,1;hk} = 0 \quad \text{for } h \neq k \quad \text{and} \quad F_{1,2;hh} = -F_{2,1;hh}.$$

Moreover, these functions are independent of P_h and Q_h respectively. These remarks applied to the first equation of (12.2) give

$$F_{1,2;kk}\frac{\partial H}{\partial Q_k} = -\frac{\partial}{\partial Q_k}(\mathcal{K}_{\mathcal{H}} - \mathcal{K}_o).$$

Write this for two indices h and k. Then take the derivative of the one corresponding to the index h with respect to Q_k, and take the derivative of the one corresponding to the index k with respect to Q_h. Subtracting the resulting expressions gives the identities

$$\frac{\partial}{\partial Q_h}\Big(F_{1,2;kk}\frac{\partial H}{\partial Q_k}\Big) = \frac{\partial}{\partial Q_k}\Big(F_{1,2;hh}\frac{\partial H}{\partial Q_h}\Big) \quad \text{for } h,k = 1,\dots,N, \tag{12.4}$$

valid for every smooth function $(Q,P;t) \to H(Q,P;t)$. Therefore $F_{1,2;hh} = F_{1,2;kk}$ for all $h,k = 1,\dots,N$, and $(F_{1,2}) = f(t)\mathbb{I}$. ∎

Problems and Complements

4c Constructing Canonical Transformations by Other Pairs of Independent Variables

Write (4.1)–(4.4) with $F = F_1$, where F_1 is defined in (3.1). By (4.1), the functions $Q \to F_1(q,Q;t)$ and $P \to F_2(q,P;t)$, for fixed $q \in \mathbb{R}^N$ and $t \in \mathbb{R}$, are affine and therefore convex. Prove that

$$F_2(q,P;t) = \sup_{\xi\in\mathbb{R}^N}[P_h\xi_h + F_1(q,\xi;t)].$$

That is, $P \to F_2(q,P;t)$ is the Legendre transform of $Q \to -F_1(q,Q;t)$ (see §5c of the Complements of Chapter 6). The latter is not coercive nor strictly convex. Prove that nevertheless, the supremum is achieved for some Q at finite distance from the origin. Prove that for such a Q, one has $P = -\nabla_Q F_1(q,Q;t)$, which is precisely the second equation of (3.2). This connection between $F_1(q,Q;t)$ and $F_2(q,P;t)$ does not imply that if one of them generates a canonical transformation the other does. Give an example of a canonical transformation for which the variables (q, P) can be taken as independent and (q, Q) cannot. In such a case F_2 generates a canonical transformation and F_1 does not. Neverthless, the relation (4.2) between them continues to hold.

Prove that the generators F_j can be constructed as the Legendre transform of some F_i, for $i \neq j$, with respect to suitable variables.

4.1. Find the canonical transformation generated by $F_2(p, Q) = \sqrt{e^{-2Q} - p^2}$ $(N = 1)$.

4.2. Find the canonical transformation generated by $F_3(p, Q) = p(1 - e^Q)$, for $N = 1$. Answer: $P = p(1 + q)$ and $Q = \ln(1 + q)$.

4.3. Let $F_2(p, Q; \varepsilon) = qP + \varepsilon q^2(1 + \cos P)$ for $N = 1$. Find the values of ε for which F_2 generates a canonical transformation. Verify that such transformations are

$$p = P + 2\varepsilon q(1 + \cos P), \qquad q = \frac{1}{2\varepsilon \sin P}\left(1 - \sqrt{1 - 4\varepsilon Q \sin P}\right).$$

4.4. Verify that the transformation $(N = 1)$

$$Q = \ln(1 + \sqrt{p} \sin q), \quad P = 2(1 + \sqrt{p} \sin q)\sqrt{p} \cos q$$

satisfies $2(PdQ - pdq) = d(p \sin 2q)$. Therefore it is completely canonical with generator $F_3(p, Q) = -(e^Q - 1)^2 \cot q$.

4.5. Verify that the transformation $(N = 1)$

$$p = 2e^t \sqrt{PQ} \ln P, \qquad q = e^{-t}\sqrt{PQ}$$

is canonical, by verifying that the differential form $q\bar{d}q + Q\bar{d}P$ is exact. Compute the generator F_2 as the primitive of such a form.

5c Examples of Canonical Transformations

5.1. Prove that any t-dependent canonical transformation is the flow map (5.2) for some $\mathcal{H}$.

5.2. Verify that the transformation $P = p$ and $Q = q - pt$ is canonical and the Hamiltonian $\mathcal{K}$ associated to any Hamiltonian $\mathcal{H}$ is

$$\mathcal{K}(P, Q; t) = \mathcal{H}(P, Q + Pt; t) - \tfrac{1}{2}P^2.$$

5.3. Let $N = 2$, assume that in (1.2) the variables $\{p_2, q_1, P_1, Q_2\}$ can be taken as independent, and seek a primitive $\bar{F}$ dependent on these variables and t. If F is a primitive of $\bar{d}\omega$, rewrite (4.1) for $N = 2$ as

$$p_1\bar{d}q_1 - q_2\bar{d}p_2 + Q_1\bar{d}P_1 - P_2\bar{d}Q_2 = \bar{d}\bar{F}, \tag{5.1c}$$

where $\bar{F} = F + (P_1Q_1 - p_2q_2)$ and where $\{p_2, q_2, P_1, Q_1\}$ are expressed in terms of the independent variables $\{p_2, q_1, P_1, Q_2\}$. Then set

$$p_1 = \bar{F}_{q_1}, \quad q_2 = -\bar{F}_{p_2}, \quad P_2 = -\bar{F}_{Q_2}, \quad Q_1 = \bar{F}_{P_1}.$$

In practice, having identified $\{p_2, q_1, P_1, Q_2\}$ as independent, one writes (5.1c) and finds a primitive $\bar{F}$ of the differential form on the left-hand side.

5.4. Verify that the transformation ($N = 2$)

$$p_1 = -P_1Q_1, \qquad q_1 = \frac{P_1P_2 - Q_1Q_2}{P_1^2 + Q_2^2},$$
$$p_2 = \frac{1}{2}(P_1^2 - Q_2^2), \qquad q_2 = \frac{P_2Q_2 - P_1Q_1}{P_1^2 + Q_2^2},$$

is canonical, by verifying that the differential form $\bar{d}\omega$ is exact.

5.5. For the indicated transformation, verify that $\bar{d}\omega$ is exact and that the functions $F(p, q; t)$ and $F_1(q, Q; t)$ are primitives:

$$P = -\arctan\frac{q}{p}, \qquad Q = \frac{1}{2}(p^2 + q^2),$$

$$F(p, q) = \frac{1}{2}(p^2 + q^2)\arctan\frac{q}{p} + \frac{1}{2}pq,$$
$$F_1(q, Q) = Q\arcsin\frac{q}{\sqrt{2Q}} + \frac{1}{2}q\sqrt{2Q - q^2}.$$

5.6. For the indicated transformation, verify that $\bar{d}\omega$ is exact and that the functions $F(p, q; t)$ and $F_1(q, Q; t)$ are primitives:

$$P = \sqrt{2q}\,e^t \sin p, \qquad Q = \sqrt{2q}\,e^{-t}\cos p.$$

$$F(p, q) = pq - q\sin p\cos p,$$
$$F_1(q, Q) = Q\arccos\frac{rQe^{-t}}{\sqrt{2q}} - \frac{1}{2}Qe^{-t}\sqrt{2q - Q^2e^{-2t}}.$$

5.7. For the indicated transformation, verify that $\bar{d}\omega$ is exact and that the functions F_j below, for $j = 1, 2, 3, 4$, are all generators:

$$P = q\cot p, \qquad Q = \ln\frac{\sin p}{q},$$

$$F_1(q, Q) = q\arccos\sqrt{1 - q^2e^{2Q}} + e^{-Q}\sqrt{1 - q^2e^{2Q}},$$
$$F_2(q, P) = q\arctan\frac{q}{P} + P\left(1 - \ln\sqrt{q^2 + P^2}\right),$$
$$F_3(p, Q) = e^{-Q}\cos p, \quad F_4(p, P) = P + P\ln\left(\frac{\cos p}{P}\right).$$

5.8c Linear Canonical Transformations

Given constant invertible $N \times N$ matrices (α_{hk}) and (β_{hk}), consider the linear transformation on $\mathbb{R}^{2N}$

$$P_h = \alpha_{hk} p_k, \qquad Q_h = \beta_{hk} q_k. \tag{5.2c}$$

Prove that (5.2c) is canonical if and only if it preserves the bilinear form pq, that is, if

$$p_h q_h = P_h Q_h \quad \Longrightarrow \quad p_h q_h = \alpha_{hk} \beta_{h\ell} p_k q_\ell.$$

Therefore (α_{hk}) must be the transpose of the inverse of (β_{hk}).

5.8.1c An Example Related to the n-Body Problem

Transform the Lagrangian coordinates q into

$$Q_1 = q_1, \quad Q_h = q_h - q_1, \quad h = 2, \dots, N. \tag{5.3c}$$

Transform now p so that the resulting transformation is linear and completely canonic. This occurs for

$$(\beta_{hk}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad (\alpha_{hk}) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 1 \\ \dots & \dots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The corresponding p transformations are (see §9 of Chapter 6, and in particular (9.1))

$$P_1 = p_1 + p_2 + \cdots + p_N, \qquad P_h = p_h, \quad h = 2, \dots, N. \tag{5.4c}$$

6c Symplectic Product in Phase Space and Symplectic Matrices

Let $\mathcal{T}$ be a linear transformations of $\mathbb{R}^{2N}$ into itself, identified with its representative matrix, and up to an isomorphism write

$$\mathcal{T} = \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{pmatrix}, \tag{6.1c}$$

where $(\mathcal{T}_{ij})$ are $N \times N$ matrices.

6.1. Prove that $\mathcal{T}$ is symplectic if and only if

$$\begin{aligned} &\mathcal{T}_{21}^t \mathcal{T}_{11} - \mathcal{T}_{11}^t \mathcal{T}_{21} = \mathbb{O}, \quad \mathcal{T}_{21}^t \mathcal{T}_{12} - \mathcal{T}_{11}^t \mathcal{T}_{22} = -\mathbb{I}, \\ &\mathcal{T}_{22}^t \mathcal{T}_{11} - \mathcal{T}_{12}^t \mathcal{T}_{21} = \mathbb{I}, \quad \mathcal{T}_{22}^t \mathcal{T}_{12} - \mathcal{T}_{12}^t \mathcal{T}_{22} = \mathbb{O}. \end{aligned}$$

6.2. Prove that if $\mathcal{T}$ is symplectic, then

$$\mathcal{T}^{-1} = \begin{pmatrix} \mathcal{T}_{22}^t & -\mathcal{T}_{12}^t \\ -\mathcal{T}_{21}^t & \mathcal{T}_{11}^t \end{pmatrix}.$$

6.3. The symplectic group of order 2 consists of 2×2 matrices whose determinant is 1. Rotation matrices of $\mathbb{R}^2$ are symplectic transformations.

7c Characterizing Canonical Transformations by Symplectic Jacobians

7.1c Linear Canonical Transformations by Symplectic Matrices

Consider the linear transformation with constant coefficients

$$\begin{aligned} P_h &= \alpha_{p,hk} p_k + \alpha_{q,hk} q_k, \\ Q_h &= \beta_{p,hk} p_k + \beta_{q,hk} q_k, \end{aligned} \qquad \text{of the Jacobian} \qquad J = \begin{pmatrix} (\alpha_{p,hk}) & (\alpha_{q,hk}) \\ (\beta_{p,hk}) & (\beta_{q,hk}) \end{pmatrix}.$$

Therefore this transformation is canonical if and only if

$$\begin{aligned} (\beta_{p,hk})^t(\alpha_{p,hk}) - (\alpha_{p,hk})^t(\beta_{p,hk}) &= \mathbb{O}, \\ (\beta_{p,hk})^t(\alpha_{q,hk}) - (\alpha_{p,hk})^t(\beta_{q,hk}) &= -\mathbb{I}, \\ (\beta_{q,hk})^t(\alpha_{p,hk}) - (\alpha_{q,hk})^t(\beta_{p,hk}) &= \mathbb{I}, \\ (\beta_{q,hk})^t(\alpha_{q,hk}) - (\alpha_{q,hk})^t(\beta_{q,hk}) &= \mathbb{O}. \end{aligned}$$

Verify that if $(\alpha_{q,hk}) = (\beta_{p,hk}) = \mathbb{O}$, one finds the same condition of canonical transformation as that in §5.8c. Set

$$(\alpha_{p,hk}) = (\beta_{q,hk}) = \mathbb{O}; \quad (\alpha_{q,hk}) = (\alpha_{hk}), \quad (\beta_{p,hk}) = (\beta_{hk}).$$

Find a condition on these coefficients for the resulting transformation to be canonical. In particular, for $N = 1$ justify why the transformation in (2.7) is not canonical unless $\lambda = 1$.

7.2c The Poincaré Recurrence Theorem

Assume that the Hamiltonian system (1.1) is *autonomous*, and that it admits an invariant region E. Specifically, $\mathcal{H}$ is independent of time and there exists a region $E \subset \mathbb{R}^{2N}$ such that for all $(p_o, q_o) \in E$, the unique solution

$$t \to S^t[(p_o, q_o)] = (p(t), q(t))$$

of (1.1) originating at (p_o, q_o) remains in E for all times. Here we have denoted by S^t the solution operator at time t. In particular, $S^o = \mathbb{I}$.

A set $B \subset E$ is mapped at time t into

$$S^t[B] = \bigcup_{(p_o,q_o)\in B} S^t[(p_o,q_o)].$$

If B is measurable, also $S^t[B]$ is measurable, and by Liouville's theorem, $|B| = |S^t[B]|$.

Theorem 7.1c (Poincaré [127 Vol. 3 Chap. 26]). *Assume that E is of finite measure. For every measurable $B \subset E$ of positive measure and every $t_o \in \mathbb{R}$,*

(a) *There exists $t > t_o$ such that $S^t[B] \cap B \neq \emptyset$.*
(b) *There exists a set of measure zero $B_o \subset B$ such that all orbits originating at $B - B_o$ return to intersect $B - B_o$ infinitely many times.*

Remark 7.1c The first part of the theorem asserts that among all orbits originating in B, there are some that return to B after a sufficiently long time. The theorem does not assert that all orbits originating in B return to B. However, the second part of the theorem asserts that this occurs for all orbits originating at all points of B, except possibly for a set B_o of measure zero.

Remark 7.2c The set B could be a ball of arbitrarily small radius. In such a case all orbits originating at almost all points in B return, to be arbitrarily close to their initial position, after a sufficiently long time. For this reason this is referred to as the recurrence theorem.

Proof (of **(a)***).* Pick $t_o > 0$, set $B_o = B$, and for $j \in \mathbb{N}$, set $B_j = S^{jt_o}[B]$. There exist two indices $j_o \geq 0$ and $j_o + i$ for some $i > 0$ such that $B_{j_o} \cap B_{j_o+i} \neq \emptyset$. Indeed, otherwise the sets B_j would be disjoint and contained in E, contradicting that $|E| < \infty$. If $j_o = 0$, the assertion is proven. If $j_o > 0$, then $B_{j_o-1} \cap B_{j_o-1+i} \neq \emptyset$. If $j_o = 1$, this proves the assertion; otherwise, $B_{j_o-2} \cap B_{j_o-2+i} \neq \emptyset$. Repeating this argument j_o times proves **(a)**. ■

Proof (of **(b)***).* Denote by B_o the subset of B such that every orbit originating at B_o never intersects B. If B_o had positive measure, by the same argument of **(a)**, this would contradict that $|E| < \infty$. The second part of the assertion is proved by iteration. ■

7.3c A Note on Liouville's Theorem

Liouville's theorem could be proved more directly from the definition of flow map. Assume that a bounded open set $E(t) \subset \mathbb{R}^N$ with smooth boundary $\partial E(t)$ is mapped by a smooth velocity field $t \to \mathbf{v}(\cdot;t)$ into $E(t + dt)$ in an infinitesimal time interval dt. Then by elementary calculus considerations and the Gauss–Green theorem,

$$\frac{d}{dt}|E(t)| = \int_{\partial E(t)} \mathbf{v}\cdot\mathbf{n}d\sigma = \int_{E(t)} \operatorname{div}\mathbf{v}dv,$$

where $\mathbf{n}$ is the unit normal to $\partial E(t)$ pointing outside $E(t)$ and $\mathrm{d}\sigma$ and dv are respectively the surface measure on $\partial E(t)$ and the volume measure on $E(t)$. For the flow map in phase space,

$$\mathbf{v} = (\dot{p},\dot{q}) = (-\nabla_q\mathcal{H}, \nabla_p\mathcal{H}) \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0.$$

More generally, the flow map generated by a smooth irrotational vector field $t \to \mathbf{v}(\cdot;t)$ is volume-preserving.

7.4c Integral Invariants

Let γ be a smooth closed curve in $\mathbb{R}^{2N}$ of parametric equations

$$\gamma = \left\{ \begin{array}{c} p = p(\tau),\ q = q(\tau)\ \text{ for }\ \tau \in [\tau_1,\tau_2] \\ \text{with }\ p(\tau_1) = p(\tau_2),\ q(\tau_1) = q(\tau_2) \end{array} \right\}.$$

Denote by Γ the image of γ by the transformation (1.2) of the parametric equations

$$\Gamma = \left\{ \begin{array}{l} P = P(\tau) = P(p(\tau),q(\tau);t)\ \text{ with }\ P(\tau_1) = P(\tau_2) \\ Q = Q(\tau) = Q(p(\tau),q(\tau);t)\ \text{ with }\ Q(\tau_1) = Q(\tau_2) \end{array} \right\}.$$

If (1.2) is canonical with generator F, then $\bar{d}\omega = \bar{d}F$. Therefore

$$\int_\gamma p\bar{d}q = \int_\Gamma P\bar{d}Q + \int_{\tau_1}^{\tau_2} \frac{d}{d\tau}F(q(\tau),Q(\tau);t)d\tau.$$

The last integral vanishes, since γ and Γ are closed, and we have the following result.

Proposition 7.1c *The integral of the differential form pdq in $\mathbb{R}^{2N}$ along a smooth closed curve $\gamma \subset \mathbb{R}^{2N}$ is invariant by canonical transformations.*

7.4.1c Poincaré Surface Invariants [20, 127]

Denote by $\mathbf{p}_i$ and $\mathbf{q}_i$ the coordinate unit vectors along the axes of the p_i and q_i respectively. To the differential form pdq there is associated the vector $\mathbf{F} = p_i\mathbf{q}_i + 0_i\mathbf{p}_i$. If $\gamma \subset \mathbb{R}^{2N}$ is a smooth closed curve as above, denote by σ a smooth surface whose boundary is γ and denote by σ_h the projection of σ on the hyperplane $\{p_h, q_h\}$, whose unit normal is $\mathbf{p}_h \wedge \mathbf{q}_h$. Let also $\mathbf{n}$ be the unit normal to σ chosen so that the orientation on γ is counterclockwise with respect to the orientation of $\mathbf{n}$. By Stokes's theorem,

$$\begin{aligned}\int_\gamma pdq &= \int_\sigma \operatorname{curl}\mathbf{F}\cdot\mathbf{n}d\sigma = \int_\sigma \nabla_{p,q}\wedge\mathbf{F}\cdot\mathbf{n}d\sigma\\ &= \int_\sigma \frac{\partial}{p_h}p_k\mathbf{p_h}\wedge\mathbf{q_k}\cdot\mathbf{n}d\sigma = \sum_{h=1}^N\int_{\sigma_h} dp_h dq_h.\end{aligned}$$

The right-hand side is the sum of the oriented areas of the projections of a smooth surface $\sigma\subset\mathbb{R}^{2N}$ on the coordinate planes $\{p_h,q_h\}$ and is called the Poincaré surface invariant. Let Γ and Σ be the transforms of γ and σ by (1.2) and denote by Σ_h the projection of Σ on the coordinate plane $\{P_h,Q_h\}$. If (1.2) are canonical, then by Proposition 7.1c,

$$\sum_{h=1}^N\int_{\sigma_h} dp_h dq_h = \sum_{h=1}^N\int_{\Sigma_h} dP_h dQ_h.$$

8c Poisson Brackets

8.1c Lagrange Brackets [103]

Let $(\mathbf{u},\mathbf{v})\to\mathbf{F}(\mathbf{u},\mathbf{v}),\mathbf{G}(\mathbf{u},\mathbf{v})$ be smooth vector-valued functions of two N-tuples of variables $(\mathbf{u},\mathbf{v})$. Pick a scalar variable u out of $\mathbf{u}$ and a scalar variable v out of $\mathbf{v}$. The Lagrange brackets of (u,v) with respect to $\mathbf{F}$ and $\mathbf{G}$ are defined by

$$[\![u,v]\!]_{\mathbf{F},\mathbf{G}} = \frac{\partial F_i}{\partial u}\frac{\partial G_i}{\partial v} - \frac{\partial F_i}{\partial v}\frac{\partial G_i}{\partial u}.$$

One verifies that

$$[\![u,v]\!]_{\mathbf{F},\mathbf{G}} = -[\![v,u]\!]_{\mathbf{F},\mathbf{G}} = [\![v,u]\!]_{\mathbf{G},\mathbf{F}}.$$

Let $\mathbf{z}$ be a smooth, locally invertible, $2N$-valued function of (p,q) so that the transformations

$$(p,q)\to\mathbf{z}(p,q) = \big(z_1(p,q),\dots,z_{2N}(p,q)\big),\quad \mathbf{z}\to(p(\mathbf{z}),q(\mathbf{z})),$$

are well defined. There exists a remarkable relation between the Lagrange brackets of any pair of variables (z_h,z_ℓ) with respect to the pair of vector functions q and p and the Poisson brackets of any pair of functions (z_ℓ,z_k) with respect to the variables (p,q).

Proposition 8.1c $[\![z_h,z_\ell]\!]_{q,p}\{z_\ell,z_k\}_{p,q} = \delta_{hk}$.

Proof. from the definitions,

$$
\begin{aligned}
[\![z_h, z_\ell]\!]_{q,p}\{z_\ell, z_k\}_{p,q} &= \Big(\frac{\partial q_i}{\partial z_h}\frac{\partial p_i}{\partial z_\ell} - \frac{\partial q_i}{\partial z_\ell}\frac{\partial p_i}{\partial z_h}\Big)\Big(\frac{\partial z_\ell}{\partial p_j}\frac{\partial z_k}{\partial q_j} - \frac{\partial z_\ell}{\partial q_j}\frac{\partial z_k}{\partial p_j}\Big) \\
&= \frac{\partial p_i}{\partial z_\ell}\frac{\partial z_\ell}{\partial p_j}\frac{\partial z_k}{\partial q_j}\frac{\partial q_i}{\partial z_h} + \frac{\partial q_i}{\partial z_\ell}\frac{\partial z_\ell}{\partial q_j}\frac{\partial z_k}{\partial p_j}\frac{\partial p_i}{\partial z_h} \\
&\quad - \frac{\partial p_i}{\partial z_\ell}\frac{\partial z_\ell}{\partial q_j}\frac{\partial q_i}{\partial z_h}\frac{\partial z_k}{\partial p_j} - \frac{\partial q_i}{\partial z_\ell}\frac{\partial z_\ell}{\partial p_j}\frac{\partial p_i}{\partial z_h}\frac{\partial z_k}{\partial q_j}.
\end{aligned}
$$

Since p and q are independent, the last two terms are zero. For the first two,

$$
\frac{\partial p_i}{\partial z_\ell}\frac{\partial z_\ell}{\partial p_j} = \frac{\partial p_i}{\partial p_j} = \delta_{ij}, \qquad \frac{\partial q_i}{\partial z_\ell}\frac{\partial z_\ell}{\partial q_j} = \frac{\partial q_i}{\partial q_j} = \delta_{ij}.
$$

Therefore

$$
[\![z_h, z_\ell]\!]_{q,p}\,\{z_\ell, z_k\}_{p,q} = \frac{\partial z_k}{\partial q_j}\frac{\partial q_i}{\partial z_h}\delta_{ij} + \frac{\partial z_k}{\partial p_j}\frac{\partial p_i}{\partial z_h}\delta_{ij} = \frac{\partial z_k}{\partial z_h} = \delta_{hk}. \qquad \blacksquare
$$

Remark 8.1c The proposition asserts that the two matrices $([\![z_h, z_\ell]\!]_{q,p})$ and $(\{z_\ell, z_k\}_{p,q})$ are inverses of each other. This is based only on the definitions of Lagrange and Poisson brackets and on the mutual invertibility of the variables $\mathbf{z}$ and (p, q). In particular, it is not required that these transformations be canonical.

8.2c Invariance of Lagrange Brackets by Canonical Transformations

Given a smooth invertible transformation as in (1.2), construct $\mathbf{z}(p, q; t)$ by setting

$$
z_i(q, p; t) = \begin{cases} P_i(q, p; t), & i = 1, 2, \dots, N, \\ Q_i(q, p; t), & i = N+1, N+2, \dots, 2N. \end{cases}
$$

With this notation,

$$
([\![z_h, z_\ell]\!]_{q,p}) = \begin{pmatrix} ([\![P_h, P_k]\!]_{q,p}) & ([\![P_h, Q_k]\!]_{q,p}) \\ ([\![Q_h, P_k]\!]_{q,p}) & ([\![Q_h, Q_k]\!]_{q,p}) \end{pmatrix},
$$

$$
(\{z_h, z_\ell\}_{p,q}) = \begin{pmatrix} (\{P_h, P_k\}_{p,q}) & (\{P_h, Q_k\}_{p,q}) \\ (\{Q_h, P_k\}_{p,q}) & (\{Q_h, Q_k\}_{p,q}) \end{pmatrix}.
$$

By Proposition 8.1, the transformation (1.2) is canonical if and only if

$$
\begin{pmatrix} (\{Q_h, Q_k\}_{q,p}) & (\{Q_h, P_k\}_{q,p}) \\ (\{P_h, Q_k\}_{q,p}) & (\{P_h, P_k\}_{q,p}) \end{pmatrix} = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix}.
$$

It follows from Proposition 8.1c that (1.2) is canonical if and only if

$$
\begin{pmatrix} ([\![Q_h, Q_k]\!]_{q,p}) & ([\![Q_h, P_k]\!]_{q,p}) \\ ([\![P_h, Q_k]\!]_{q,p}) & ([\![P_h, P_k]\!]_{q,p}) \end{pmatrix} = \begin{pmatrix} \mathbb{O} & -\mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}.
$$

Proposition 8.2c *The transformations (1.2) are canonical if and only if*

$$\llbracket p_h, p_k \rrbracket_{P,Q} = \llbracket q_h, q_k \rrbracket_{P,Q} = 0; \quad \llbracket p_h, q_k \rrbracket_{P,Q} = \delta_{hk},$$
$$\llbracket P_h, P_k \rrbracket_{p,q} = \llbracket Q_h, Q_k \rrbracket_{p,q} = 0; \quad \llbracket P_h, Q_k \rrbracket_{p,q} = \delta_{hk}. \tag{8.1c}$$

Corollary 8.1c *The transformation (1.2) is canonical if and only if it preserves the Lagrange brackets.*

11

INTEGRATING HAMILTON–JACOBI EQUATIONS AND CANONICAL SYSTEMS

1 Complete Integrals of Hamilton–Jacobi Equations

A complete integral of the Hamilton–Jacobi equation

$$F_t + \mathcal{H}(q, \nabla_q F; t) = 0 \tag{1.1}$$

is a smooth function F of q and t that depends on N arbitrary parameters $Q = (Q_1, , \dots, Q_N)$ and that satisfies (1.1) as q and Q range within the domain of definition of F. The dependence on the parameters Q is required to be "essential" in the sense that

$$\det \left(\frac{\partial^2 F(q, Q; t)}{\partial q_h \partial Q_k} \right) \neq 0 \tag{1.2}$$

within its domain of definition. Finding a complete integral of (1.1) resolves the motion, as indicated in §5.3 of Chapter 10.

1.1 Examples of Complete Integrals of (1.1)

Since F appears in (1.1) only through its derivatives, from a complete integral $F(q, Q; t)$ one can construct a family of integrals by adding a constant. If $\mathcal{H}$ depends only on p, then (1.1) has the complete integral

$$F(q, Q; t) = q \cdot Q - t\mathcal{H}(Q) + C. \tag{1.3}$$

A complete integral of the eikonal equation $\|\nabla_q F\| = 1$ is

$$F(q, Q) = q \cdot Q + C \quad \text{for } C \in \mathbb{R} \quad \text{and } \|Q\| = 1.$$

A complete integral of Clairaut's equation

$$q \cdot \nabla_q F + f(\nabla_q F) = F \tag{1.4}$$

for a smooth $f : \mathbb{R}^N \to \mathbb{R}$ is

$$F(q, Q) = q \cdot Q + f(Q). \tag{1.5}$$

E. DiBenedetto, *Classical Mechanics: Theory and Mathematical Modeling*, Cornerstones, DOI 10.1007/978-0-8176-4648-6_11,

1.2 Hamiltonian Independent of t

If $\mathcal{H}$ is independent of t, we seek a complete integral of (1.1) of the form

$$F(q,Q;t) = F_o(q,Q) - E(Q)t \tag{1.6}$$

for a smooth $E : \mathbb{R}^N \to \mathbb{R}$, where $F_o(q,Q)$ is a complete integral of the *stationary* Hamilton–Jacobi equation

$$\mathcal{H}(q, \nabla_q F_o) = E(Q). \tag{1.7}$$

If such a complete integral can be determined, the corresponding canonical transformations take the form

$$p = \nabla_q F_o(q,Q), \quad P + \boldsymbol{\omega} t = \nabla_Q F_o(q,Q), \quad \boldsymbol{\omega} = \nabla_Q E(Q). \tag{1.8}$$

Since $\mathcal{H}$ is independent of time, it coincides with the energy of the system, which is conserved. On the other hand, (1.7), written along the motion, gives

$$t \to \mathcal{H}(q(t), p(t)) = E(Q).$$

Therefore the constant $E(Q)$ is the energy of the system determined through the N parameters Q, which remain constant along the motion.

2 Separation of Variables

Denote by $q' = (q_2, \dots, q_N)$ the last $N-1$ of the variables q. Assume that the Hamiltonian is of the form

$$\mathcal{H}(q, \nabla_q F; t) = \mathcal{H}\big(q', \nabla_{q'} F, f_1(q_1, F_{q_1}); t\big), \tag{2.1}$$

where $f_1(\cdot,\cdot)$ is a smooth function of two scalar variables only and is independent of t. Thus $\mathcal{H}$ depends on q_1 and F_{q_1} only through a function f_1 of these, independent of time. Examples of such Hamiltonians are

$$\mathcal{H}(q, \nabla_q F; t) = \sum_{j=1}^N f_j(q_j, F_{q_j}), \qquad \mathcal{H}(q, \nabla_q F; t) = \prod_{j=1}^N f_j(q_j, F_{q_j}),$$

where $f_j(\cdot,\cdot)$ are smooth functions each of two scalar variables and independent of t. More generally, the Hamiltonian could be of the form

$$\mathcal{H}(q, \nabla_q V) = \mathcal{H}\left(f_1\left(q_1, F_{q_1}; f_2\left(q_2, F_{q_2}; f_3\left(\cdots; f_N(q_N, V_{q_N}; \cdots)\right)\right)\right)\right). \tag{2.2}$$

Hamiltonians that do not exhibit the structure (2.1) include

$$\mathcal{H}(q, \nabla_q F) = \frac{\operatorname{div}_q F}{1 + \|q\|}, \qquad \mathcal{H}(q, \nabla_q F; t) = \frac{F_{q_1}}{q_1^2 + t^2}.$$

Assume that $\mathcal{H}$ has the structure (1.1) and seek a solution of (1.1) of the form

$$F(q;t) = F_{N-1}(q';t) + h_1(q_1)$$

for smooth functions $F_{N-1} : \mathbb{R}^N \to \mathbb{R}$ and $h_1 : \mathbb{R} \to \mathbb{R}$. If a solution of this form exists, then we must have

$$\frac{\partial}{\partial t} F_{N-1} + \mathcal{H}\left(q', \nabla_{q'} F_{N-1}, f_1(q_1, h_{1,q_1}); t\right) = 0 \tag{2.3}$$

identically as q_1 ranges within its domain of definition. Therefore $f_1(\cdot,\cdot)$ must be constant, that is,

$$(q_1, h_{1,q_1}) \to f_1(q_1, h_{1,q_1}) = Q_1, \tag{2.4}$$

for some constant Q_1. The latter is a first-order partial differential equation, whose complete integral $F_1(q_1; Q_1)$ is computed by quadratures. Putting now (2.4) into (2.3) gives

$$\frac{\partial}{\partial t} F_{N-1} + \mathcal{H}\left(q', \nabla_{q'} F_{N-1}, Q_1; t\right) = 0. \tag{2.5}$$

This is a Hamilton–Jacobi equation involving only $N-1$ independent variables. Thus if the Hamiltonian has the structure (2.1), then the variable q_1 can be *separated* from the remaining variables q'.

Denote by $q'' = (q_3, \dots, q_N)$ the last $N-2$ of the variables q and assume that the Hamiltonian in (2.5) is of the form

$$\mathcal{H}(q', \nabla_{q'} F_{N-1}, Q_1; t) = \mathcal{H}\left(q'', \nabla_{q''} F_{N-1}, f_2(q_2, F_{N-1,q_2}; Q_1)\right),$$

where $f_2(\cdot,\cdot;Q_1)$ is a smooth function of two scalar variables and is independent of t. Then we seek a solution of (2.4) of the form

$$F_{N-1}(q';t) = F_{N-2}(q'';t) + h_2(q_2; Q_1).$$

If a solution of this form exists, then

$$(q_2, h_{2,q_2}) \to f_2(q_2, h_{2,q_2}; Q_1) = Q_2,$$

for some constant Q_2. The latter can be integrated by quadratures to give the complete integral $F_2(q_2; Q_1, Q_2)$. Therefore an integral of (1.1) has the form

$$F(q;t) = F_{N-2}(q'';t) + F_1(q_1; Q_1) + F_2(q_2; Q_1, Q_2),$$

where $F_{N-2}(q'';t)$ is a solution of the Hamilton–Jacobi equation in $N-2$ independent variables

$$\frac{\partial}{\partial t} F_{N-2} + \mathcal{H}\left(q'', \nabla_{q''} F_{N-2}, Q_1, Q_2; t\right) = 0.$$

If this process can be repeated N times, all the variables q are separated, and the Hamilton–Jacobi equation (1.1) is called *completely separable*.

2.1 An Example of Complete Separability

Assume that the Hamiltonian $\mathcal{H}$ in (1.1) is independent of t and has the general structure (2.2). Then set

$$F_o(q,Q) = F_1(q_1,Q_1) + F_2(q_2;Q_1,Q_2) + \cdots + F_N(q_N;Q_1,\dots,Q_N),$$

where for $j = 1,\dots,N$, the function F_j is the complete integral of the first-order equation

$$f_j(q_j, F_{j,q_j}; Q_1,\dots,Q_{j-1}) = Q_j$$

in the single variable q_j. Regarding now t as the $(N+1)$st variable, we seek a complete integral of (1.1) of the form

$$(q,Q;t) \to F(q,Q;t) = F_o(q,Q) + F_{N+1}(t,Q).$$

Putting this into (1.1) and taking into account the construction of the f_j and the structure (2.2) of the Hamiltonian gives

$$\frac{\partial}{\partial t}F_{N+1}(t,Q) = -\mathcal{H}(q,\nabla_q F_o) = -E(Q).$$

Thus $F_{N+1} = -tE(Q)$, and the complete integral of (1.1) is

$$(q,Q) \to F(q;Q) = F_o(q,Q) - tE(Q),$$

where $(q,Q) \to F_o(q,Q)$ is the complete integral of the corresponding stationary Hamilton–Jacobi equation.

2.2 Cyclic Variables

If $\mathcal{H}$ has a cyclic variable, say for example q_1, then it is of the form (2.1) with $f_1(q_1, F_{q_1}) = F_{q_1}$. Now F_{q_1} is the kinetic moment p_1 corresponding to the Lagrangian coordinate q_1 (see (3.2) of Chapter 10 and §5.3 of the same chapter). Since q_1 is cyclic, $F_{q_1} = Q_1$ for a constant Q_1. Therefore a complete integral of (1.1) can be sought of the form

$$F(q,Q;t) = F'(q';t) + Q_1 q_1,$$

where F' is a complete integral of the Hamilton–Jacobi equation (2.5) in $N-1$ independent variables. Further cyclic variables permit one to reduce further the number of independent variables.

3 Reducing the Rank of a Canonical System

The rank of the canonical system

$$\dot{p} = -\nabla_q \mathcal{H}(q,p;t), \qquad \dot{q} = \nabla_p \mathcal{H}(q,p;t), \tag{3.1}$$

is defined as the nonnegative integer

$$\text{rank of the matrix } \left(\frac{\partial^2 \mathcal{H}(p,q;t)}{\partial p_h \partial q_k}\right).$$

The general theory of systems of first-order differential systems ensures that the knowledge of a first integral of (3.1) lowers the rank of the system from $2N$ to $(2N-1)$. However, for canonical systems as in (3.1), the knowledge of a first integral lowers the rank from $2N$ to $2(N-1)$.

In the next sections we will verify this property for specific physical integrals, such as the energy integral and the integrals of the kinetic momenta.

3.1 Canonical Systems with the Energy Integral

If $\mathcal{H}$ is independent of t, the mechanical system has the energy integral

$$t \to \mathcal{H}\big(p(t), q(t)\big) = E = \text{const}$$

for t about some initial t_o. Without loss of generality we may assume that for at least one of the variables q, say for example q_N,

$$|\dot{q}_N| > 0 \quad \text{and} \quad |\mathcal{H}_{p_N}(p,q)| > 0 \tag{3.2}$$

for all t about t_o. Then by the implicit function theorem t can be locally resolved in terms of q_N, which in turn may be taken as an independent variable. To stress such a choice of independent variables, set

$$q_N = \eta, \qquad t = t(\eta) \qquad \text{in a neighborhood of } t_o.$$

For such a choice, along the motion $t \to \big(p(t), q(t)\big)$, a solution of (3.1), set

$$\begin{aligned} \eta &\to \bar{p}(\eta) = (p_1(t(\eta)), \dots, p_{N-1}(t(\eta))), \\ \eta &\to \bar{q}(\eta) = (q_1(t(\eta)), \dots, q_{N-1}(t(\eta))). \end{aligned}$$

The second inequality of (3.2) and the implicit function theorem permit one also to compute

$$\eta \to p_N(\eta) = \varphi(\bar{p}(\eta), \bar{q}(\eta); \eta)$$

for a smooth $(\bar{p}, \bar{q}; \eta) \to \varphi(\bar{p}, \bar{q}; \eta)$. Next for $h = 1, \dots, N-1$, compute

$$\begin{aligned} \frac{d}{d\eta}\bar{p}_h &= \frac{\dot{p}_h}{\dot{q}_N} = -\left(\frac{\partial \mathcal{H}}{\partial q_h}\right) / \left(\frac{\partial \mathcal{H}}{\partial p_N}\right) \stackrel{\text{def}}{=} G_h\left(\bar{q}, \bar{p}, \eta, \varphi(\bar{q}, \bar{p}, \eta)\right), \\ \frac{d}{d\eta}\bar{q}_h &= \frac{\dot{q}_h}{\dot{q}_N} = \left(\frac{\partial \mathcal{H}}{\partial p_h}\right) / \left(\frac{\partial \mathcal{H}}{\partial p_N}\right) \stackrel{\text{def}}{=} F_h\left(\bar{q}, \bar{p}, \eta, \varphi(\bar{q}, \bar{p}, \eta)\right). \end{aligned}$$

This is a system whose rank does not exceed $2(N-1)$.

3.2 Cyclic Variables

Assume that $\mathcal{H}$ in (3.1) is independent of ℓ components of the Lagrangian configuration q, say, for example after a possible reordering, $(q_1, \dots, q_\ell)$. In such a case the first ℓ kinetic momenta are constant along the motion, say for example $t \to p_j(t) = p_{o,j}$ for $j = 1, \dots, \ell$. Therefore $\mathcal{H}$ might be considered as independent of the first ℓ Lagrangian positions q_j as well as the corresponding kinetic momenta p_j. Setting

$$\begin{aligned} \tilde{p} &= (p_{\ell+1}, \dots, p_N), \\ \tilde{q} &= (q_{\ell+1}, \dots, q_N), \end{aligned} \qquad \tilde{\mathcal{H}}(\tilde{p}, \tilde{q}; t) = \mathcal{H}(\tilde{p}, \tilde{q}, p_{o,1}, \dots, p_{o,\ell}; t),$$

the canonical system (3.1) is transformed into the system

$$\dot{p}_h = -\frac{\partial \tilde{\mathcal{H}}(\tilde{q}, \tilde{p}; t)}{\partial q_h}, \quad \dot{q}_h = \frac{\partial \tilde{\mathcal{H}}(\tilde{q}, \tilde{p}; t)}{\partial p_h} \quad \text{for } \; h = \ell + 1, \dots, N,$$

of rank not exceeding $2(N - \ell)$. If $\mathcal{H}$ is independent of all the N components of the Lagrangian positions q, then all N kinetic momenta p are constant. These are N integrals of motion, and the resulting system is of rank zero. Equivalently, if all the Lagrangian coordinates are cyclic, the N integrals of the corresponding kinetic momenta resolve the motion. This is a special case of the next Liouville theorem.

4 The Liouville Theorem of Quadratures

Assume that the canonical system (3.1) has N independent first integrals

$$(p, q; t) \to F_i(p, q; t) = Q_i \qquad i = 1, \dots, N, \tag{4.1}$$

where $Q = (Q_1, \dots, Q_N)$ are constants. These are independent in the sense that the $N \times 2N$ matrix

$$\left(\frac{\partial F_i(p, q; t)}{\partial p_j} \; \frac{\partial F_\ell(p, q; t)}{\partial q_k} \right)_{i,j,\ell,k=1,\dots,N} \tag{4.2}$$

has maximum rank in a neighborhood of some fixed initial configuration. Assume, moreover, that these first integrals are in *involution* with respect to the Poisson brackets, that is,

$$\{F_i, F_j\} = \sum_{h=1}^{N} \left(\frac{\partial F_i}{\partial p_h} \frac{\partial F_j}{\partial q_h} - \frac{\partial F_i}{\partial q_h} \frac{\partial F_j}{\partial p_h} \right) = 0, \quad i, j = 1, \dots, N. \tag{4.3}$$

Theorem 4.1 (Liouville [116,117]). *N independent first integrals of (3.1), in involution, resolve the motion.*

Remark 4.1 This is a special case of a more general theorem of Lie ([115]; also in [64, Chap.VIII]), which asserts that ℓ independent first integrals in involution reduce (3.1) into another one, still canonical and of rank $2(N-\ell)$. Liouville's theorem follows from this for $\ell = N$.

The integration of (3.1) hinges upon finding a complete integral of the Hamilton–Jacobi equation (1.1). The proof of Liouville's theorem consists in constructing such a complete integral from the independent first integrals F_j in involution.

4.1 Poisson Brackets in $2(N+1)$ Variables

Denote the time t by q_o and introduce the two $(N+1)$-tuples of variables

$$(p_o, p) = (p_o, p_1, \dots, p_N), \qquad (q_o, q) = (q_o, q_1, \dots, q_N),$$

where p_o is a scalar auxiliary variable. Set also

$$(p_o, p, q_o, q) \to F_o(p_o, p, q_o, q) = p_o + \mathcal{H}(p, q; q_o)$$

and regard the first integrals F_j as functions of the $2(N+1)$ variables (p_o, p, q_o, q), independent of p_o.

Lemma 4.1 *The $(N+1)$ functions $\{F_o, F_1, \dots, F_N\}$ are in involution with respect to the two $(N+1)$-tuples of variables (p_o, p) and (q_o, q), i.e., denoting by $\{\{\cdot,\cdot\}\}$ the Poisson brackets with respect to these two $(N+1)$-tuples of variables*

$$\{\{F_i, F_j\}\} = \frac{\partial F_i}{\partial p_o}\frac{\partial F_j}{\partial q_o} - \frac{\partial F_i}{\partial q_o}\frac{\partial F_j}{\partial p_o} + \{F_i, F_j\} = 0$$

for all $i, j = 0, 1, \dots, N$.

Proof. If $i, j \neq 0$, the assertion follows since F_j are independent of p_o and are in involution with respect to the two N-tuples of variables p and q. If $i = 0$ and $j \neq 0$, then

$$\{\{F_o, F_j\}\} = F_{j,t} + \{\mathcal{H}, F_j\} = 0$$

by Proposition 10.1 of Chapter 10. ∎

4.2 Proof of Liouville's Theorem

Without loss of generality, we assume that the independence of the first integrals $\{F_1, \dots, F_N\}$ as expressed in (4.2) is realized with respect to the kinetic momenta p_j, that is,

$$\det\left(\frac{\partial F_i(q, p; t)}{\partial p_j}\right)_{i,j=1,\dots,N} \neq 0 \tag{4.4}$$

in a neighborhood of some fixed initial configuration. Then by the implicit function theorem, each of the p_j for $j = 1, \dots, N$ can be expressed in terms of $(q, t; Q)$ on the set $\sigma = \bigcap_{i=1}^{N} [F_i = Q_i]$. Such a set is nonempty, since the F_j are first integrals of (3.1). Now on σ, also p_o can be expressed in terms of $(q, Q; t)$ by setting $F_o = 0$. Thus on σ,

$$p_i = p_i(q, Q; t) \quad \text{for } \; i = 1, \dots, N$$

and

$$p_o = -\mathcal{H}\big(q, p_1(q, Q; t), \dots, p_N(q, Q; t); t\big).$$

Lemma 4.2 *The differential form*

$$pdq = p_o(q, Q; q_o)dq_o + \sum_{i=1}^{N} p_i(q, Q; q_o)dq_i = -\mathcal{H}dt + \sum_{i=1}^{N} p_i dq_i$$

is exact on σ.

Proof. Consider the $(N+1) \times (N+1)$ matrices

$$A = -\left(\frac{\partial F_i}{\partial q_j}\right), \qquad B = \left(\frac{\partial F_i}{\partial p_j}\right), \qquad C = \left(\frac{\partial p_i}{\partial q_j}\right).$$

It suffices to show that C is symmetric in σ. Indeed, in such a case

$$\frac{\partial p_i}{\partial q_j} = \frac{\partial p_j}{\partial q_i} \qquad \text{for all } \; i, j = 0, 1, \dots, N,$$

which is the characteristic condition for pdq to be exact. On σ,

$$0 = \frac{dF_i}{dq_j} = \frac{\partial F_i}{\partial q_j} + \frac{\partial F_i}{\partial p_\ell}\frac{\partial p_\ell}{\partial q_j}.$$

Therefore $A = BC$ on σ, and since B is nondegenerate, $C = B^{-1}A$. Thus C is symmetric if

$$B^{-1}A = A^t(B^t)^{-1} \iff \quad B^tA = BA^t \iff B^tA - BA^t = \mathbb{O}.$$

The latter, written in components, is precisely the condition $\{\{F_i, F_j\}\} = 0$ for all $i, j \in \{0, 1, \dots, N\}$, ensured by Lemma 4.1. ■

Let $(q, Q; t) \to F(q, Q; t)$ be a primitive of pdq whose existence is ensured by the previous lemma. Then

$$p_i(q, Q; t) = \frac{\partial F(q, Q; t)}{\partial q_i} \qquad \text{for } \; i = 0, 1, \dots, N.$$

For $i \neq 0$ this gives $p = \nabla_q F$, and for $i = 0$,

$$\frac{\partial}{\partial t} F(q, Q; t) = -\mathcal{H}(q, \nabla_q F(q, Q; t); t).$$

Therefore F is a family of solutions of the Hamilton–Jacobi equation (1.1), depending on the N-dimensional parameter Q. To show that it is a complete integral, it remains to show that it satisfies (1.2). Since $F_h = Q_h$ on σ,

$$\frac{\partial^2 F(q, Q; t)}{\partial q_h \partial Q_k} = \frac{\partial F_h(q; t)}{\partial Q_k} = \delta_{hk}. \qquad \blacksquare$$

Problems and Complements

1c Complete Integrals of Hamilton–Jacobi Equations

1.1c Envelopes of Solutions

Let $(q, Q; t) \to F(q, Q; t)$ be a complete integral of (1.1) and seek a smooth function $(q; t) \to U(q; t)$ whose graph, for t fixed, is tangent at each of its points $(\eta, U(\eta; t))$ to the graph of $(q, Q(\eta); t) \to F(q, Q(\eta; t); t)$ for some η within the range of the parameters Q. Such a function, if it exists, is the *envelope* of the family of solutions $F(\cdot, Q; t)$ of (1.1) parameterized by Q. From the definition,

$$U(q; t) = F(q, Q(q; t); t) \quad \text{and} \quad \nabla U(q) = \nabla_q F(q, Q; t) \Big|_{Q=Q(q;t)} .$$

Therefore finding the envelope of a complete integral $F(q; Q; t)$ reduces to finding $Q = Q(q; t)$ and putting this in the expression for F.

From the first of these, for fixed t,

$$\nabla U = \nabla_q F = \nabla_q F + \nabla_Q F \cdot \nabla_q Q.$$

Therefore

$$F_{Q_i}(q, Q(q; t); t) Q_{i,q_j} = 0, \qquad j = 1, \dots, N. \tag{1.1c}$$

This is a linear homogeneous system in the unknowns $F_{Q_i}(q, Q(q; t); t)$ that admits only the zero solution if $\det(Q_{i,q_j}) \neq 0$. In such a case the functions $Q = Q(q; t)$ can be computed from

$$F_{Q_i}(q, Q(q; t); t) = 0, \qquad i = 1, \dots, N.$$

1.1.1c Examples of Envelopes

(a) In the Clairaut equation (1.4) take $f(\nabla_q F) = -\frac{1}{2}\|\nabla_q F\|^2$, and show that the corresponding envelope is $U(q) = \frac{1}{2}\|q\|^2$.

(b) For the eikonal equation show that the envelope of its complete integral is $U(q) = \|q\|$. This is a case when the determinant of the coefficients in (1.1c) vanishes. Discuss how the calculation $Q = Q(q)$ is actually done.

(c) If $\mathcal{H} = \|p\|^2$, the envelope of the complete integral of corresponding Hamilton–Jacobi equation is $U(q) = \|q\|^2/4t$.

2c Separation of Variables

Consider a point mass $\{P; m\}$ acted upon by a potential $x \to V(x, \|x\|)$ with nontrivial dependence on $\|x\|$. The Lagrangian is

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\|\dot{x}\|^2 + V(x, \|x\|). \tag{2.1c}$$

The kinetic momenta are $p_i = m\dot{x}_i$ and the Hamiltonian is

$$\mathcal{H}(x, p) = \frac{1}{2m}\|p\|^2 - V(x, \|x\|). \tag{2.2c}$$

The corresponding Hamilton–Jacobi equation is

$$F_t + \frac{1}{2m}\|\nabla_x F\|^2 - V(x, \|x\|) = 0. \tag{2.3c}$$

Since V has a nontrivial dependence on $\|x\|$, the Hamiltonian does not have the structure (2.1) with respect to any of the pairs of variables (x_i, F_{x_i}). As an example, consider the case of a gravitational potential $V(x) = \|x\|^{-1}$. However, having fixed V, at times it might be possible to choose Lagrangian coordinates for which $\mathcal{H}$ has the structure (2.1). In the next sections we will indicate some of these choices.

2.1c Spherical Coordinates

For these coordinates the Lagrangian has the form (§1.3c of the Complements of Chapter 2)

$$\mathcal{L} = \frac{1}{2}m\big(\dot{\rho}^2 + \dot{\varphi}^2\rho^2\sin^2\theta + \dot{\theta}^2\rho^2\big) + V(\rho, \varphi, \theta). \tag{2.4c}$$

From this one computes the kinetic momenta and the Hamiltonian,

$$p_\rho = m\dot{\rho}, \qquad p_\varphi = m\dot{\varphi}\rho^2\sin^2\theta, \qquad p_\theta = m\dot{\theta}\rho^2,$$

$$\mathcal{H} = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2\sin^2\theta} + \frac{p_\theta^2}{\rho^2}\right) - V(\rho, \varphi, \theta). \tag{2.5c}$$

Assume that the potential V is of the form

$$-2mV(\rho,\varphi,\theta) = g(\rho) + \frac{h(\theta)}{\rho^2}$$

for smooth functions $g(\cdot)$ and $h(\cdot)$. Then

$$\mathcal{H} = \frac{1}{2m}\Big(p_\rho^2 + g(\rho) + \frac{p_\theta^2 + h(\theta)}{\rho^2} + \frac{p_\varphi^2}{\rho^2\sin^2\theta}\Big),$$

and the corresponding Hamilton–Jacobi equation is

$$F_t + \frac{1}{2m}\Big[F_\rho^2 + g(\rho) + \frac{1}{\rho^2}\Big(F_\theta^2 + h(\theta) + \frac{1}{\sin^2\theta}F_\varphi^2\Big)\Big] = 0. \tag{2.6c}$$

Since $\mathcal{H}$ is cyclic with respect to the variable φ, we seek a solution of the form

$$F(\rho,\varphi,\theta;t) = F'(\rho,\theta,Q_\varphi;t) + Q_\varphi\varphi,$$

where Q_φ is a scalar constant and F' is a solution of the Hamilton–Jacobi equation

$$F'_t + \frac{1}{2m}\Big[F'^2_\rho + g(\rho) + \frac{1}{\rho^2}\Big(F'^2_\theta + h(\theta) + \frac{Q_\varphi^2}{\sin^2\theta}\Big)\Big] = 0$$

in the sole variables ρ and θ. Since this has the structure (2.1) with respect to θ, we seek a solution of the form

$$F'(\rho,\theta,Q_\varphi;t) = F''(\rho,Q_\varphi,Q_\theta;t) + W(\theta,Q_\varphi,Q_\theta),$$

where Q_θ is a constant and W is a complete integral of the first-order partial differential equation

$$W_\theta^2 + h(\theta) + \frac{Q_\varphi^2}{\sin^2\theta} = Q_\theta \implies W = \int\sqrt{Q_\theta - h(\theta) - \frac{Q_\varphi^2}{\sin^2\theta}}\,d\theta.$$

The function $\rho\to F''(\rho,Q_\varphi,Q_\theta;t)$ is a solution of the Hamilton–Jacobi equation

$$F''_t + \frac{1}{2m}\Big(F''^2_\rho + g(\rho) + \frac{Q_\theta}{\rho^2}\Big) = 0$$

in the sole variable ρ. This last equation is separable with respect to ρ and t and gives an integral of the form

$$F''(\rho,Q_\varphi,Q_\theta;t) = W'(\rho,Q) - tE(Q)t, \quad Q = (Q_\rho,Q_\varphi,Q_\theta),$$

where $Q \to E(Q)$ is a smooth function of Q, and W' is a complete integral of the stationary Hamilton–Jacobi equation

$$W_\rho'^2 + g(\rho) + \frac{Q_\theta}{\rho^2} = 2mE(Q) \implies W' = \int \sqrt{2mE(Q) - g(\rho) - \frac{Q_\theta}{\rho^2}}\, d\rho.$$

Combining these remarks, the complete integral of (2.6c) is given by

$$\begin{aligned} F(\rho, \varphi, \theta, Q_\rho, Q_\varphi, Q_\theta; t) &= -tE(Q_\rho, Q_\varphi, Q_\theta)t + Q_\varphi \varphi \\ &+ \int \sqrt{Q_\theta - h(\theta) - \frac{Q_\varphi^2}{\sin^2\theta}}\, d\theta + \int \sqrt{2mE(Q) - g(\rho) - \frac{Q_\theta}{\rho^2}}\, d\rho. \end{aligned}$$

Using this complete integral resolves the motion.

2.2c Parabolic Coordinates

Lagrangian, kinetic momenta, and Hamiltonian are (§1.2c of the Complements of Chapter 2)

$$\mathcal{L} = \frac{1}{2} m \Big(\frac{u+v}{4} \Big(\frac{\dot{u}^2}{u} + \frac{\dot{v}^2}{v} \Big) + \dot{\varphi}^2 uv \Big) + V(u, v, \varphi),$$

$$p_u = m\frac{u+v}{4u}\dot{u}, \qquad p_v = m\frac{u+v}{4v}\dot{v}, \qquad p_\varphi = muv\dot{\varphi},$$

$$\mathcal{H} = \frac{2}{m}\frac{up_u^2 + vp_v^2}{u+v} + \frac{1}{2m}\frac{p_\varphi^2}{uv} - V(u, v, \varphi).$$

Assume that V is of the form

$$V(u, v, \varphi) = -\frac{g(u) + h(v)}{u+v}$$

for smooth functions $g(\cdot)$ and $h(\cdot)$. Then

$$\mathcal{H} = \frac{1}{m(u+v)} \Big(2(up_u^2 + vp_v^2) + \frac{(u+v)}{2uv} p_\varphi^2 + m\left(g(u) + h(v)\right) \Big),$$

and the corresponding Hamilton–Jacobi equation is

$$F_t + \frac{1}{m(u+v)} \Big[2uF_u^2 + 2vF_v^2 + \frac{(u+v)}{2uv} F_\varphi^2 + m\left(g(u) + h(v)\right) \Big] = 0.$$

Since $\mathcal{H}$ is cyclic in the variables φ and t, we seek a complete integral of the form

$$F(\rho, \varphi, \theta, Q, E) = F'(u, v, Q) + Q_\varphi \varphi - tE(Q),$$

where $Q = (Q_u, Q_v, Q_\varphi)$ and $E(Q)$ are constants and F' is a complete integral of the stationary Hamilton–Jacobi equation

$$\frac{1}{m(u+v)}\left[2uF_u'^2 + 2vF_v'^2 + \frac{(u+v)}{2uv}Q_\varphi^2 + m\left(g(u)+h(v)\right)\right] = E(Q)$$

in the sole variables u and v. Multiplying by $(u+v)$ and collecting homologous terms gives the completely separable equation

$$\left(2uF_u'^2 + mg(u) + \frac{Q_\varphi^2}{2u} - mE(Q)u\right) + \left(2vF_v'^2 + mh(v) + \frac{Q_\varphi^2}{2v} - mE(Q)v\right) = 0.$$

The complete integral of this can be sought in the separated form

$$(u,v,Q) \to F'(u,v,Q) = F_1'(u,Q) + F_2'(v,Q),$$

$$F_1'(u,Q) = \int \frac{1}{2u}\sqrt{2mu\left[uE(Q) - g(u)\right] - Q_\varphi^2 + 2uQ_v}\, du,$$
$$F_2'(v,Q) = \int \frac{1}{2v}\sqrt{2mv\left[vE(Q) - h(v)\right] - Q_\varphi^2 - 2vQ_u}\, dv.$$

Write down the complete integral and resolve the motion.

2.3c Elliptic Coordinates

Lagrangian, kinetic momenta, and Hamiltonian are (§1.1c of the Complements of Chapter 2)

$$\mathcal{L} = \frac{1}{2}m\left[\ell^2(\dot{u}^2 + \dot{\theta}^2)(\sinh u^2 + \sin^2\theta) + 2\ell^2\dot{\varphi}^2\sinh^2 u\sin^2\theta\right] + V(u,\varphi,\theta),$$

$$\begin{aligned} p_u &= m\ell^2(\sinh^2 u + \sin^2\theta)\dot{u},\\ p_\varphi &= 2m\ell^2\left(\sinh u\sin\theta\right)^2\dot{\varphi},\\ p_\theta &= m\ell^2(\sinh^2 u + \sin^2\theta)\dot{\theta}, \end{aligned}$$

$$\mathcal{H} = \frac{1}{2m\ell^2}\left(\frac{p_u^2 + p_\theta^2}{\sinh^2 u + \sin^2\theta} + \frac{p_\varphi^2}{2(\sinh u\sin\theta)^2}\right) - V(u,\varphi,\theta).$$

Assume that V is the gravitational potential generated by two masses situated at two points P_1 and P_2. Take first a Cartesian coordinate system such that the two points are on the x_3-axis and symmetric with respect to the origin, say for example $P_1 = (0,0,\ell)$ and $P_2 = (0,0,-\ell)$ for some $\ell > 0$. Such a potential has the form

$$P \to V(P) = -\frac{\gamma_1}{\|P - P_1\|} - \frac{\gamma_2}{\|P - P_2\|},$$

where γ_i for $i = 1, 2$ are given positive constants. Expressing P first in Cartesian coordinates and then in terms of elliptic coordinates gives

$$\begin{aligned} -V(P) &= \frac{\gamma_1}{\sqrt{(x_1^2 + x_2^2) + (x_3 - \ell)^2}} + \frac{\gamma_2}{\sqrt{(x_1^2 + x_2^2) + (x_3 + \ell)^2}} \\ &= \frac{\ell\gamma_1}{\cosh u - \cos\theta} + \frac{\ell\gamma_2}{\cosh u + \cos\theta}. \end{aligned}$$

From this we obtain

$$\begin{aligned} \mathcal{H} = \frac{1}{2m\ell^2}\Big(&\frac{p_u^2 + p_\theta^2}{\sinh^2 u + \sin^2\theta} + \frac{p_\varphi^2}{2(\sinh u \sin\theta)^2}\Big) \\ &+ \frac{\ell\gamma_1}{\cosh u - \cos\theta} \quad + \frac{\ell\gamma_2}{\cosh u + \cos\theta}, \end{aligned}$$

and the associated Hamilton–Jacobi equation is

$$\begin{aligned} F_t + \frac{1}{2m\ell^2}\Big(&\frac{F_u^2 + F_\theta^2}{\sinh^2 u + \sin^2\theta} + \frac{F_\varphi^2}{2(\sinh u \sin\theta)^2}\Big) \\ &+ \frac{\ell\gamma_1}{\cosh u - \cos\theta} + \frac{\ell\gamma_2}{\cosh u + \cos\theta} = 0. \end{aligned}$$

Since φ and t are cyclic, we seek an integral of the form

$$F(u, \varphi, \theta) = F'(u, \theta, Q) + Q_\varphi \varphi - tE(Q),$$

where $Q = (Q_u, Q_v, Q_\varphi)$ and $E(Q)$ are constants and F' is a complete integral of the stationary Hamilton–Jacobi equation

$$\begin{aligned} \frac{1}{2m\ell^2}\Big(&\frac{F_u'^2 + F_\theta'^2}{\sinh^2 u + \sin^2\theta} + \frac{Q_\varphi^2}{2(\sinh u \sin\theta)^2}\Big) \\ &+ \frac{\ell\gamma_1}{\cosh u - \cos\theta} + \frac{\ell\gamma_2}{\cosh u + \cos\theta} = E(Q) \end{aligned}$$

in the sole variables u and θ. Multiply both sides by

$$\sinh^2 u + \sin^2\theta = \cosh^2 u - \cos^2\theta = (\cosh u - \cos\theta)(\cosh u + \cos\theta)$$

and collect homologous terms to get

$$\begin{aligned} &\Big(F_u'^2 + \frac{Q_\varphi^2}{2\sinh^2 u} - 2m\ell^2 E(Q)\cosh^2 u + 2m\ell^2(\gamma_1 + \gamma_2)\cosh u\Big) \\ &\quad + \Big(F_\theta'^2 + \frac{Q_\varphi^2}{2\sin^2\theta} + 2m\ell^2 E(Q)\cos^2\theta + 2m\ell^2(\gamma_1 - \gamma_2)\cos\theta\Big) = 0. \end{aligned}$$

This is a completely separable stationary Hamilton–Jacobi equation. Compute a complete integral and resolve the motion.

2.4c Force-Free $\{P; m\}$ in $\mathbb{R}^3$

If $V = 0$, then (2.3c) has the complete integral

$$F(x, Q; t) = Q \cdot x - \frac{1}{2m}\|Q\|^2 t, \qquad Q = (Q_1, Q_2, Q_3).$$

From this and the integration method of §5.3 of Chapter 10, we have

$$m\dot{x}_i = Q_i, \quad P_i + \omega_i t = -x_i, \quad \omega_i = \frac{Q_i}{m}, \quad i = 1, 2, 3,$$

where Q_i and P_i are determined by the initial data.

2.5c The Cycloidal Pendulum

The Lagrangian of a harmonic oscillator, its kinetic moment, and its Hamiltonian are (§7.2c of Chapter 3; here q is the arc length on the cycloid and $\omega^2 = g/4R$)

$$\mathcal{L} = \frac{m}{2}(\dot{q}^2 + \omega^2 q^2), \quad p = m\dot{q}, \quad \mathcal{H} = \frac{1}{2m}(p^2 + m^2\omega^2 q^2),$$

The corresponding Hamilton–Jacobi equation is

$$F_t + \frac{1}{2m}(F_q^2 + m^2\omega^2 q^2) = 0,$$

whose complete integral is

$$F(q, Q; t) = \int \sqrt{2mE(Q) - m^2\omega^2 q^2}\, dq - tE(Q).$$

Resolve the motion starting from this integral.

12

INTRODUCTION TO FLUID DYNAMICS

1 Geometry of Deformations

A bounded open connected set in $E_o \subset \mathbb{R}^N$ deforms in time to E in the sense that points $y \in E_o$ are in one-to-one correspondence with points $x \in E$ through smooth, nonintersecting trajectories $t \to x(t)$ such that $x(0) = y$ and $x(t) = x$. This defines a flow map and a velocity field

$$x = \Phi(y,t), \qquad \mathbf{v}(x,t) = \Phi_t(y,t). \tag{1.1}$$

The functions $\Phi(\cdot,t)$ may be regarded as a family of transformations defined in E_o and parameterized with t. These transformation will be assumed to be smooth and invertible independent of t. In the Lagrangian formalism, kinematic information on $x(t) \in E$ is provided by the trajectories $t \to x(t)$, independently of their membership to E, as an open connected subset of $\mathbb{R}^N$, this bearing a role only in the determination of such paths [100]. In the Eulerian formalism, kinematic information on points $x \in E$ is provided by the flow map $\Phi(\cdot,t)$, which bears the "globality" of E_o and E [47, 48]. In both formalisms these quantities must coincide. Therefore $\dot{x} = \mathbf{v}(x,t)$ and

$$\ddot{x} = \frac{d}{dt}\dot{x} = \frac{\partial}{\partial t}\mathbf{v}(x,t) + \dot{x}\cdot\nabla_x\mathbf{v} = D_t\mathbf{v}, \tag{1.2}$$

where the operator D_t formally defined by

$$D_t = \frac{\partial}{\partial t} + \mathbf{v}\nabla_x \tag{1.3}$$

is the *total* or *material derivative* along Lagrangian paths. For t fixed, the Jacobian of the transformation $\Phi(\cdot,t)$ is

$$J(x,t) = J[\Phi(y,t),t)] = \det\Big(\frac{\partial\Phi_i(y,t)}{\partial y_j}\Big) = A_{ij}\frac{\partial\Phi_k(y,t)}{\partial y_j} = J\delta_{ik},$$

where A_{ij} is the determinant of the algebraic complement of the (ij)th entry of the Jacobian matrix $\nabla\Phi$.

E. DiBenedetto, *Classical Mechanics: Theory and Mathematical Modeling*, Cornerstones, DOI 10.1007/978-0-8176-4648-6_12,

Proposition 1.1 (Euler [47]) $D_t J = J \operatorname{div} \mathbf{v}$.

Proof. From the previous expression of J, we have[1]

$$\begin{aligned} D_t J(x,t) &= \frac{\partial}{\partial t} \det \Big(\frac{\partial \Phi_i(y,t)}{\partial y_j} \Big) = A_{ij} \frac{\partial}{\partial y_j} \frac{\partial \Phi_i(y,t)}{\partial t} \\ &= A_{ij} \frac{\partial v_i}{\partial y_j} = A_{ij} \frac{\partial v_i}{\partial x_k} \frac{\partial \Phi_k(y,t)}{\partial y_j} = \frac{\partial v_i}{\partial x_i} J. \end{aligned}$$ ∎

1.1 Incompressible Deformations

If an infinitesimal portion about any $y \in E_o$ moves by possibly changing its shape and/or configuration, but keeping fixed its infinitesimal volume, then $D_t J = 0$ and consequently $\operatorname{div} \mathbf{v} = 0$, and the deformation is called incompressible. Conversely, a deformation is incompressible if and only if $\operatorname{div} \mathbf{v} = 0$.

1.2 The Equation of Continuity

Let $G \subset \mathbb{R}^N$ be open and let $E(t) \subset G$ be a deforming subdomain of G with smooth boundary $\partial E(t)$. For a smooth function $(x,t) \to \rho(x,t)$ defined in a neighborhood of $E(t)$, by the previous proposition,

$$\begin{aligned} \frac{d}{dt} \int_{E(t)} \rho(x,t) dx &= \frac{d}{dt} \int_{E_o} \rho(\Phi(y,t),t) J dy \\ &= \int_{E_o} \frac{d}{dt} \rho[\Phi(y,t),t) J] dy \\ &= \int_{E_o} [(\Phi_t \cdot \nabla_x \rho + \rho_t) J + \rho J_t] dy \\ &= \int_{E_o} [\rho_t + \operatorname{div}(\rho \mathbf{v})] J dy \\ &= \int_{E(t)} [\rho_t + \operatorname{div}(\rho \mathbf{v}) J dx. \end{aligned}$$

If $\rho(x,t)$ is the material density of a body occupying the domain G, then for every deforming subset $E(t) \subset G$,

$$\int_{E(t)} \rho(x,t) dx = \text{mass of the body in } E(t).$$

[1]The derivative of the determinant of $n \times n$ matrix is the sum of n determinants obtained from the original matrix upon substitution of each row (column) by the row (column) of the corresponding derivatives.

If elements of G evolve conserving their mass, then

$$\frac{d}{dt}\int_{E(t)} \rho(x,t)dx = 0 \qquad \text{for all deforming subdomains } E(t) \subset G.$$

Since $E(t) \subset G$ is arbitrary, local deformations of G preserve the mass if and only if

$$\rho_t + \operatorname{div}(\rho\mathbf{v}) = 0 \qquad \text{pointwise in } G. \tag{1.4}$$

This is the *continuity equation*, and it expresses conservation of mass.

2 Cardinal Equations

Along the motion, points $x \in E \subset G$ are acted upon by a material distributions of forces $\mathbf{f}(x,\dot{x};t)\rho(x,t)dx$, and by reactions acting on ∂E due to the remaining portion $G - E$ that opposes the possible deformation of E. These are a priori unknown, depend on the material structure of G, and should not depend on the particular subdomain $E \subset G$. In the Cauchy formalism they are represented by a smooth vector-valued function

$$G \times S_1 \times \mathbb{R} \ni (x,\mathbf{n},t) \to \mathbf{T}(x,\mathbf{n},t) \in \mathbb{R}^3,$$

where S_1 is the unit sphere in $\mathbb{R}^3$. Then, assuming that ∂E is smooth, reaction forces of $G - E$ acting on ∂E are described by

$$\{\text{reactions opposing deformations of } E\} = \int_{\partial E} \mathbf{T}(x,\mathbf{n},t)d\sigma,$$

where $d\sigma$ is the surface measure on ∂E and $\mathbf{n}$ is the outward unit normal to ∂E at $x \in \partial E$. The component $(\mathbf{T}\cdot\mathbf{n})\mathbf{n}$ of $\mathbf{T}$ along $\mathbf{n}$ is the *traction* or *compression force*, whereas the component $\mathbf{T} - (\mathbf{T}\cdot\mathbf{n})\mathbf{n}$ tangent to ∂E at x is the *shear force*. By d'Alembert principle, the motion of any subdomain $E \subset G$ is a sequence of instantaneous equilibrium states, parameterized with time, of all forces acting on that portion, including the reactions to deformation. Thus

$$\int_E [\ddot{x} - \mathbf{f}(x,\dot{x},t)]\rho dx = \int_{\partial E} \mathbf{T}(x,\mathbf{n},t)d\sigma, \tag{2.1}$$

$$\int_E x \wedge [\ddot{x} - \mathbf{f}(x,\dot{x},t)]\rho dx = \int_{\partial E} x \wedge \mathbf{T}(x,\mathbf{n},t)d\sigma, \tag{2.2}$$

for all subdomains $E \subset G$.

Lemma 2.1 $\mathbf{T}(\cdot,\mathbf{n},t) = -\mathbf{T}(\cdot,-\mathbf{n},t)$.

Proof. Fix $P \in G$ and $\mathbf{n} \in S_1$. For $0 < \varepsilon, \delta \ll 1$ consider the disk $D_\varepsilon(P)$ centered at P with radius ε, normal to $\mathbf{n}$, and the right cylinder $C_\delta(P)$ of base $D_\varepsilon(P)$ and height δ. Write (2.1) over $C_\delta(P)$ and let $\delta \to 0$ by keeping $\varepsilon > 0$ fixed, to obtain

$$\int_{D_\varepsilon(P)} \mathbf{T}(x, \mathbf{n}, t) d\sigma = - \int_{D_\varepsilon(P)} \mathbf{T}(x, -\mathbf{n}, t) d\sigma.$$

Divide both sides by $|D_\varepsilon(P)|$ and let $\varepsilon \to 0$. ■

3 The Stress Tensor and Cauchy's Theorem

Having fixed a triad $\Sigma = \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, represent $\mathbf{n} \in S_1$ by its director cosines $\mathbf{n} = (\alpha_1, \alpha_2, \alpha_3)$ with respect to the coordinate axes of Σ.

Theorem 3.1 (Cauchy). *For all* $\mathbf{n} = (\alpha_1, \alpha_2, \alpha_3) \in S_1$,

$$\mathbf{T}(\cdot, \mathbf{n}, t) = \alpha_i \mathbf{T}(\cdot, \mathbf{e}_i, t).$$

Proof. Fix $P \in G$ and $\mathbf{n} \in S_1$ and write down (2.1), where E is the tetrahedron with vertex in P, height $0 < \varepsilon \ll 1$, base ΔABC normal to $\mathbf{n}$, and faces ΔAPB, ΔBPC, ΔCPA, parallel to the coordinate planes. By setting $\Delta\sigma = |\Delta ABC|$, one has $|E| = \frac{1}{3}\varepsilon \Delta\sigma$, and

$$|\Delta APB| = \alpha_3 \Delta\sigma, \quad |\Delta BPC| = \alpha_1 \Delta\sigma, \quad |\Delta APC| = \alpha_2 \Delta\sigma.$$

For these choices, (2.1) takes the form

$$\begin{aligned}\int_E [\ddot{x} - \mathbf{f}(x, \dot{x}; t)] \rho dx = &\int_{\Delta ABC} \mathbf{T}(x, \mathbf{n}, t) d\sigma + \int_{\Delta BPC} \mathbf{T}(x, -\mathbf{e}_1, t) d\sigma \\ &+ \int_{\Delta APC} \mathbf{T}(x, -\mathbf{e}_2, t) d\sigma + \int_{\Delta APB} \mathbf{T}(x, -\mathbf{e}_3, t) d\sigma.\end{aligned}$$

Dividing both sides by $\Delta\sigma$ gives

$$\begin{aligned}\frac{\varepsilon}{3|E(t)|} \int_E [\ddot{x} - \mathbf{f}(x, \dot{x}; t)] \rho dx = &\frac{1}{|\Delta ABC|} \int_{\Delta ABC} \mathbf{T}(x, \mathbf{n}, t) d\sigma \\ &+ \frac{\alpha_1}{|\Delta BPC|} \int_{\Delta BPC} \mathbf{T}(x, -\mathbf{e}_1, t) d\sigma \\ &+ \frac{\alpha_2}{|\Delta APC|} \int_{\Delta APC} \mathbf{T}(Q, -\mathbf{e}_2, t) d\sigma \\ &+ \frac{\alpha_3}{|\Delta APB|} \int_{\Delta APB} \mathbf{T}(x, -\mathbf{e}_3, t) d\sigma.\end{aligned}$$

Let $\varepsilon \to 0$ by keeping the vertex P of the tetrahedron fixed and the base ΔABC normal to $\mathbf{n}$. ■

While computed at $\mathbf{e}_i$, the vectors $\mathbf{T}(\cdot, \mathbf{e}_i, t)$, need not be directed along the homologous coordinate axes. The components $\tau_{ij}(\cdot, t) = \mathbf{T}(\cdot, \mathbf{e}_j, t) \cdot \mathbf{e}_i$ of $\mathbf{T}(\cdot, \mathbf{e}_j, t)$ along $\mathbf{e}_i$ define a matrix

$$\mathbb{T} = (\tau_{ij}) = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix},$$

called a *stress tensor*. The entries τ_{ii} are traction or compression stresses, and τ_{ij}, for $i \neq j$, are shear stresses. The shear force acting on an infinitesimal plane surface normal to $\mathbf{e}_1$ is $\tau_{21}\mathbf{e}_2 + \tau_{31}\mathbf{e}_3$. In general,

$$\{\text{shear force relative to } \mathbf{e}_i\} = \sum_{j \neq i} \tau_{ij}\mathbf{e}_j.$$

Corollary 3.1 $\mathbf{T}(\cdot, \mathbf{n}, t) = \mathbb{T} \cdot \mathbf{n} = (\tau_{ij})\mathbf{n}$.

Proof. From the definitions and Theorem 3.1,

$$\mathbf{T}(\cdot, \mathbf{n}, t) = \alpha_j \mathbf{T}(\cdot, \mathbf{e}_j, t) = \alpha_j [\mathbf{T}(\cdot, \mathbf{e}_j; t) \cdot \mathbf{e}_i]\mathbf{e}_i$$

$$= \alpha_1 \begin{pmatrix} \tau_{11} \\ \tau_{21} \\ \tau_{31} \end{pmatrix} + \alpha_2 \begin{pmatrix} \tau_{12} \\ \tau_{22} \\ \tau_{32} \end{pmatrix} + \alpha_3 \begin{pmatrix} \tau_{13} \\ \tau_{23} \\ \tau_{33} \end{pmatrix} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \mathbb{T} \cdot \mathbf{n}.$$

■

Corollary 3.2 *Let $G \subset \mathbb{R}^3$ be an open set identified with a material system of density $\rho(\cdot, t)$ whose points $x \in G$ are in motion under the external force density $\mathbf{f}(x, \dot{x}, t)$ and the internal stress tensor $\mathbb{T}$. Then*

$$[\ddot{x} - \mathbf{f}(x, \dot{x}, t)]\rho = \operatorname{div} \mathbb{T} \qquad \text{in } G. \tag{3.1}$$

Proof. Let E be any portion of G with smooth boundary ∂E. By the Gauss–Green theorem and Corollary 3.1,

$$\int_{\partial E} \mathbf{T}(x, \mathbf{n}, t) d\sigma = \int_{\partial E} \mathbb{T} \cdot \mathbf{n} d\sigma = \int_E \operatorname{div} \mathbb{T} dx.$$

Therefore (2.1) takes the form

$$\int_E [\ddot{x} - \mathbf{f}(x, \dot{x}, t)]\rho dx = \int_E \operatorname{div} \mathbb{T} dx$$

for all subdomains $E \subset G$. ■

3.1 Symmetry of the Stress Tensor

Proposition 3.1 $(\tau_{ij}) = (\tau_{ji})$.

Proof. By the Gauss–Green theorem,

$$\begin{aligned}\int_{\partial E} x \wedge \mathbf{T}(x, \mathbf{n}, t) d\sigma &= \int_{\partial E} x_h \tau_{ij} \mathbf{e}_h \wedge \mathbf{e}_i \alpha_j d\sigma \\ &= \int_E \frac{\partial}{\partial x_j} (\tau_{ij} x_h) \mathbf{e}_h \wedge \mathbf{e}_i dx \\ &= \int_E \frac{\partial \tau_{ij}}{\partial x_j} x_h \mathbf{e}_h \wedge \mathbf{e}_i dx + \int_E \tau_{ij} \delta_{hj} \mathbf{e}_h \wedge \mathbf{e}_i dx \\ &= \int_E x \wedge \operatorname{div} \mathbb{T} dx - \int_E \tau_{ij} \mathbf{e}_i \wedge \mathbf{e}_j dx.\end{aligned}$$

Put this in the second cardinal equation (2.2) and take into account (3.1) to obtain

$$\int_E \tau_{ij} \mathbf{e}_i \wedge \mathbf{e}_j dx = \int_E [(\tau_{23} - \tau_{32}) \mathbf{e}_1 + (\tau_{31} - \tau_{13}) \mathbf{e}_2 + (\tau_{12} - \tau_{21}) \mathbf{e}_3] dx = 0$$

for all subdomains $E \subset G$. ■

3.2 Miscellaneous Remarks

The matrix $\mathbb{T}$ is intrinsic to the system and independent of its representations in the following sense. Let $\Sigma' = \{O; \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ be a new triad obtained by Σ by a rotation of the coordinate axes realized by a unitary matrix U, so that in particular $\mathbf{e}_j' = U\mathbf{e}_j$. By Corollary 3.1,

$$\tau_{ij}' = \mathbf{T}(\cdot, \mathbf{e}_j', t) \cdot \mathbf{e}_i' = (\tau_{hk}) U \mathbf{e}_j \cdot U \mathbf{e}_i = \mathbf{e}_i^t [U^t (\tau_{hk}) U] \mathbf{e}_j = [U^t (\tau_{hk}) U]_{ij}.$$

The tensor $\mathbb{T}$ is a linear map in $\mathbb{R}^3$ whose matrix (τ_{ij}) is a representative. We will call both $\mathbb{T}$ and its matrix representations stress tensors.

The unknowns of the motion are the trajectories $t \to x(t)$ of the points of G, the density function $\rho(\cdot, t)$, and the nine components τ_{ij} of $\mathbb{T}$. The second cardinal equation (2.2), which amounts to three scalar equations, has been used to establish the symmetry of $\mathbb{T}$ and thus reduce by three the unknowns of the motions. The remaining first cardinal equation, in the pointwise form (3.1), amounts to three scalar equations, which alone are insufficient to resolve the motion. One needs to provide additional information on the material structure and on the tensorial state of the system both in the interior of G and on its boundary ∂G. For example, for rigid systems $\rho =$ const, and the $\mathbf{T}$ is the rigidity constraint.

4 Perfect Fluids and Cardinal Equations

A fluid is a continuum material system whose equilibrium configurations are possible if and only if the stress tensor $\mathbb{T}$ is proportional to the identity $\mathbb{I}$, that is, if

$$\mathbf{T}(\cdot, \mathbf{n}) = \mathbb{T} \cdot \mathbf{n} = -p(\cdot)\mathbf{n} \qquad \text{in steady state,}$$

where $p(\cdot)$ is a smooth function defined in G, called *pressure*. This formula is a mathematical rendering of *Pascal's principle*, by which the pressure in any point of the fluids exerts equal force by unit surface in all directions [125]. This mathematical definition of fluid reflects the intuitive idea of a material continuum system that does not oppose the mutual sliding of its ideal internal layers. If in the fluid at rest the shear components of its stress tensor were not zero, these would generate an incipient shearing of internal layers, since the system does not have a mechanism to oppose it. Likewise, an ideal material surface traced in the fluid at rest remains in equilibrium only if acted upon by forces normal to it. In a *real* fluid in motion, the *kinematic viscosity* generates shear stresses that oppose layer sliding (see §3.2c of the Complements of Chapter 3). Then real fluids are classified as more viscous (oil, paraffin, etc.) or less viscous (alcohol, ether, gas, etc.) according to the size of these shear stresses. A real fluid is *ideal* of *perfect* if the shear stresses are negligible even in dynamic regime, that is, if

$$\mathbf{T}(\cdot, \mathbf{n}, t) = \mathbb{T} \cdot \mathbf{n} = -p(\cdot, t)\mathbf{n} \quad \text{in } G \text{ and for all times.} \tag{4.1}$$

In such a case $\operatorname{div} \mathbb{T} = -\nabla p(\cdot, t)$, and (3.1) takes the form

$$\rho[\ddot{x} - \mathbf{f}(x, \dot{x}, t)] + \nabla p = 0 \quad \text{in } G \text{ for all } t. \tag{4.2}$$

Equation (4.1) is the *constitutive law* of ideal fluids, and (4.2) is the cardinal or momentum equation of an ideal fluid.

4.1 Barotropic Fluids

These are fluids for which a link is known between pressure and density, say, for example,

$$\rho = \rho(p) \in C^1(\mathbb{R}^+), \qquad \rho(\cdot) \geq \rho_o \;\text{ for some }\; \rho_o > 0. \tag{4.3}$$

Relations of this kind, called *equations of state*, are experimental and include, for example, homogeneous fluids for which $\rho(\cdot, t) = \rho_o$. In view of the continuity equation (1.4) these are also incompressible. Adiabatic fluids are those for which heat transfer is far slower than variations of pressure. Thermodynamic considerations lead to the equation of state [138, §16, page 150, and §30, pages 172–176]

$$p = c\rho^{1+\alpha} \quad \text{for given positive constants } c \text{ and } \alpha. \tag{4.4}$$

The constant $1+\alpha$ is the ratio between the heat capacity at constant volume and heat capacity at constant pressure. If $\alpha = 0$, these heat capacities are the same, and both pressure and density are essentially constant. For barotropic fluids, (4.2) takes the form

$$\ddot{x} = \mathbf{f}(x, \dot{x}, t) - \nabla \int^{p(x,t)} \frac{ds}{\rho(s)}. \tag{4.5}$$

If the fluid is in equilibrium, $\dot{x} = \ddot{x} = 0$ identically and

$$\mathbf{f}(x) = \nabla \int^{p(x)} \frac{ds}{\rho(s)}.$$

If $\mathbf{f}$ is conservative with potential V, then

$$\nabla\left(V - \int^{p} \frac{ds}{\rho(s)}\right) = 0, \qquad \text{which implies} \qquad V - \int^{p} \frac{ds}{\rho(s)} = \text{const.}$$

Summarizing, we may state the following.

Proposition 4.1 *A barotropic fluid admits an equilibrium configuration if and only if* $\mathbf{f}$ *is conservative. Moreover, the equipotential surfaces are also curves of constant pressure (isobaric) and surfaces of constant pressure.*

As an example consider a barotropic fluid subject only to gravity. Then $\mathbf{f}(x) = -g\mathbf{e}_3$, where $\mathbf{e}_3$ is the unit ascending vertical, and $V(x) = -gx_3 + \text{const}$. If the fluid is adiabatic and satisfies (4.4) with $\alpha = 0$ (that is, both pressure and density are essentially constant), then

$$-gx_3 = c \ln \frac{p}{p_o}, \qquad \text{where } p_o \text{ is the pressure at level } x_3 = 0.$$

From this follows the *barometric formula*

$$p(x) = p_o e^{-(g/c)x_3}.$$

This implies that the equipotential and isobaric surfaces are horizontal planes. If the fluid is separated from the air by a surface, its pressure along such a surface must equal that of the air, which is constant. Therefore the separation surface is isobaric and therefore is a horizontal plane. This is known as Stevin's law.

4.2 Fluid in Uniform Rotation

A homogeneous fluid confined in a right circular cylinder is in uniform rotation about the axis of the cylinder with angular speed ω and is subject to its weight. Introduce a triad $\Sigma = \{O; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, fixed with the fluid, with origin at the axis of the cylinder and $\mathbf{u}_3$ ascending vertically. The generic point x of the fluid is acted upon by gravity and the Coriolis forces, so that

$$\mathbf{f}(x) = \rho(\omega^2 x_1,\, \omega^2 x_2,\, -g).$$

Therefore equilibrium is possible only if $\nabla p = \left[\rho\omega^2(x_1, x_2), -\rho g\right]$. From this by integration,

$$p(x) = p_o + \rho\left[\tfrac{1}{2}\omega^2(x_1^2 + x_2^2) - g\, x_3\right].$$

Therefore the isobaric surfaces are the paraboloids

$$x_3 = \frac{1}{2\,g}\omega^2\left(x_1^2 + x_2^2\right) + \text{const.}$$

In particular, the free surface in contact with air is a paraboloid.

5 Rotations and Deformations

Let $\mathbf{v}(\cdot, t)$ be the velocity field generated by the flow map in (1.1) and assume that the fluid at time t undergoes an elemental rigid motion of characteristics $blv(x_o, t)$ and $\boldsymbol{\omega}$, where x_o is an arbitrary but fixed point in the instantaneously rigid fluid. By the Poisson formula (6.1) of Chapter 1,

$$\mathbf{v}(x, t) = \mathbf{v}(x_o.t) + \boldsymbol{\omega} \wedge (x - x_o).$$

Since the motion is instantaneously rigid, $\boldsymbol{\omega}$ does not depend on the variables x of the generic point in the fluid. Taking the curl of both sides gives

$$\operatorname{curl}\mathbf{v} = (v_{3,x_2} - v_{2,x_3})\mathbf{e}_1 + (v_{1,x_3} - v_{3,x_1})\mathbf{e}_2 + (v_{1,x_2} - v_{1,x_2})\mathbf{e}_3 = 2\boldsymbol{\omega}. \quad (5.1)$$

Therefore $\operatorname{curl}\mathbf{v}(x, t)$ gives, apart the factor 2, the angular velocity of the infinitesimal element of fluid about x, regarded as instantaneously rigid. For this reason $\operatorname{curl}\mathbf{v}(\cdot, t)$ is called a *vorticity field*. If $\operatorname{curl}\mathbf{v}(\cdot, t) = 0$, the field is *irrotational*. If G is simply connected, an irrotational field is also *potential*, that is, there exists a function $\varphi(\cdot, t) \in C^1(G)$, called a *kinetic potential*, such that $\mathbf{v}(\cdot, t) = \nabla\varphi(\cdot, t)$. The flow is called *potential* and the velocity field $\mathbf{v}(\cdot, t)$ is normal to the instantaneous equipotential surfaces $[\varphi(\cdot, t) = \text{const}(t)]$. If the velocity field is stationary, the kinetic potential is independent of t and the trajectories of the fluid particles are normal to the equipotential surfaces.

Next expand $\mathbf{v}(\cdot, t)$ in a Taylor series about a point x_o in the fluid, to obtain

$$\mathbf{v}(x, t) = \mathbf{v}(x_o, t) + [\nabla\mathbf{v}(x_o, t)] \cdot (x - x_o) + \mathbf{O}(|x - x_o|^2).$$

Therefore up to terms of higher order,

$$v_i(x, t) = v_i(x_o, t) + v_{i,x_j}(x_j - x_{o,j}), \qquad i = 1, \ldots, N.$$

For fixed indices i, j,

$$v_{i,x_j} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big) + \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\Big) = \mathcal{D}_{ij} + \mathcal{R}_{ij}. \tag{5.2}$$

The entries $\mathcal{D}_{ij}$ and $\mathcal{R}_{ij}$ define two tensors $\mathcal{D}$ and $\mathcal{R}$. The first is symmetric and is called a *deformation tensor*. The second is skew-symmetric and is called a *rotation tensor*. With this notation, the previous Taylor expansion takes the approximate form

$$\mathbf{v}(x,t) = \mathbf{v}(x_o,t) + \mathcal{D}\cdot(x-x_o) + \mathcal{R}\cdot(x-x_o). \tag{5.3}$$

Consider an infinitesimal arc $d(x-x_o)$ within the fluid and along its motion, of length $d\ell = \sqrt{d(x-x_o)^2}$. Then

$$\begin{aligned}\frac{d}{dt}d\ell^2 &= 2d(x-x_o)\cdot d(\dot{x}-\dot{x_o})\\ &= d(x_i - x_{o,i})v_{i,x_j}d(x_j - x_{o,j})\\ &= 2d(x-x_o)P^t\cdot\mathcal{D}\cdot d(x-x_o).\end{aligned}$$

Therefore $\mathcal{D}$ tracks the deformations of infinitesimal lengths along the motion. In a rigid motion lengths are preserved and $\mathcal{D} = 0$. If $\mathcal{D} = \lambda\mathbb{I}$, then the deformation occurs uniformly along the coordinate axes and the fluid expands if $\lambda > 0$ and contracts if $\lambda < 0$. From the definition of $\mathcal{R}$,

$$\mathcal{R}\cdot(x-x_o) = \tfrac{1}{2}\operatorname{curl}\mathbf{v}\wedge(x-x_o) = \boldsymbol{\omega}\wedge(x-x_o).$$

Therefore $\mathcal{R}$ gives the angular velocity of the system as if it were in instantaneous rigid motion. These remarks and (5.3) suggest we regard the infinitesimal motion of a fluid as the sum of (i) an infinitesimal translation along $\mathbf{v}(x_o,t)$, (ii) an infinitesimal deformation along the coordinate axes, and (iii) an infinitesimal rigid rotation about the axis through x_o and directed as $\operatorname{curl}\mathbf{v}(x_o,t)$. This is known as Cauchy's theorem.

6 Vortex Filaments and Vortex Sheets

A vortex filament is a curve in G tangent, for fixed t, to the vortex field $\operatorname{curl}\mathbf{v}(\cdot,t)$. A point y where $\operatorname{curl}\mathbf{v}(y,t) \neq 0$ is a *vortex point*. Through a vortex point y passes a unique vortex filament $\tau \to x(\tau)$, a solution of

$$\begin{aligned}&\operatorname{curl}\mathbf{v}(\cdot,t)\wedge\frac{dx(\tau)}{d\tau} = 0, && \tau\in\mathbb{R} \text{ a parameter,}\\ &x(0) = y, && (t \text{ fixed}).\end{aligned}$$

If $\operatorname{curl}\mathbf{v}(y,t)=0$, the vortex filament through y is not defined. Let $\gamma=\{s\to x(s)\}$ be a smooth curve in G whose points are vortex points. From each of the points of γ trace the corresponding vortex filament

$$\begin{aligned}&\operatorname{curl}\mathbf{v}[x(s),t]\wedge\frac{dx(\tau;s)}{d\tau}=0, \qquad (t \text{ fixed})\\ &x(0;s)=x(s),\end{aligned} \tag{6.1}$$

to obtain, locally in τ, a surface called a *vortex sheet.* If γ is closed, then (6.1) generates tubelike surfaces, called *vortex tubes.* Let $\mathcal{T}$ be the portion of a vortex tube cut by two smooth nonintersecting sections Σ_1 and Σ_2. The oriented boundary $\partial\mathcal{T}$ consists of Σ_1 and Σ_1, with unit normal $\mathbf{n}_i$ oriented outward $\mathcal{T}$ and the portion Σ_o, of $\partial\mathcal{T}$ lying on the vortex tube, with outward unit normal $\mathbf{n}$. By the Gauss–Green theorem,

$$\begin{aligned}\int_{\mathcal{T}}\operatorname{div}\operatorname{curl}\mathbf{v}dx=&\int_{\Sigma_o}\operatorname{curl}\mathbf{v}\cdot\mathbf{n}d\sigma\\ &+\int_{\Sigma_1}\operatorname{curl}\mathbf{v}\cdot\mathbf{n}_1 d\sigma\\ &+\int_{\Sigma_2}\operatorname{curl}\mathbf{v}\cdot\mathbf{n}_2 d\sigma.\end{aligned}$$

Since Σ_o is made out of portions of vortex filaments, $\operatorname{curl}\mathbf{v}\cdot\mathbf{n}=0$ on Σ_o. Moreover, $\operatorname{div}\operatorname{curl}\mathbf{v}=0$. Therefore

$$-\int_{\Sigma_1}\operatorname{curl}\mathbf{v}\cdot\mathbf{n}_1 d\sigma=\int_{\Sigma_2}\operatorname{curl}\mathbf{v}\cdot\mathbf{n}_2 d\sigma. \tag{6.2}$$

Since the sections Σ_1 and Σ_2 are arbitrary, the flow of vortex entering a vortex tube across a section Σ equals the flow of vortex exiting the tube through any other section.

6.1 Circulation and Helmholtz Theorem

Let $\gamma\subset G$ be a smooth closed curve, and let Σ be any smooth oriented surface with unit normal $\mathbf{n}$ whose boundary is γ. By Stokes's theorem,

$$\int_{\Sigma}\operatorname{curl}\mathbf{v}(x,t)\cdot\mathbf{n}(x)d\sigma=\int_{\gamma}\mathbf{v}(x,t)\cdot dx=C_\gamma(t). \tag{6.3}$$

The quantity $C_\gamma(t)$ is the *circulation* of the vector field $\mathbf{v}(\cdot,t)$ along γ. If G is simply connected, (6.3) implies that the motion is potential if and only if $C_\gamma(t)=0$ for any smooth closed curve $\gamma\subset G$. Likewise, if G is simply connected, the motion is irrotational if and only if is potential. A smooth closed curve traced on a vortex tube *surrounds* it if it cannot be deformed with continuity while remaining on the tube to be shrunk to a point. On a vortex sheet one can draw curves homotopic to a point. If Σ is a portion of the vortex sheet included by γ, then by (6.3), $C_\gamma(t) = 0$, since on a vortex sheet $\operatorname{curl}\mathbf{v}\cdot\mathbf{n} = 0$.

Theorem 6.1 (Helmholtz). *Let γ_1 and γ_2 any two smooth closed curves traced on the same vortex tube and surrounding it. Then*

$$C_{\gamma_1}(t) = \int_{\gamma_1} \mathbf{v}(x,t)\cdot dx = \int_{\gamma_2} \mathbf{v}(x,t)\cdot dx = C_{\gamma_2}(t).$$

Proof. Combine (6.2) and (6.3). ■

The circulation along a smooth closed curve γ surrounding a vortex tube is called the *intensity of the vortex tube.* The intensity of a vortex tube is independent of the curve γ lying on the tube surrounding it.

7 Equations of Motion of Ideal Fluids

The motion is driven by the momentum equation (4.2) and by the assumption that mass is conserved. Then recalling the form (1.1) of $\ddot{x}$ in terms of $\mathbf{v}$, the equations of motion of an ideal fluid are

$$\begin{aligned} &\rho[\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mathbf{f}] = -\nabla p, && \text{momentum equation,}\\ &\rho_t + \operatorname{div}(\rho\mathbf{v}) = 0, && \text{conservation of mass,}\\ &\text{some equation of state} && \text{or a derivative of it.} \end{aligned} \tag{7.1}$$

The equation of state could be, for example, (4.4). Another example is that of a link of the type $p = F(\mathcal{E})\rho$ between pressure p and density ρ. here $\mathcal{E} = \mathcal{E}(x,t)$ is the entropy of the elemental mass of fluid about x at time t, and $F(\cdot)$ is a smooth experimentally given function of the entropy. For adiabatic transformations the entropy is constant along the Lagrangian path of $t \to x(t)$, or equivalently along the flow map [146]. Therefore

$$D_t\left(\frac{p}{\rho}\right) = 0 \qquad \text{in } G. \tag{7.2}$$

This might take the place of the last of equation (7.1). The system (7.1)–(7.2) is called the system of Euler equations for ideal isentropic flows.

These systems, irrespective of the form of the last equation of (7.1), consist of five scalar equations from which one seeks to resolve the motion by determining the five scalar unknowns $\mathbf{v}, \rho, p$. The domain G might or might not be bounded, and solvability requires some information on the behavior of the fluid at the boundary of G. Even so, these equations, while not amenable to resolving the motion, may be used to derive some specific dynamic issues, as indicated by the next example.

7.1 Flow past an Obstacle

A rigid impermeable body occupies a bounded, simply connected domain $C \subset \mathbb{R}^3$ and is immersed in a fluid occupying $\mathbb{R}^3 - C$. One seeks to compute the force exerted by the fluid on C. Combining the momentum equation and the continuity equation in (7.1) gives

$$\rho \mathbf{f} - \nabla p = \rho \mathbf{v}_t + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = (\rho \mathbf{v})_t + (\rho v_j \mathbf{v})_{x_j}.$$

Integrate this over $B_R - C$, where B_R is the ball of radius R centered at the origin and R is so large that $\bar{C} \subset B_R$. Using the Gauss–Green theorem and taking into account that C is impermeable and therefore $\mathbf{v} \cdot \mathbf{n} = 0$ on ∂C, this gives

$$-\int_{\partial C} p\mathbf{n} d\sigma = \int_{|x|=R} \left[p\frac{x}{R} + \rho \mathbf{v}\, \mathbf{v} \cdot \frac{x}{R} \right] d\sigma + \int_{B_R - C} (\rho \mathbf{v})_t dx - \int_{B_R - C} \mathbf{f} dx,$$

where $\mathbf{n}$ is the unit normal internal to C. On the other hand, by the constitutive law (4.1) of ideal fluids, the force acting on ∂C is

$$\mathbf{F} = \int_{\partial C} \mathbf{T}(x, \mathbf{n}, t) d\sigma = -\int_{\partial C} p\mathbf{n} d\sigma.$$

Putting this in the previous relation and letting $R \to \infty$ gives

$$\begin{aligned} \mathbf{F} = & \lim_{R\to\infty} \int_{|x|=R} \left[p\frac{x}{R} + \rho \mathbf{v}\, \mathbf{v} \cdot \frac{x}{R} \right] d\sigma \\ & + \lim_{R\to\infty} \int_{B_R - C} (\rho \mathbf{v})_t dx - \lim_{R\to\infty} \int_{B_R - C} \mathbf{f} dx, \end{aligned}$$

provided the limits exist. If the motion of the fluid is stationary and there are not external forces, then

$$\mathbf{F} = \lim_{R\to\infty} \int_{|x|=R} \left[p\frac{x}{R} + \rho \mathbf{v}\, \mathbf{v} \cdot \frac{x}{R} \right] d\sigma. \tag{7.3}$$

8 Barotropic Flows with Conservative Forces

The acceleration of a point x in motion inside the fluid is computed from (1.1). Also by standard vector calculus,

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \tfrac{1}{2}\nabla \mathbf{v}^2 - \mathbf{v} \wedge \operatorname{curl} \mathbf{v}.$$

Therefore the acceleration of a generic $x \in G$ can be written in any of the equivalent forms

$$\ddot{x} = \begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}, \\ \mathbf{v}_t - \mathbf{v} \wedge \operatorname{curl} \mathbf{v} + \tfrac{1}{2}\nabla \mathbf{v}^2. \end{cases} \tag{8.1}$$

Using the second of these in the momentum equation (7.1), the latter takes the form

$$\mathbf{v}_t - \mathbf{v} \wedge \operatorname{curl} \mathbf{v} = -\nabla \mathcal{B}, \qquad \mathcal{B} = \tfrac{1}{2}\mathbf{v}^2 + \int^p \frac{ds}{\rho(s)} - V. \tag{8.2}$$

The first term in the expression of $\mathcal{B}$ is the specific (i.e., per unit mass) kinetic energy of the particle about x. The last term is the specific potential energy of such a particle. The middle term is the specific internal energy due to pressure. Therefore $\mathcal{B}$ is the total specific energy of the particle about x. The quantity $\mathcal{B}$ is the *Bernoulli trinomial* [7].

Computing the material derivative of $\mathcal{B}$ along Lagrangian paths yields

$$\begin{aligned} D_t \mathcal{B}(P,t) &= \mathcal{B}_t + \mathbf{v} \cdot \nabla \mathcal{B} \\ &= \mathbf{v} \cdot \Big[\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{\rho} \nabla p - \nabla V \Big] + \frac{1}{\rho} \frac{\partial}{\partial t} p \\ &= \mathbf{v} \cdot \Big(\ddot{x} - \mathbf{f} + \frac{1}{\rho} \nabla p \Big) + \frac{1}{\rho} \frac{\partial}{\partial t} p. \end{aligned}$$

Enforcing now the continuity equation (7.1) gives

$$D_t \mathcal{B} = \frac{1}{\rho} \frac{\partial}{\partial t} p. \tag{8.3}$$

Therefore in general, the specific energy is not conserved along material paths, due to the variation of its internal energy. If the motion is stationary, then $p_t = 0$ and every particle conserves its specific energy along its material trajectory. Moreover,

$$\nabla \mathcal{B} = \mathbf{v} \wedge \operatorname{curl} \mathbf{v}. \tag{8.4}$$

Therefore, for stationary motions, the energy is conserved along streamlines of the velocity field $\mathbf{v}$. Since the motion is stationary, the streamlines of $\mathbf{v}$ are actually *material paths*, since they are trajectories of elemental particles along their stationary motion. By (8.4), the energy is also conserved along streamlines of the vorticity field $\operatorname{curl} \mathbf{v}$. Such an energy, however, might change from point to point. If the motion is stationary and irrotational, then the energy is constant for all fluid particles.[2]

[2]More generally, the energy is constant for motions that while not irrotational, satisfy $\mathbf{v} \wedge \operatorname{curl} \mathbf{v} = 0$. In such a case the particles move along $\mathbf{v}$ and effect infinitesimal rotations about their material path. They are called *helicoidal or Beltrami* flows.

8.1 Incompressible Barotropic Fluids Subject to Gravity

If the external forces are due only to the weight, then $V(x) = -gx_3 + \text{const}$. Therefore if the fluid is incompressible ($\rho = \text{const}$), the Bernoulli trinomial becomes

$$\frac{1}{g}\mathcal{B} = \frac{\mathbf{v}^2}{2\,g} + x_3 + \frac{p}{\rho\,g} \quad \text{in } G.$$

The first term is called the *kinetic level*, since it gives the height from which a point mass has to fall for an impact speed of $|\mathbf{v}|$ (see §6c of the Complements of Chapter 3). The second term is the actual level of a point x. The third is the *piezometric level* or *head*, since it gives the height of a column of fluid that exerts pressure p on x. By (8.3), variations of the total energy of a fluid particle along its Lagrangian path are due only to variations of the piezometric head. If the motion is stationary, the piezometric head is constant and the energy is preserved along material paths of the fluid and streamlines of the vorticity field curl $\mathbf{v}$. This occurrence is referred to as the theorem of the three levels.

9 Material Lines and Surfaces

Continue to assume that the fluid in G is barotropic and acted upon only by conservative forces, so that the momentum equation (4.5) has the form

$$\ddot{x} = -\nabla\Big(\int^p \frac{ds}{\rho(s)} + V\Big). \tag{9.1}$$

Let $\gamma(\cdot, t_o)$ be a smooth curve in G at some fixed instant t_o. Identify the points of $\gamma(\cdot, t_o)$ with the actual material particles lying on it, and follow their motion following the parameter t. The flow map transforms $\gamma(\cdot, t_o)$ into a smooth curve $\gamma(\cdot, t) \subset G$.[3]

The points of $\gamma(\cdot, t)$ are the same material points of $\gamma(\cdot, t_o)$, and for this reason $\gamma(\cdot, t)$ are called *material lines*. Similarly, one defines smooth *material surfaces* $\Sigma(t) \subset G$ as images by the flow map of smooth surfaces $\Sigma(t_o) \subset G$, so that $\Sigma(t)$ contains the same material particles as $\Sigma(t_o)$.

Theorem 9.1 (Kelvin). *For a barotropic fluid subject only to conservative forces, the circulation of the velocity field $\mathbf{v}(\cdot, t)$ along closed material lines $\gamma(\cdot, t)$ is constant in t. Equivalently,*

$$\int_{\gamma(\cdot, t_o)} \mathbf{v}(x, t_o) \cdot dx = \int_{\gamma(\cdot, t_1)} \mathbf{v}(x, t_1) \cdot dx$$

for any two smooth closed material lines $\gamma(\cdot, t_o)$ mutually connected by the flow map.

[3] While this occurs for $|t - t_o| < \varepsilon$ for sufficiently small ε, we will avoid specifying such a locality.

Proof. Let $\gamma(\cdot,t)$ be smooth closed material lines, parameterized by $\tau \in [-\delta,\delta]$ for some $\delta > 0$, for all $|t - t_o| < \varepsilon$ for sufficiently small ε. Then by the momentum equation (9.1),

$$\begin{aligned}
\frac{d}{dt}\int_{\gamma(\cdot,t)} \mathbf{v}(x,t)\cdot dx &= \frac{d}{dt}\int_{-\delta}^{\delta} \mathbf{v}\big[x(\tau,t),t\big]\cdot\frac{dx(\tau,t)}{d\tau}d\tau \\
&= \int_{-\delta}^{\delta} \ddot{x}(\tau,t)\cdot\frac{dx(\tau,t)}{d\tau}d\tau + \frac{1}{2}\int_{-\delta}^{\delta}\frac{d}{d\tau}\dot{x}^2(\tau,t)d\tau \\
&= -\int_{-\delta}^{\delta}\nabla\Big(\int^p \frac{ds}{\rho(s)} + V\Big)\cdot\frac{dx(\tau,t)}{d\tau}d\tau \\
&= -\int_{\gamma(\cdot,t)}\nabla\Big(\int^p\frac{ds}{\rho(s)} + V\Big)\cdot dx = 0.
\end{aligned}$$ ■

Proposition 9.1 *For a barotropic fluid subject only to conservative forces:*

i. Vortex sheets are material surfaces. Equivalently, if $\Sigma(t_o)$ is a vortex sheet, then all material surfaces $\Sigma(t)$ generated from $\Sigma(t_o)$ by the flow map are vortex sheets.
ii. Vortex filaments are material lines. Equivalently, if $\gamma(\cdot,t_o)$ is a vortex filament, then all material lines $\gamma(\cdot,t)$ generated from $\gamma(\cdot,t_o)$ by the flow map are vortex filaments.
iii. The intensity of a vortex tube is constant in time.
iv. If the motion is irrotational at some instant t_o, it remains irrotational for all times.

Proof. Let $\Sigma(t)$ be a material surface generated by a vortex sheet $\Sigma(t_o)$. If $\Sigma(t)$ were not a vortex sheet, it would contain a smooth closed curve $\gamma(\cdot,t) \subset \Sigma(t)$, where the circulation of $\mathbf{v}(\cdot,t)$ is not zero. By definition of material surface, $\gamma(\cdot,t)$ is generated by a smooth closed curve $\gamma(\cdot,t_o) \subset \Sigma(t_o)$. Since $\Sigma(t_o)$ is a vortex sheet, the circulation of $\mathbf{v}(\cdot,t_o)$ about $\gamma(\cdot,t_o)$ is zero, and it remains zero in time by Kelvins's theorem. The contradiction proves (i). The statement (ii) for vortex filaments follows from (i), since vortex filaments lie on vortex sheets. Statement (iii) follows from (ii) when applied to vortex filaments on vortex tubes. To establish (iv) it suffices to observe that if the motion were not irrotational at some instant t, there would exist a curve $\gamma(\cdot,t)$ traced on a material surface $\Sigma(t)$ about which the circulation of $\mathbf{v}(\cdot,t)$ would not be zero. ■

10 Transport of the Vorticity in Barotropic Fluids Subject to Conservative Forces

Take the curl of both sides of (9.1) to get

$$\mathbf{w}_t - \operatorname{curl}(\mathbf{v}\wedge\mathbf{w}) = 0, \qquad \text{where} \quad \mathbf{w} = \operatorname{curl}\mathbf{v}.$$

The first component of curl($\mathbf{v} \wedge \mathbf{w}$) is

$$\begin{aligned}[\operatorname{curl}(\mathbf{v} \wedge \mathbf{w})]_1 &= (v_1 w_2 - v_2 w_1)_{x_2} - (v_3 w_1 - v_1 w_3)_{x_3} \\ &= (\mathbf{w} \cdot \nabla) v_1 - (\operatorname{div} \mathbf{v}) w_1 - (\mathbf{v} \cdot \nabla) w_1.\end{aligned}$$

Computing the remaining equations similarly, we arrive at the vector equation

$$\mathbf{w}_t - (\mathbf{w} \cdot \nabla)\mathbf{v} + (\operatorname{div} \mathbf{v})\mathbf{w} + (\mathbf{v} \cdot \nabla)\mathbf{w} = 0.$$

Transform the various terms by the continuity equation and by introducing the *vorticity*

$$\boldsymbol{\omega} = \frac{\operatorname{curl} \mathbf{v}}{\rho} = \frac{\mathbf{w}}{\rho} \qquad \text{(angular velocity per unit mass).}$$

With this notation compute

$$\begin{aligned}\mathbf{w} &= \rho \boldsymbol{\omega}_t + \boldsymbol{\omega} \rho_t, & (\operatorname{div} \mathbf{v})\mathbf{w} &= \rho (\operatorname{div} \mathbf{v}) \boldsymbol{\omega}, \\ (\mathbf{w} \cdot \nabla) \cdot \mathbf{v} &= \rho (\boldsymbol{\omega} \cdot \nabla) \mathbf{v}, & (\mathbf{v} \cdot \nabla)\mathbf{w} &= \rho (\mathbf{v} \cdot \nabla) \boldsymbol{\omega} + \boldsymbol{\omega} (\mathbf{v} \cdot \nabla) \rho.\end{aligned}$$

Putting these in the previous vector equation gives

$$\rho[\boldsymbol{\omega}_t + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}] = -\boldsymbol{\omega}[\rho_t + (\mathbf{v} \cdot \nabla)\rho + \rho(\operatorname{div} \mathbf{v})].$$

The term on the right-hand side is zero by the continuity equation, whereas the first two terms in brackets on the left-hand side represent the total derivative of the vorticity. Therefore

$$D_t \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}. \tag{10.1}$$

This is the law by which the vorticity is transported along the material trajectories, or equivalently by the flow map.

Proposition 10.1 *Let $x = \Phi(y,t)$ be the flow map defined, say, for $|t| < \delta$ for some $\delta > 0$. Then*

$$\boldsymbol{\omega}\big(\Phi(y,t), t\big) = \big(\nabla \Phi(y,t)\big) \cdot \boldsymbol{\omega}(y,0). \tag{10.2}$$

Proof. Consider the two vector-valued functions

$$G \times \mathbb{R}^+ \ni (y,t) \longrightarrow \begin{cases} \mathbf{A}(y,t) = \boldsymbol{\omega}\Big(\Phi(y,t), t\Big), \\ \mathbf{B}(y,t) = \nabla \Phi(y,t) \cdot \boldsymbol{\omega}(y,0). \end{cases}$$

They coincide for $t = 0$, since

$$\mathbf{B}(y,0) \nabla \Phi(y,0) \cdot \boldsymbol{\omega}(y,0) = \mathbb{I} \boldsymbol{\omega}(\Phi(y,0),0) = \mathbf{A}(y,0).$$

The transport law (10.1), applied to $\mathbf{A}$ and written in components is

$$\frac{\partial}{\partial t} A_i = A_j \frac{\partial}{\partial \Phi_j} v_i(\Phi(y,t),t).$$

Also one verifies that

$$\begin{aligned}\frac{\partial}{\partial t} B_i &= \frac{\partial}{\partial y_h} \Phi_{i,t} \omega_h(y,0)\\ &= \frac{\partial}{\partial y_h} v_i\big(\Phi(y,t),t\big)\omega_h(y,0)\\ &= \frac{\partial}{\partial \Phi_j} v_i\big(\Phi(y,t),t\big)\frac{\partial}{\partial y_h}\Phi_j(y,t)\omega_h(y,0)\\ &= B_j \frac{\partial}{\partial \Phi_j} v_i\big(\Phi(y,t),t\big).\end{aligned}$$

Therefore $\mathbf{A}$ and $\mathbf{B}$ satisfy the same differential first-order equation. Since they are the same at $t = 0$, they coincide for all times. ∎

Corollary 10.1 (Lagrange–Cauchy)[4] *If the motion of a barotropic fluid subject only to conservative forces is irrotational at some time, it remains irrotational at all times.*

11 Barotropic Potential Flows

A potential barotropic fluid with equation of state (4.4) moves in a domain $G \subset \mathbb{R}^3$, with the velocity field $\mathbf{v}(\cdot, t)$ generated by a kinetic potential $u(\cdot, t)$, so that $\mathbf{v} = \nabla u$. If there are no exterior forces acting on the fluid ($\mathbf{f} = 0$), the momentum equation (7.1) takes the form

$$\frac{\partial}{\partial t} u_{x_i} + u_{x_j} u_{x_i x_j} = -\frac{p_{x_i}}{\rho}, \quad i = 1,2,3. \tag{11.1}$$

More concisely,

$$\nabla\left(u_t + \frac{1}{2}|\nabla u|^2 + \int_0^p \frac{ds}{\rho(s)}\right) = 0. \tag{11.1$'$}$$

Using the equation of state, compute

$$\int_0^p \frac{ds}{\rho(s)} = \frac{1+\alpha}{\alpha}\left(\frac{p}{\rho}\right).$$

[4] [21, Vol. 1, 5–318]; [97, Vol. 4, 695–748].

Combining these remarks gives the *Bernoulli law* for barotropic potential fluids

$$u_t + \frac{1}{2}|\nabla u|^2 + \frac{1+\alpha}{\alpha}\left(\frac{p}{\rho}\right) = h, \tag{11.2}$$

where $h(\cdot)$ is a function only of t. The positive quantity

$$G \times \mathbb{R}^+ \ni (x,t) \to c(x,t)^2 = \frac{dp}{d\rho} = (1+\alpha)\left(\frac{p}{\rho}\right)$$

has the dimensions of a square velocity, and $c(x,t)$ is the *speed of sound* in the fluid at x at time t. Multiply the ith equation in (11.1) by u_{x_i} and sum over $i = 1, 2, 3$, to get

$$\frac{1}{2}\frac{\partial}{\partial t}|\nabla u|^2 + \frac{1}{2}\nabla u \cdot \nabla|\nabla u|^2 = -\frac{1}{\rho}\nabla p \cdot \nabla u. \tag{11.3}$$

Using the continuity equation yields

$$\begin{aligned}
-\frac{1}{\rho}\nabla p \cdot \nabla u &= -\nabla\left(\frac{p}{\rho}\right)\nabla u - \left(\frac{p}{\rho}\right)\frac{1}{\rho}\nabla\rho \cdot \nabla u \\
&= -\nabla\left(\frac{p}{\rho}\right) \cdot \nabla u + \left(\frac{p}{\rho}\right)\frac{1}{\rho}\rho_t + \left(\frac{p}{\rho}\right)\Delta u.
\end{aligned}$$

From the equation of state (4.4) compute

$$D_t \frac{p}{\rho^{1+\alpha}} = \frac{1}{\rho^\alpha} D_t\left(\frac{p}{\rho}\right) + \frac{p}{\rho} D_t \frac{1}{\rho^\alpha} = 0.$$

Expand the total derivative and recall that $\mathbf{v} = \nabla u$ to obtain

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho}\right) + \nabla u \cdot \nabla\left(\frac{p}{\rho}\right) - \alpha\frac{p}{\rho^2}[\rho_t + \nabla u \cdot \nabla\rho] = 0.$$

By the continuity equation,

$$\rho_t + \nabla u \cdot \nabla\rho = \rho\Delta u.$$

Therefore

$$-\nabla\left(\frac{p}{\rho}\right) \cdot \nabla u = \frac{\partial}{\partial t}\left(\frac{p}{\rho}\right) + \alpha\left(\frac{p}{\rho}\right)\Delta u.$$

Putting this in (11.3) gives

$$c^2\Delta u - \frac{1}{2}\nabla u \cdot \nabla|\nabla u|^2 = \frac{1}{2}\frac{\partial}{\partial t}|\nabla u|^2 - \frac{1}{\rho}\frac{\partial}{\partial t}p. \tag{11.4}$$

11.1 Barotropic, Potential, Stationary Flows

If in addition the flow is stationary, rewrite (11.4) in the form

$$A_{ij} u_{x_i x_j} = 0, \qquad \text{where} \quad (A_{ij}) = \Big(\delta_{ij} - \frac{1}{c^2} u_{x_i} u_{x_j}\Big). \tag{11.5}$$

the matrix (A_{ij}) admits the two eigenvalues

$$\lambda_1 = 1 - \Big(\frac{|\nabla u|}{c}\Big)^2, \qquad \lambda_2 = 1.$$

The ratio $M = |\nabla u|/c$ is called the *Mach number*. This equation can be regarded also in terms of *relative motion*. For example, one might think of a rigid body in stationary motion in a potential barotropic fluid with no further solicitations, say for example an airplane or a bullet. With respect to an observer fixed with that rigid body, the body is at rest and the fluid moves around it. If $M < 1$, the flow is subsonic, the matrix (A_{ij}) has two positive eigenvalues, and the partial differential equation in (11.5) is called *elliptic*. If $M > 1$, then the speed of the fluid exceeds the speed of sound and the flow is supersonic; the matrix (A_{ij}) has one positive and one negative eigenvalue and the corresponding equation is *hyperbolic*. If $M = 1$, the flow is *sonic*, the matrix (A_{ij}) has one zero eigenvalue, and the equation is *parabolic* [41, Preliminaries]. Mach 1 is the speed of sound, Mach 2 is twice the speed of sound, and so on.

11.2 Stationary, Potential, Incompressible Flows

The velocity field $\mathbf{v}(\cdot, t)$ in G is given by a kinetic potential $u(\cdot, t)$ defined in G. Since the flow is stationary, both are independent of t and $\mathbf{v} = \nabla u$. Since the fluid is incompressible, $\operatorname{div} \mathbf{v} = 0$ (§1.1), and therefore $\operatorname{div} \nabla u = 0$ in G. Some information is given on the behavior of the fluid near the boundary ∂G, which itself is assumed to be smooth. For example, denoting by $\mathbf{n}(x)$ the outward unit normal to ∂G and $x \in \partial G$, one may assume that the quantity of fluid crossing ∂G at x per unit surface is known, that is,

$$\mathbf{v} \cdot \mathbf{n} = \nabla u \cdot \mathbf{n} = h \quad \text{for some given } h \in C(\partial G).$$

Resolving the motion hinges on finding the kinetic potential u. From the previous remarks, such a potential is a solution of the Neumann problem

$$u \in C^2(G) \cap C^1(\bar{G}), \qquad \Delta u = 0 \ \text{ in } \ G, \qquad \nabla u \cdot \mathbf{n} = h \ \text{ on } \ \partial G. \tag{11.6}$$

Necessary and sufficient conditions of solvability, solution techniques, and a discussion on uniqueness are in [40, Chapters III and IV]. Here we report a physical characterization of such solutions observed by Kelvin.

Theorem 11.1 (Kelvin [143]). *The motion corresponding to (11.1) is the one that minimizes the kinetic energy among all irrotational vector fields* $\mathbf{v}$ *with the same flux* h *on* ∂G. *Equivalently, if* u *is a solution of (11.1), then*

$$\frac{2}{\rho}T_{\nabla u} = \int_G |\nabla u|^2 dx \le \int_G |\mathbf{v}|^2 dx = \frac{2}{\rho}T_{\mathbf{v}}$$

for all divergence-free vector fields defined in $\bar{G}$ *and such that* $\mathbf{v}\cdot\mathbf{n} = h$ *on* ∂G.

Proof. The difference between the kinetic energy due to ∇u and that due to $\mathbf{v}$ is

$$\begin{aligned}\frac{2}{\rho}(T_{\nabla u} - T_{\mathbf{v}}) &= \int_G \left(|\nabla u|^2 - |\mathbf{v}|^2\right) dx \\ &= -\int_G |\nabla u - \mathbf{v}|^2 dx + \int_G (\nabla u - \mathbf{v})\cdot\nabla u dx \\ &\le \int_G (\nabla u - \mathbf{v})\cdot\nabla u dx = 0.\end{aligned}$$

■

12 Stationary, Incompressible Potential Flow past an Obstacle

Let C be a bounded simply connected domain in $\mathbb{R}^3$ with no cavities, which will be identified with a material rigid impermeable body immersed in a fluid in motion and occupying $\mathbb{R}^3 - \bar{C}$. By a change of reference system we may assume that C is still and the fluid moves around it. Assuming that the fluid is stationary, potential, and incompressible, the potential u satisfies

$$\Delta u = 0 \ \text{ in } \ \mathbb{R}^3 - \bar{C}, \qquad \nabla u\cdot\mathbf{n} = 0 \ \text{ on } \ \partial C, \qquad \lim_{|x|\to\infty} \nabla u = \mathbf{v}_\infty, \tag{12.1}$$

where $\mathbf{v}_\infty$ is a given vector and $\mathbf{n}$ is the unit normal to ∂C, interior to C. The next proposition gives precise information on the behavior of u and ∇u as $|x| \to \infty$.

Proposition 12.1 *For any arbitrary constant* γ, *for* $|x| \gg 1$,

$$u(x) = \gamma + x\cdot\mathbf{v}_\infty + O(|x|^{-2}), \qquad \nabla u(x) = \mathbf{v}_\infty + O(|x|^{-3}). \tag{12.2}$$

The proof of this proposition is given in §12c of the Complements.

12.1 The Paradox of D'Alembert

The formulation (12.1) requires only that the fluid be potential and incompressible. Assume in addition that the fluid if barotropic and there are no forces, external or internal. Since the flow is stationary $u_t = 0$ and incompressible (ρ = const), and there are no external forces, the Bernoulli law (11.2) gives

$$p = \gamma_o + \gamma_1 \mathbf{v}^2 \qquad \text{for given positive constants } \gamma_o \text{ and } \gamma_1.$$

The asymptotic behavior of Proposition 12.1 then gives for $|x| \gg 1$,

$$p(x) = \gamma_o + \gamma_1 \big(\mathbf{v}_\infty + O(|x|^{-3})\big)^2 = \gamma_* + O(|x|^{-3}),$$

where $\gamma_* = \gamma_o + \gamma_1 \mathbf{v}^2$. Compute now the force exerted by the fluid on C. By (7.3) and the indicated asymptotic behavior of p one computes

$$\mathbf{F} = \lim_{R\to\infty} \Big(\gamma_* \int_{|x|=R} \frac{x}{R} d\sigma + \rho \mathbf{v}_\infty \int_{|x|=R} \mathbf{v}_\infty \cdot \frac{x}{R} d\sigma + O(R^{-3}) 4\pi R^2\Big) = 0.$$

Therefore, having assumed that the fluid is perfect, incompressible, potential, and that there are no forces internal and external yields the paradox that the fluid exerts no force on C. This suggests that we formulate more physically reasonable assumptions on the fluid, such as, for example, that the fluid is viscous, and as a consequence the presence of internal friction.

13 Friction Tensor for Newtonian Viscous Fluids

In real fluids the friction generated by the mutual sliding of infinitesimal layers generates shear forces that oppose the motion. The stress tensor $\mathbb{T}$ takes the more general form

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}, \tag{13.1}$$

where σ_{ij} are due to friction. Two infinitesimal layers slide over one another if their velocities are different. Therefore the $\sigma_{ij} = \sigma_{ij}(\nabla \mathbf{v})$ depend on the gradient of the velocity. Moreover, $\sigma_{ij} = 0$ if $\nabla \mathbf{v} = 0$. Assuming that the $\sigma_{ij}(\cdot)$ are smooth functions of their arguments, they can be expanded in Taylor series about the origin of their arguments to give

$$\sigma_{ij}(\nabla \mathbf{v}) = \gamma_{ijhk} v_{h,x_k} + O_{ij}(\|\nabla \mathbf{v}\|^2), \qquad \text{where } \gamma_{ijhk} = \frac{\partial \sigma_{ij}}{\partial v_{h,x_k}}\Big|_{\nabla \mathbf{v}=0}$$

for $i,j = 1,2,3$, where $O_{ij}(\cdot)$ are infinitesimal of higher order in $|\nabla \mathbf{v}|$. A fluid is *Newtonian* if (σ_{ij}) depends linearly on $\nabla \mathbf{v}$, so that the higher-order terms in the previous Taylor expansions are negligible. Water and alcohol are Newtonian, whereas paints and gels are not.

The numbers γ_{ijhk} as the indices i,j,h,k run over 1, 2, 3, represent a fourth-order tensor that quantifies the stresses due to the presence of internal friction in a fluid. By its physical nature such a tensor must be isoptropic, that is, must be independent of rotations of the Cartesian system of its representation.

Lemma 13.1 *Let* (γ_{ijhk}) *for* $i, j, h, k = 1, 2, 3$ *be a representation of an isotropic tensor* σ*. Then there exist numbers* λ *and* μ_1*,* μ_2 *such that*

$$\gamma_{ijhk} = \lambda\delta_{ij}\delta_{hk} + \mu_1\delta_{ih}\delta_{jk} + \mu_2\delta_{ik}\delta_{jh}.$$

The lemma is established in §13.1c of the Complements.[5] Assuming it for the moment, it implies that σ_{ij} must be of the form

$$\sigma_{ij} = \lambda\delta_{ij}v_{h,x_h} + \mu_1 v_{i,x_j} + \mu_2 v_{j,x_i}.$$

Since (σ_{ij}) must also be symmetric (Proposition 3.1),

$$\sigma_{ij} = \lambda\delta_{ij}v_{h,x_h} + \mu_1 v_{j,x_i} + \mu_2 v_{i,x_j}.$$

Adding these two expressions for σ_{ij} gives

$$\sigma_{ij} = \tfrac{1}{2}\lambda \operatorname{div}\mathbf{v}\delta_{ij} + \tfrac{1}{2}\bar{\mu}(v_{i,x_j} + v_{j,x_i}) = \tfrac{1}{2}\lambda \operatorname{div}\mathbf{v}\delta_{ij} + \tfrac{1}{2}\bar{\mu}\mathcal{D}_{ij},$$

where $\bar{\mu} = \mu_1 + \mu_2$ and $(\mathcal{D}_{ij})$ is the deformation tensor introduced in (5.2). This representation of the friction stress tensor in Newtonian fluids is due to Stokes [141]. If the fluids are also incompressible, then

$$\sigma_{ij} = \tfrac{1}{2}\bar{\mu}\mathcal{D}_{ij}. \tag{13.2}$$

The constant $\frac{1}{2}\bar{\mu}$ is called the *kinematic viscosity*, and is determined experimentally (§3.2c). By thermodynamic considerations, $\mu > 0$ [138, page 213] and [108, Chapter V].

14 The Navier–Stokes Equations

A Newtonian, viscous, incompressible fluid moves in a domain $G \subset \mathbb{R}^3$. The momentum equations for such a fluid are those in (3.1). Taking into account the form (8.1) of the acceleration $\ddot{x}$ and the form (13.1)–(13.2) of the stress tensor $\mathbb{T}$, these equations take the form

$$[\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} - f]\rho - \nabla p = \tfrac{1}{2}\bar{\mu}\operatorname{div}(\mathcal{D}_{ij}) = \bar{\mu}\Delta\mathbf{v} + \nabla \operatorname{div}\mathbf{v}.$$

Therefore, since the fluid is incompressible,

$$\left.\begin{aligned} \mathbf{v}_t - \mu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p &= f, \\ \operatorname{div}\mathbf{v} &= 0, \end{aligned}\right\} \quad \text{in } G\times\mathbb{R}^+, \tag{14.1}$$

[5] A more general stress–deformation relation is due to Serrin [139].

where $\mu = \bar{\mu}/2\rho$. The constant μ is the *kinematic viscosity*, and its physical dimensions are length squared over time (§3.2c of the Complements of Chapter 3).

14.1 Conservation and Dissipation of Energy

Assume that there are no external forces, so that $\mathbf{f} = 0$. Multiply (14.1) by $\mathbf{v}$ and perform standard vector calculus operations. Recalling the definition (8.2) of the Bernoulli trinomial, we get

$$D_t\mathcal{B} = \frac{p_t}{\rho} + \mu\Delta\tfrac{1}{2}\mathbf{v}^2 - \mu\sum_{i=1}^{3}|\nabla v_i|^2, \qquad \text{where } \mathcal{B} = \tfrac{1}{2}\mathbf{v}^2 + \frac{p}{\rho}. \tag{14.2}$$

For $\mu = 0$ this coincides with Bernoulli's equation (8.3) for $\mathbf{f} = 0$ and $\rho =$ const. As indicated in §8, the term $\mathcal{B}$ is the specific energy of a material particle about x. Therefore the left-hand side of (14.2) is the material derivative of such a specific energy. The first term on the right-hand side is the time variation of the internal energy about x. The second term can be regarded as a dissipation of kinetic energy due to viscosity. The last term is the energy dissipation due to the rough mutual sliding of infinitesimal layers over one another. Thus the variation of energy along Lagrangian paths is balanced by the time variation of the internal energy and the dissipation of energy due to viscosity.

14.2 Dimensionless Formulation, Reynolds Number, and Similarities

The Navier–Stokes equations (14.1) are written in their physical dimensions. To render them dimensionless, select length and time units ℓ and τ and introduce dimensionless variables and quantities[6]

$$x' = \frac{x}{\ell}, \qquad t' = \frac{t}{\tau}, \qquad \mathbf{v}' = \frac{\tau}{\ell}\mathbf{v}, \qquad p' = \frac{\tau^2}{\ell^2}\frac{p}{\rho}, \qquad \mathbf{f}' = \frac{\tau^2}{\ell}\mathbf{f}.$$

Then (14.1) becomes

$$\begin{aligned} \mathbf{v}'_{t'} - \frac{1}{R}\Delta'\mathbf{v}' + (\mathbf{v}'\cdot\nabla')\mathbf{v}' + \nabla' p' = \mathbf{f}', \\ \operatorname{div}'\mathbf{v}' = 0, \end{aligned} \qquad \text{in } G'\times\mathbb{R}'^{+}, \tag{14.3}$$

where

$$R = \frac{1}{\mu}\frac{\ell^2}{\tau} = \frac{\rho}{\bar{\mu}}\frac{\ell^2}{\tau}.$$

[6] Recall that $\mathbf{f}$ is a specific force, that is, force per unit mass.

Here Δ', ∇', and div' denote the analogous differential operations with respect to the variables x', and G' is the dimensionless description of G. The number R is called the *Reynolds number*. From the dimensions of μ it follows that R is dimensionless.

Two motions are *similar* if they take place in homothetic domains with the same Reynolds number. Roughly speaking, the two domains have the same geometry and are mutually rescaled by a given length scale. The length scale being fixed, one then rescales the time to obtain the same Reynolds number. For example, in building a vessel one is interested in investigating a priori how the shape of the hollow impacts the motion of the surrounding fluid. One builds a model vessel, to be used in a limited laboratory environment, of the same shape but of reduced size, by rescaling the geometry by a fixed length. Experiments are performed with such a model in the same fluid where the vessel is intended to operate, so that the two fluids have the same viscosity. Finally, having fixed the length scale, one introduces a new time scale so that the Reynolds number remains the same. The two motions are then similar, and experimental laboratory operations correspond to those of the real fluid up to inverse length and time scales.

These remarks imply that the mathematical investigation of motions modeled by the Navier–Stokes equations reduces to an investigation of (14.3) with $R = 1$, since space and time scales can always be chosen so that $R = 1$. Denoting again by $\mathbf{v}$, $\mathbf{p}$, and $\mathbf{f}$ the indicated dimensionless quantities, and by Δ, ∇, div the homologous operations with respect to the indicated rescaled, dimensionless variables, the mathematical problem consists in finding a velocity field $\mathbf{v}$ defined in $G \times \mathbb{R}^+$ such that

$$\left.\begin{aligned} \mathbf{v}_t - \Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}, \\ \mathrm{div}\,\mathbf{v} = 0, \end{aligned}\right\} \quad \text{in } G \times \mathbb{R}^+, \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_o, \quad \text{in } G \text{ for } t = 0. \tag{14.4}$$

Here $\mathbf{v}_o$ is the initial velocity field defined in G and assumed to be known. The determination of $\mathbf{v}$ hinges on further information on its behavior on the boundary of G. For example, since the fluid is viscous, it adheres to the boundary of its container G, so that $\mathbf{v} = 0$ on ∂G. This is a *Dirichlet datum* of $\mathbf{v}$ on ∂G. On the other hand, the container might be impermeable, so that no fluid flows out of it at ∂G, that is, $\mathbf{v}\cdot\mathbf{n} = 0$ on ∂G for all times. This is a Neumann datum of $\mathbf{v}$ on ∂G. This boundary information need not be homogeneous or could be intertwined, so that for example, a Dirichlet datum is given on a portion $\partial_1 G$ of ∂G and a Neumann datum is prescribed on the remaining portion $\partial_2 G = \partial G - \partial_1 G$. While the physical formulation is simple, the corresponding mathematical problems are still not well understood and are the object of current investigations [59, 78].

Problems and Complements

7c Equation of Motion of Ideal Fluids

7.1c Transmission of Sound Waves

An ideal compressible fluid moves in a region $G \subset \mathbb{R}^3$ satisfying the momentum equation and the conservation of mass (7.1). Assume the following physical modeling assumptions:

(a) The fluid moves with small relative velocity and small time variations of density. Therefore second-order terms of the type $v_i v_{j,x_h}$ and $\rho_t v_i$ are negligible with respect to first-order terms.
(b) Heat transfer is slower than pressure drops, i.e., the process is adiabatic and $\rho = h(p)$ for some $h \in C^2(\mathbb{R})$.

Expanding $h(\cdot)$ about the equilibrium pressure p_o, renormalized to be zero, gives

$$\rho = a_o p + a_1 p^2 + \cdots .$$

Assume further that the pressure is close to the equilibrium pressure, so that all terms of order higher than one are negligible when compared to $a_o p$. These assumptions in the momentum equation yield

$$(\rho \mathbf{v})_t = -\nabla p + \mathbf{f} \quad \text{in } G \times \mathbb{R}.$$

Now take the divergence of both sides to obtain

$$\frac{\partial}{\partial t} \operatorname{div}(\rho \mathbf{v}) = -\Delta p + \operatorname{div} \mathbf{f} \quad \text{in } G \times \mathbb{R}.$$

From the continuity equation,

$$\operatorname{div}(\rho \mathbf{v}) = -\rho_t = -a_o p_t.$$

Combining these remarks gives the equation of the pressure in the propagation of sound waves in a fluid, in the form [130]

$$\frac{\partial^2 p}{\partial t^2} - c^2 \Delta p = f \quad \text{in } G \times \mathbb{R}, \tag{7.1c}$$

where $c^2 = 1/a_o$ and $f = -\operatorname{div} \mathbf{f}/a_o$. This is the wave equation in three space dimensions [41, Chapter VI].

7.2c Continuity Equation in Cylindrical Coordinates

Refer back to the cylindrical coordinates of §1.4c of the Complements of Chapter 2 and deduce the formulas of formal differentiation

$$\frac{\partial}{\partial x_1} = \cos\varphi \frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}, \qquad \frac{\partial}{\partial x_2} = \sin\varphi \frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi}. \tag{7.2c}$$

From these compute

$$\begin{aligned}
\operatorname{div}(\rho\dot{P}) &= \frac{\partial}{\partial x}\rho(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi) + \frac{\partial}{\partial y}\rho(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi) + \frac{\partial}{\partial x_3}(\rho\dot{z}) \\
&= \cos\varphi\frac{\partial}{\partial r}\rho(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi) - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}\rho(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi) \\
&\quad + \sin\varphi\frac{\partial}{\partial r}\rho(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi) + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi}\rho(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi) + \frac{\partial}{\partial x_3}(\rho\dot{z}) \\
&= \frac{1}{r}\Big[\frac{\partial}{\partial r}(\rho r\dot{r}) + \frac{\partial}{\partial\varphi}(\rho r\dot{\varphi}) + \frac{\partial}{\partial x_3}(\rho r\dot{z})\Big].
\end{aligned}$$

Therefore the continuity equation takes the form

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\Big(\frac{\partial}{\partial r}\rho r v_r + \frac{\partial}{\partial\varphi}\rho r v_\varphi + \frac{\partial}{\partial z}\rho r v_z\Big) = 0, \tag{7.3c}$$

where $(v_r, v_\varphi, v_{x_3})$ are the Lagrangian components of the velocity of the system in cylindrical coordinates. The derivative along $\mathbf{v}$ is computed starting from (7.2c) as

$$\begin{aligned}
\mathbf{v}\cdot\nabla &= v_{x_1}\frac{\partial}{\partial x_1} + v_{x_2}\frac{\partial}{\partial x_2} + v_{x_3}\frac{\partial}{\partial x_3} \\
&= (\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi)\big(\cos\varphi\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}\big) \\
&\quad + (\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi)\big(\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi}\big) + \dot{x}_3\frac{\partial}{\partial x_3} \\
&= \dot{r}\frac{\partial}{\partial r} + \dot{\varphi}\frac{\partial}{\partial\varphi} + \dot{x}_3\frac{\partial}{\partial x_3}.
\end{aligned}$$

Therefore the operation of total derivative in terms of these coordinates is

$$D_t = \frac{\partial}{\partial t} + (v_r, v_\varphi, v_{x_3})\cdot\nabla_{(r,\varphi,x_3)}.$$

8c Barotropic Flows with Conservative Forces

8.1c Torricelli's Theorem

Consider the outflow of an ideal barotropic fluid subject to gravity from a small orifice on the bottom of a container of large horizontal cross section.

The free surface exposed to air is taken as a reference level $x_3 = 0$. The outflow is so small that the flow is taken as stationary with good approximation, and as a consequence the height h from the free surface $x_3 = 0$ to the orifice is constant. Also, the level of the free surface is lowered so slowly that the velocity of the points on the plane $x_3 = 0$ can be taken as zero. It is assumed also that the horizontal planes within the container are isobaric surfaces (see the final example of §4.1). Let p_o be the atmospheric pressure at the free surface $x_3 = 0$. Since the orifice is in contact with air, the atmospheric pressure there must also be p_o and such a value of the pressure is the same on the horizontal plane through the orifice and within the container. Since the Bernoulli trinomial is constant $\mathcal{B}(0) = \mathcal{B}(-h)$, we have

$$\frac{p_o}{\rho g} = \frac{\mathbf{v}^2}{2g} - h + \frac{p_o}{\rho g}, \qquad \text{which implies} \qquad |\mathbf{v}| = \sqrt{2gh}.$$

Therefore the outflow speed depends only on the height of the fluid above the orifice. In particular, it is independent of the orientation of the nozzle.

11c Barotropic Potential Flows

11.1c Bidimentional Incompressible Flows

The fluid moves in a simply connected domain $G \subset \mathbb{R}^2$ with smooth boundary ∂G and it does not flow out of G, so that $\mathbf{v} \cdot \mathbf{n} = 0$ on ∂G, where $\mathbf{n}$ is the unit outward normal at ∂G. Assume also that $\rho = 1$, so that $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$. Incompressibility $\operatorname{div} \mathbf{v} = 0$ implies $v_{1,x_1} = -v_{2,x_2}$. Therefore, for fixed t, the differential form $v_2(x,t)dx_1 - v_1(x,t)dx_2$ is exact and there exists a smooth function φ defined in G such that

$$\mathbf{v} = (\varphi_{x_2}, -\varphi_{x_1}) = \nabla^* \varphi, \qquad \text{which implies} \qquad \omega = -\Delta\varphi,$$

where ω is the only nonzero element of $\boldsymbol{\omega}$. If $\mathbf{n} = (n_1, n_2)$ is the unit normal exterior to ∂G, the vector $\boldsymbol{\tau} = (-n_2, n_1)$ is the unit vector tangent to ∂G oriented counterclockwise. Using these remarks, we compute on ∂G

$$0 = \mathbf{v} \cdot \mathbf{n} = \nabla\varphi \cdot \boldsymbol{\tau}, \qquad \text{which implies} \qquad \varphi \,\big|_{\partial G} = c(t),$$

where $c(t)$ is constant in the space variables $x \in \partial G$ and depends at most on time. Notice also that the right-hand side of (10.1) is zero, since the only nonzero component of $\boldsymbol{\omega}$ is along the x_3-axis, and $\mathbf{v}$ is independent of x_3. The flow is then determined by the system

$$\Delta\varphi = -\omega, \quad D_t\omega = 0, \quad \mathbf{v} = \nabla^*\varphi \quad \text{in } G \times \mathbb{R}^+, \tag{11.1c}$$

$$\varphi \,\big|_{\partial G} = \text{const}(t) \quad \text{on } \partial G \times \mathbb{R}^+. \tag{11.2c}$$

From the second of these one computes

$$-\omega_t = \det(\nabla\varphi, \nabla\omega)^t.$$

Therefore the motion is stationary, that is, $\omega_t = 0$, if and only if ω and φ are functionally independent. Prove that if $\omega_t = 0$ at some instant, then $\omega_t \equiv 0$.

12c Stationary, Incompressible Potential Flow past an Obstacle

12.1c Proof of Proposition 12.1

Look for solutions of (12.1) of the form

$$u(y) = y \cdot \mathbf{v}_\infty + w(y),$$

where w satisfies

$$\begin{gathered} \Delta w = 0 \ \text{ in } \ \mathbb{R}^3 - C, \qquad \nabla w \cdot \mathbf{n} = -\mathbf{v}_\infty \cdot \mathbf{n} \ \text{ on } \ \partial C, \\ \lim_{|y|\to\infty} \nabla w(y) = 0, \qquad \lim_{|y|\to\infty} w(y) = 0. \end{gathered} \tag{12.1c}$$

These follow from (12.2), and the last equation corresponds to taking $\gamma = 0$ in (12.2). Let B_R denote the ball of radius R about the origin, and let $R > 0$ be so large that $C \subset B_R$. Introduce the cutoff function

$$r \to \zeta(r) = \begin{cases} 1 & \text{for } \ 0 \le r < R, \\ \cos^2\left(\dfrac{\pi}{2}\dfrac{r-R}{R}\right) & \text{for } \ R \le r < 2R, \\ 0 & \text{for } \ r > 2R. \end{cases}$$

By definition $\zeta \in C^2(\mathbb{R}^+)$, and

$$\zeta'(r) = \begin{cases} 0 & \text{for } \ 0 < r \le R, \\ \dfrac{1}{R}\sin\pi\left(\dfrac{r-R}{R}\right) & \text{for } \ R < r \le 2R, \\ 0 & \text{for } \ r > 2R, \end{cases}$$

$$\zeta''(r) = \begin{cases} 0 & \text{for } \ 0 < r \le R, \\ \dfrac{\pi}{R^2}\cos\pi\left(\dfrac{r-R}{R}\right) & \text{for } \ R < r \le 2R, \\ 0 & \text{for } \ r > 2R. \end{cases}$$

Fix $x \in B_R - C$, multiply the first equation of (12.1c) by

$$y \to \frac{\zeta(|y|)}{|y-x|}, \qquad y \ne x,$$

and integrate by parts over $G_\varepsilon = B_{2R} - \big(C \cup B_\varepsilon(x)\big)$, where $0 < \varepsilon \ll 1$ is so small that $B_\varepsilon(x) \subset B_{2R} - C$. The boundary of G_ε consists of ∂C, with exterior normal $\mathbf{n}$, and the two spheres $|y - x| = \varepsilon$ and $|y| = 2R$. The unit normals exterior to G on ∂C and $|x - y| = \varepsilon$ are interior to C and $B_\varepsilon(x)$ respectively. By the Gauss–Green theorem,

$$-\int_{G_\varepsilon} \nabla w \nabla \cdot \frac{\zeta(|y|)}{|y-x|} dy = \int_{|y-x|=\varepsilon} \nabla w \cdot \frac{y-x}{|y-x|} d\sigma + \int_{\partial C} \frac{\mathbf{v}_\infty \cdot \mathbf{n}}{|y-x|} d\sigma.$$

The integral on the left-hand side is further integrated by parts to give

$$\begin{aligned}\int_{|y-x|=\varepsilon} w \nabla \frac{1}{|y-x|} \cdot \frac{y-x}{|y-x|} d\sigma &= -\int_{G_\varepsilon} w \Delta \frac{\zeta(|y|)}{|y-x|} dy \\ = \int_{|y-x|=\varepsilon} \nabla w \cdot \frac{y-x}{|y-x|} d\sigma &+ \int_{\partial C} \frac{\mathbf{v}_\infty \cdot \mathbf{n}}{|y-x|} d\sigma + \int_{\partial C} w \nabla \frac{1}{|y-x|} \cdot \mathbf{n} d\sigma.\end{aligned} \tag{12.2c}$$

The various integrals on the right-hand side are transformed and are estimated as follows. First one verifies that $\Delta|y-x|^{-1} = 0$ for $y \neq x$; then

$$\begin{aligned}\Delta \frac{\zeta(|y|)}{|y-x|} &= 2\nabla \frac{1}{|y-x|} \cdot \nabla \zeta(|y|) + \frac{1}{|y-x|} \Delta\zeta\,(|y|) \\ &= 2\zeta'(|y|) \frac{y-x}{|y-x|^3} \frac{y}{|y|} + \frac{1}{|y-x|} \Big(\zeta''(|y|) + \zeta'(|y|) \frac{2}{|y|}\Big).\end{aligned}$$

By the structure of the cutoff function $\zeta(\cdot)$, the right-hand side is nonzero only on the spherical annulus $R < |y| < 2R$. For fixed x, we may take R so large that $x \in B_{\frac{1}{2}R}$, so that $|y - x| \geq \frac{1}{2}R$. For these choices,

$$\Big|\Delta \frac{\zeta(|y|)}{|y-x|}\Big| \leq \frac{\text{const}}{R^3} \qquad \text{for } R \leq |y| \leq 2R.$$

Therefore the first integral on the right-hand side of (12.2c) is estimated as

$$\Big|\int_{G_\varepsilon} w \Delta \frac{\zeta(|y|)}{|y-x|} dy\Big| \leq \text{const} \sup_{R\leq|y|\leq 2R} |w| \to 0 \qquad \text{as } |y| \to \infty.$$

The second integral on the right-hand side is estimated as

$$\Big|\int_{|y-x|=\varepsilon} \nabla w \cdot \frac{y-x}{|y-x|} d\sigma\Big| \leq 4\pi\varepsilon \sup_{|y|\leq 2|x|} |\nabla w| \to 0 \qquad \text{as } \varepsilon \to 0.$$

Computing the derivative in the integral on the left-hand side gives

$$\nabla \frac{1}{|y-x|} \cdot \frac{y-x}{|y-x|} = \frac{\partial}{\partial r} \frac{1}{r}\Big|_{r=\varepsilon} = -\frac{1}{\varepsilon^2}.$$

Putting these calculations in (12.2c) and taking the limit first as $R \to \infty$ for ε fixed, and then as $\varepsilon \to 0$, yields the implicit representation formula

$$w(x) = -\frac{1}{4\pi}\int_{\partial C} \frac{\mathbf{v}_\infty \cdot \mathbf{n}}{|y-x|} d\sigma - \frac{1}{4\pi}\int_{\partial C} w\nabla \frac{1}{|y-x|} \cdot \mathbf{n} d\sigma. \tag{12.3c}$$

Proposition 12.1c *For* $|x| \gg 1$ *we have the asymptotic behavior*

$$w(x) = O\Big(\frac{1}{|x|^2}\Big), \qquad \nabla w(x) = O\Big(\frac{1}{|x|^3}\Big).$$

Proof. Such a behavior is obvious for the second integral on the right-hand side of (12.3c). To prove it for the first integral, observe that for $y \in \partial C$ and $|x| \gg 1$,

$$\frac{1}{|x|}\frac{1}{1+t}\frac{1}{|y-x|} \le \frac{1}{|x|}\frac{1}{1-t}, \qquad \text{where } t = \frac{|y|}{|x|}.$$

Therefore by Taylor's formula,

$$\frac{1}{|y-x|} = \frac{1}{|x|}\Big[1 + O\Big(\frac{|y|}{|x|}\Big)\Big].$$

Thus

$$\int_{\partial C} \frac{\mathbf{v}_\infty \cdot \mathbf{n}}{|y-x|} d\sigma = \frac{1}{|x|}\Big[\int_{\partial C} \mathbf{v}_\infty \cdot \mathbf{n} d\sigma + \int_{\partial C} O\Big(\frac{|y|}{|x|}\Big)\mathbf{v}_\infty \cdot \mathbf{n} d\sigma\Big].$$

Since $\mathbf{v}_\infty$ is a constant vector, the first integral vanishes, thereby proving the assertion. ■

13c Friction Tensor for Newtonian Viscous Fluids

13.1c Isotropic Tensors of the Fourth Order

Given a continuously differentiable velocity field $\mathbf{v}$ defined in $\mathbb{R}^3$, consider the expression

$$T_{ij} = \gamma_{ijhk} v_{hk}, \qquad \text{where } v_{hk} = \frac{\partial v_h}{\partial x_k}, \quad i,j,h,k = 1,2,3. \tag{13.1c}$$

The nine numbers T_{ij} are the representative entries of a tensor $\mathbf{T}$ of order 2, with respect to a Cartesian triad Σ. Similarly, (γ_{ijhk}) is the Σ-representative of a fourth-order tensor Γ. Let now Σ' be a new Cartesian triad obtained from Σ by a rotation, realized by a unitary matrix $A : \Sigma \to \Sigma'$. The vector field $x \to \mathbf{v}(x)$ is transformed into

$$\Sigma' \ni y \longrightarrow \mathbf{v}'(y) = A\mathbf{v}(A^{-1}y), \tag{13.2c}$$

and the representation of $\mathbf{T}$ in Σ' is

$$T'_{\ell m} = \gamma_{\ell mrs} v'_{rs}, \qquad \text{where } \; v'_{rs} = \frac{\partial v'_s}{\partial \xi_s}, \qquad \ell, m, r, s = 1, 2, 3.$$

Using (13.2c), compute

$$v'_{rs} = A_{rh} A_{sk} \frac{\partial v_h}{\partial x_k} = A_{rh} A_{sk} v_{hk}.$$

Therefore

$$T'_{\ell m} = A_{rh} A_{sk} \gamma_{\ell mrs} v_{hk}.$$

The tensor $\mathbf{T}$ is *isotropic* if its action on vectors is independent of the reference Cartesian triad, that is, if for all $\mathbf{w} \in \Sigma$,

$$(T_{ij})\mathbf{w} = A^{-1}(T'_{ij}) A\mathbf{w}, \qquad \forall \mathbf{w} \in \Sigma.$$

Since $\mathbf{w} \in \Sigma$ is arbitrary,

$$T_{ij} = A_{\ell i} T'_{\ell m} A_{mj}, \qquad i, j = 1, 2, 3.$$

Using these representations, it follows that $\mathbf{T}$ is isotropic if

$$\gamma_{ijhk} v_{hk} = A_{\ell i} A_{mj} A_{rh} A_{sk} \gamma_{\ell mrs} v_{hk}, \qquad i, j = 1, 2, 3,$$

for all unitary matrices A. This in turn implies

$$\gamma_{ijhk} = A_{\ell i} A_{mj} A_{rh} A_{sk} \gamma_{\ell mrs}, \qquad i, j, h, k = 1, 2, 3. \tag{13.3c}$$

This is the condition for a fourth-order tensor Γ to be isotropic.

Proposition 13.1c *Let Γ be a fourth-order isotropic tensor. Then its representation with respect to a Cartesian triad Σ is*

$$\gamma_{ijhk} = \lambda \delta_{ij} \delta_{hk} + \mu_1 \delta_{ih} \delta_{jk} + \mu_2 \delta_{ik} \delta_{jh}, \tag{13.4c}$$

where the constants λ, μ_1, and μ_2 are independent of Σ.

Proof. In (13.3c) take the rotation matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For such a choice, the entries in (13.3c) are nonzero only if the quadruple $ijhk$ coincides with ℓmrs, and in such a case

$$\gamma_{ijhk} = A_{ii} A_{jj} A_{hh} A_{kk} \gamma_{ijhk}.$$

From the structure of the matrix A above, one verifies that if in the quadruple $ijhk$, the index 3 occurs an odd number of times, then $\gamma_{ijhk} = -\gamma_{ijhk}$. Therefore

$$\gamma_{ijhk} = 0 \qquad \begin{array}{l}\text{if in the quadruple } ijhk \text{ the index 3} \\ \text{occurs an odd number of times.}\end{array}$$

Repeating the same arguments for the choices of rotation matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

one concludes that

$$\gamma_{ijhk} = 0 \qquad \begin{array}{l}\text{if in the quadruple } ijhk \text{ any one of the indices} \\ 1,\,2,\,3 \text{ occurs an odd number of times.}\end{array}$$

Therefore the only nonzero elements are of the form

$$\gamma_{iihh}, \qquad \gamma_{ihih}, \qquad \gamma_{ihhi},$$

where repeated indices are not meant to be added. From (13.3c) compute

$$\begin{aligned}\gamma_{1133} &= A_{\ell 1}A_{m1}A_{r3}A_{s3}\gamma_{\ell mrs} = \delta_{\ell m}\delta_{rs}\gamma_{\ell mrs}, \\ \gamma_{2233} &= A_{\ell 2}A_{m2}A_{r3}A_{s3}\gamma_{\ell mrs} = \delta_{\ell m}\delta_{rs}\gamma_{\ell mrs}.\end{aligned}$$

Therefore $\gamma_{1133} = \gamma_{2233}$, and by symmetry,

$$\gamma_{1122} = \gamma_{1133} = \gamma_{2233} = \gamma_{3311} = \gamma_{2211} = \lambda.$$

If, on the other hand, all indices are equal, (13.3c) gives the identity

$$\gamma_{iiii} = A_{\ell i}A_{mi}A_{ri}A_{si}\gamma_{\ell mrs} = \delta_{\ell mrs}\gamma_{\ell mrs} = \gamma_{\ell\ell\ell\ell}, \qquad i, \ell = 1,2,3.$$

Analogous considerations for the remaining terms imply that there exist constants λ, μ_1, μ_2, θ such that

$$\underbrace{\gamma_{iihh} = \lambda, \quad \gamma_{ihih} = \mu_1, \quad \gamma_{ihhi} = \mu_2,}_{i \neq h} \quad \gamma_{iiii} = \theta, \qquad \text{for all } \; i, h = 1,2,3.$$

Putting this in (13.3c) gives

$$\begin{aligned}\gamma_{ijhk} = &\sum_{\substack{\text{indices of the form} \\ iihh,\; i\neq h}} A_{\ell i}A_{mj}A_{rh}A_{sk}\gamma_{\ell mrs} \\ &+ \sum_{\substack{\text{indices of the form} \\ ihih,\; i\neq h}} A_{\ell i}A_{mj}A_{rh}A_{sk}\gamma_{\ell mrs}\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{\text{indices of the form} \\ ihhi,\ i\neq h}} A_{\ell i}A_{mj}A_{rh}A_{sk}\gamma_{\ell mrs} \\
&+ \sum_{\substack{\text{indices of the form} \\ iiii}} A_{\ell i}A_{mj}A_{rh}A_{sk}\gamma_{\ell mrs} \\
&= \underbrace{\lambda\delta_{ij}\delta_{hk} + \mu_1\delta_{ih}\delta_{jk} + \mu_2\delta_{ik}\delta_{jh}}_{i\neq h} + \theta\delta_{ij}\delta_{ik} \\
&= \lambda\delta_{ij}\delta_{hk} + \mu\delta_{ih}\delta_{jk} + \sigma\delta_{ik}\delta_{jh} + [\theta - (\lambda + \mu_1 + \mu_2)]\delta_{ijhk}.
\end{aligned}
$$

To conclude the proof it will be shown that this form of the tensor (γ_{ijhk}) satisfies (13.3c) for every unitary matrix A if and only if $\theta = \lambda + \mu_1 + \mu_2$. Indeed, from (13.3c),

$$
\begin{aligned}
&\lambda\delta_{ij}\delta_{hk} + \mu_1\delta_{ih}\delta_{jk} + \mu_2\delta_{ik}\delta_{jh} + [\theta - (\lambda + \mu_1 + \mu_2)]\delta_{ijhk} \\
&= A_{\ell i}A_{mj}A_{rh}A_{sk}\big\{\lambda\delta_{\ell m}\delta_{rs} + \mu_1\delta_{\ell r}\delta_{ms} + \mu_2\delta_{\ell s}\delta_{mr} + [\theta - (\lambda + \mu_1 + \mu_2)]\delta_{ijhk}\big\} \\
&= \lambda\delta_{ij}\delta_{hk} + \mu_1\delta_{ih}\delta_{jk} + \mu_2\delta_{ik}\delta_{jh} + A_{\ell i}A_{mj}A_{rh}A_{sk}[\theta - (\lambda + \mu_1 + \mu_2)]\delta_{\ell mrs}.
\end{aligned}
$$

Therefore the tensor on the left-hand side satisfies (13.3c) for all unitary matrices A if

$$
[\theta - (\lambda + \mu_1 + \mu_2)](\delta_{ijhk} - A_{\ell i}A_{mj}A_{rh}A_{sk}\delta_{\ell mrs}) = 0
$$

for all unitary matrices A. This is possible only if the coefficient independent of the indices is zero. ■

14c The Navier–Stokes Equations

14.1c A Paradox of Ideal Fluids

Denote by (x, y, z) the coordinates of $\mathbb{R}^3$. An incompressible fluid of constant density ρ_o fills the slab $0 < y < 1$ and moves so that the velocity field has constant direction, say for example along the x-axis, and is driven by a constant pressure difference between two sections normal to the direction of the velocity. Thus with obvious symbolism,

$$
\begin{gathered}
\mathbf{v}(x, y, z, t) = \big(v(x, y, t), 0, 0\big)^t, \quad p(x, y, z) = p(x), \\
p(x_1) = p_1, \quad p(x_2) = p_2, \quad \Delta p = p_2 - p_1, \quad \ell = x_2 - x_1 > 0.
\end{gathered}
$$

Assume that there are no external forces. Incompressibility $\operatorname{div}\mathbf{v} = 0$ implies $v_x = 0$ and $v(x, y, t) = v(y, t)$. If the fluid is ideal, it must satisfy the Euler equations (7.1), that is,

$$
\rho_o v_t = -p_x.
$$

Taking the x-derivative of both sides gives

$$p_{xx} = 0, \qquad \text{from which} \qquad p(x) = p_1 + \frac{\Delta p}{\ell}(x - x_1). \tag{14.1c}$$

Putting this in the previous Euler equation yields

$$v_t = -\frac{\Delta p}{\rho_o \ell}, \qquad \text{from which} \qquad v(y,t) = -\frac{\Delta p}{\rho_o \ell} t + f(y),$$

where $f(\cdot)$ is a smooth arbitrary function of y, subject only to the conditions $f'(0) = f'(1) = 0$, that is, that the fluid does not flow out of the slab. This solution of the Euler equations implies that the velocity grows to infinity as $t \to \infty$, even though the pressure difference between the sections $x = x_1$ and $x = x_2$ is finite and constant.

14.2c Viscous Fluids: Solving the Paradox

If the fluid is viscous, velocity and pressure satisfy the Navier–Stokes equations with $\mathbf{f} = 0$. Recalling that only the first component of $\mathbf{v}$ is nonzero, and that it depends only on y, the Navier–Stokes equations for such a fluid give

$$v_t - \frac{1}{R} v_{yy} + \frac{1}{\rho_o} p_x = 0 \quad \text{in } [0 < y < 1] \times \mathbb{R}^+, \tag{14.2c}$$

where R is the Reynolds number of the system. Since the fluid is viscous, it adheres to the walls $y = 0$ and $y = 1$ of the slab, so that $v(0,t) = v(1,t) = 0$ for all times. Taking the x-derivative of (14.2c), one computes the pressure as in (14.1c), which substituted into (14.2c) gives

$$v_t - \frac{1}{R} v_{yy} + \frac{\Delta p}{\rho_o \ell} = 0,$$

whose solution is independent of t and is given by

$$v(y) = \frac{R\,\Delta p}{2\rho\ell} y(1 - y).$$

Thus the fluid is in stationary regime with zero velocity at the walls and distributed along a parabolic profile for $y \in (0,1)$.

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