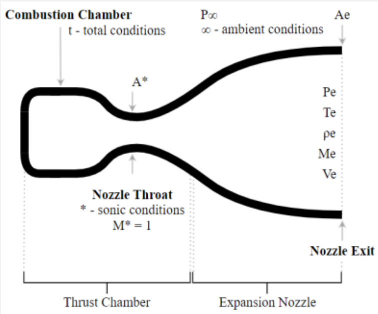
AEE465/565 – Rocket Propulsion – Professor Dr. Werner J.A. Dahm

Isentropic Nozzle Flow

Total-to-Static Relations:



$$\frac{T_t}{T} = \left[1 + \frac{\gamma - 1}{2} M^2\right]$$

$$\frac{p_t}{p} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{\rho_t}{\rho} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{1}{\gamma - 1}}$$

Mass Flow Rate:

$$\dot{m}(x) = \dot{m}^* = \rho^* u^* A^*$$

$$\dot{m}_p = \frac{p_t\, A^*}{\sqrt{R\, T_t}} \left[\gamma \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]^{\frac{1}{2}}$$

^This is valid for any choked nozzle.

Isentropic Area-Mach Number Relation:

$$\frac{A}{A^*} = \frac{1}{M}\left\{\left(\frac{2}{\gamma + 1}\right)\left[1 + \frac{\gamma - 1}{2} M^2\right]\right\}^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}$$

*Find $\frac{A_e}{A^*}$ given Exit Mach number or solve numerically to find Exit Mach from $\frac{A_e}{A^*}$

*Once M_e is known, can solve or T_e and p_e using isentropic relations (above).

Nozzle Choking Criterion:

$$\frac{p_t}{p^*} = \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma}{\gamma - 1}} \quad at\ \ M^* = 1$$

*Unchoked if,

$$\frac{p_t}{p^*} < \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma}{\gamma - 1}}$$

Exit Velocity:

$$V_e = a_e\, M_e \qquad where,\ \ a_e = \sqrt{\gamma\, R\, T_e}$$

$$V_e = \left\{\frac{2\,\gamma}{\gamma - 1} R\, T_t \left[1 - \left(\frac{p_e}{p_t}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}}$$

Where,

$$R = \frac{\tilde{R}}{MW} = \frac{\left(8314 \frac{kJ}{mol \cdot K}\right)}{MW}$$

*Therefore, lower MW gives higher V_e → higher thrust

Isentropic Model of Atmosphere:

$$\frac{p(z)}{p_s} \approx \left[1 - \frac{\gamma - 1}{2}\left(\frac{z}{z^*}\right)\right]^{\frac{\gamma}{\gamma - 1}}$$

*Where surface pressure, $p_s = 101.3\ (10^3)\ \frac{N}{m^2}$

$$z^* = 8404\ m$$

Summerfeld Separation Criterion:

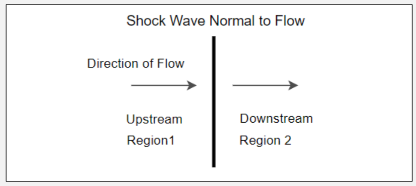
When nozzle flow is "highly" over expanded, when $\frac{p_e}{p_\infty}$ gets too small, flow in nozzle separates when,

$$\frac{p_e}{p_\infty} \lesssim K$$

for $0.25 \le K \le 0.4$

*Boundary layer separation can cause highly turbulent recirculating flow

Normal Shock Relations: (M = Mach before shock)



Total Relations:

$$\frac{p_{t_2}}{p_{t_1}} = \left[\frac{(\gamma + 1)\, M^2}{(\gamma - 1)\, M^2 + 2}\right]^{\frac{\gamma}{\gamma - 1}}\left[\frac{\gamma + 1}{2\gamma M^2 - (\gamma - 1)}\right]^{\frac{1}{\gamma - 1}}$$

$$\frac{T_{t2}}{T_{t1}} = 1$$

Shock Jump Relations:

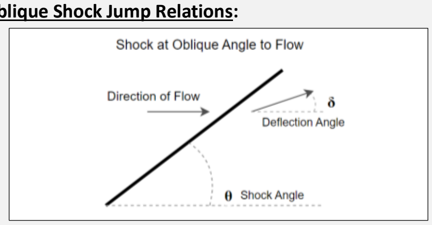
$$\frac{p_2}{p_1} = \frac{2\gamma M^2 - (\gamma - 1)}{\gamma + 1}$$

$$\frac{T_2}{T_1} = \frac{[2\gamma M^2 - (\gamma - 1)][(\gamma - 1)M^2 + 2]}{(\gamma + 1)^2 M^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M^2}{(\gamma - 1)M^2 + 2}$$

$$M_2^2 = \frac{(\gamma - 1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma - 1)}$$

Oblique Shock Jump Relations:



Angle-Mach Relations:

$$\cot(\delta) = \tan(\theta)\left[\frac{(\gamma + 1)M_1^2}{2(M_1^2 \sin^2(\theta) - 1)} - 1\right]$$

$$M_2^2 \sin^2(\theta - \delta) = \frac{(\gamma - 1)M_1^2 \sin^2(\theta) + 2}{2\gamma M_1^2 \sin^2(\theta) - (\gamma - 1)}$$

Total Relations:

$$\frac{p_{t_2}}{p_{t_1}} = \left[\frac{(\gamma + 1)\, M_1^2 \sin^2(\theta)}{(\gamma - 1)\, M_1^2 \sin^2(\theta) + 2}\right]^{\frac{\gamma}{\gamma - 1}}\left[\frac{\gamma + 1}{2\gamma M_1^2 \sin^2(\theta) - (\gamma - 1)}\right]^{\frac{1}{\gamma - 1}}$$

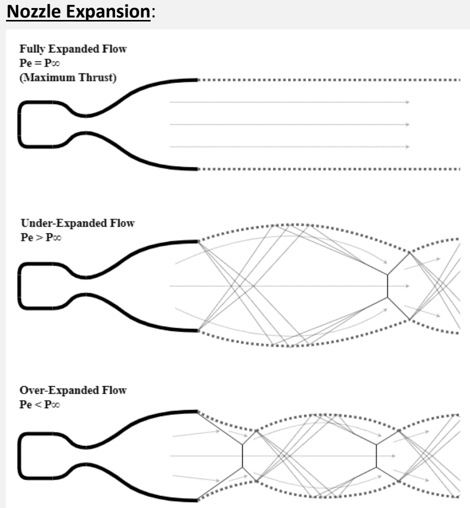
Shock Jump Relations:

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 \sin^2(\theta) - (\gamma - 1)}{\gamma + 1}$$

$$\frac{T_2}{T_1} = \frac{[2\gamma M_1^2 \sin^2(\theta) - (\gamma - 1)][(\gamma - 1)M_1^2 \sin^2(\theta) + 2]}{(\gamma + 1)^2 M_1^2 \sin^2(\theta)}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2 \sin^2(\theta)}{(\gamma - 1)M_1^2 \sin^2(\theta) + 2}$$

Nozzle Expansion:



Thrust Performance

Thrust Equation:

$$T = \dot{m}_p V_e + (p_e - P_\infty) A_e$$

*First term is jet (momentum) thrust, the second term is pressure thrust.

$$T = \dot{m}_p V_{eq}$$

*Valid in all cases

Equivalent Velocity:

$$V_{eq} = V_e + \frac{(p_e - p_\infty)\, A_e}{\dot{m}_p} = \frac{T}{\dot{m}_p}$$

*Would be the exit velocity from a fully expanded nozzle that produces same thrust as actual nozzle.

Specific Impulse:

$$I_{sp} = \frac{V_{eq}}{g_0} = \frac{T}{\dot{m}_p g_0}$$

*Equivalent time that 1 lbf of combustion products could produce 1 lbf of thrust, units of seconds

Thrust Coefficient:

$$(c_T)_{actual} = \frac{T}{p_t A^*}$$

$$(c_T)_{isentropic} = \gamma \left\{\left(\frac{2}{\gamma - 1}\right)\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1 - \left(\frac{p_e}{p_t}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} + \frac{(p_e - p_\infty)}{p_t}\frac{A_e}{A^*}$$

$$(c_T)_{ideal} = \gamma \left\{\left(\frac{2}{\gamma - 1}\right)\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1 - \left(\frac{p_e}{p_t}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}}$$

*Isentropic and fully expanded nozzle flow ($p_e = p_\infty$)

$$(c_T)_{ideal\ vacuum} = \gamma \left\{\left(\frac{2}{\gamma - 1}\right)\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\right\}^{\frac{1}{2}}$$

*Fully expanded flow in a vacuum (infinite nozzle length)

Trajectories

Tsiolkovsky Rocket Equation (velocity increment):

$$\Delta V_{burn} = V_{eq} \ln\left(\frac{M_0}{M_f}\right) - g_0 t_b$$

- Larger velocity increment due to high thrust for short burn time than due to lower thrust for long burn time.
- Short burn at high thrust reduces energy consumed lifting propellant (for vertical launch)

Aerodynamic Drag:

$$D = \frac{1}{2}\rho V^2 A C_D$$

*Circular cross-section with diameter d has area $A = \frac{\pi}{4} d^2$

Gravitational Force:

$$g(z) = g_e \left(\frac{R_e}{R_e + z}\right)^2$$

*As a function of altitude z in m,
earth radius $R_e = 6378\ (10^3)\ m$,
and acceleration of gravity at surface $g_e = 9.8\ m/s^2$

Time and Altitude at Burnout:

$$h_b = V_{eq}\left\{1 - \frac{\ln(R)}{R - 1}\right\} t_b - \frac{1}{2} g_e t_b^2$$

*First term is dependent on Mass Ratio $R = \left(\frac{M_0}{M_f}\right)$

$$t_b = \left(\frac{M_P}{\dot{m}_p}\right) \qquad t_b = (t - t_0)$$

Maximum Vehicle Altitude (vertical launch):

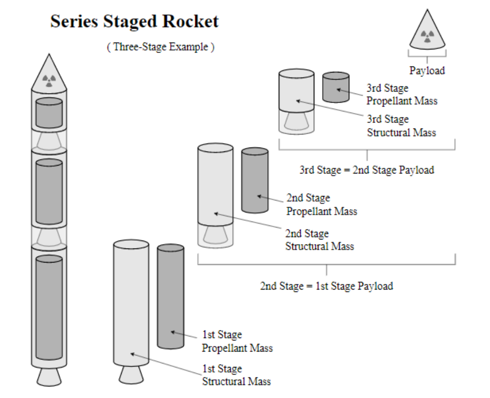
$$h_{max} = h_b + \frac{1}{2}\frac{(\Delta V_b)^2}{g_e}$$

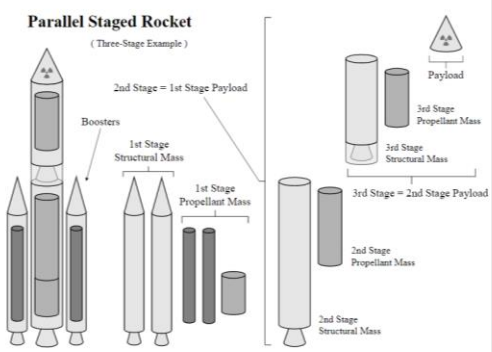
$$h_{max} = \frac{V_{eq}^2 (\ln(R))^2}{2 g_e} - V_{eq} t_b \left\{\frac{R}{R - 1}\ln(R) - 1\right\}$$

*Minimizing t_b maximizes h_{max}

Rocket Staging

Staging Methods:





1st Stage = M_0	2nd Stage = M_{02}	3rd Stage = M_{03}
M_{L_1} = Payload =	M_{L_2} = Payload =	M_L = Payload (final)
		M_{S_3} = Structural
		M_{P_3} = Propellant
	M_{S_2} = Structural	
	M_{P_2} = Propellant	
M_{S_1} = Structural		
M_{P_1} = Propellant		

*For Parallel Staging: Mass of 1st stage propellant from large tank is equal to the total mass of the large tank's propellant times the fraction of the 1st stage burn time over the total burn time of the 1st and 2nd stages.

Stage Mass Relation: N-stage vehicle, the i^{th}-stage has,

$$M_{0i} = M_{Pi} + M_{Si} + M_{Li}$$

Payload Ratio:

$$\lambda_i \equiv \frac{M_{0(i+1)}}{M_{0i} - M_{0(i+1)}} = \frac{M_{Li}}{M_{0i} - M_{Li}} = \frac{M_{Li}}{M_{Pi} + M_{Si}}$$

*Equal for all similar stages, $\lambda_i = \lambda_{(i+1)} = \cdots = \lambda_N$

$$\lambda_{optimal} = \frac{\left({M_L}/{M_0}\right)^{1/N}}{\left(1 - {M_L}/{M_0}\right)^{1/N}}$$

Structural Coefficient:

$$\varepsilon_i = \frac{M_{Si}}{M_{0i} - M_{0(i+1)}} = \frac{M_{Si}}{M_{0i} - M_{Li}} = \frac{M_{Si}}{M_{Si} + M_{Pi}}$$

*Equal for all similar stages, $\varepsilon_i = \varepsilon_{(i+1)} = \cdots = \varepsilon_N$

Mass Ratio:

$$R_i = \frac{M_{0i}}{M_{0i} - M_{Pi}} = \frac{M_{0i}}{M_{Si} + M_{Li}} = \frac{1 + \lambda_i}{\varepsilon_i + \lambda_i}$$

Velocity Increments for an Optimally Staged Rocket:

ΔV due to burnout of the i^{th}-stage,

$$(V_b)_i = (V_{eq})_i \ln(R_i) - g_0 (t_b)_i$$

Final ΔV imparted on final payload M_L of stage N,

$$T_b = \sum_{i=1}^{N} (t_b)_i$$

$$(V_b)_N = \sum_{i=1}^{N} \left[(V_{eq})_i \ln\left(\frac{1+\lambda_i}{\varepsilon_i + \lambda_i}\right) \right] - g_o T_b$$

$$(\Delta V_b)_N = V_{eq}\, N \ln \left\{ \varepsilon + (1-\varepsilon)\left[\frac{M_L}{M_0}\right]^{\frac{1}{N}} \right\}^{-1} - g_o T_b$$

Ideal Case (infinitely many stages):

$$(\Delta V_b)_{N\to\infty} = V_{eq}\,(1-\varepsilon)\ln\left[\frac{M_0}{M_L}\right] - g_o T_b$$

*Equivalent Velocity $V_{eq} = \frac{T}{\dot{m}_p} \approx$ same for all stages

Orbital Dynamics

Gravitational Parameter:

$$\mu = GM' \qquad where\ G = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg}$$

Radial Position (from vehicle to center of planet):

$$r = \frac{a(1-\varepsilon^2)}{1 + \varepsilon\cos(\theta)}$$

Semimajor Axis:

$$a = r_{min}\frac{1}{1-\varepsilon} \qquad a = \frac{1}{2}(r_a + r_p)$$

Eccentricity:

$$\varepsilon = \frac{r_a - r_p}{r_a + r_p} \qquad \varepsilon = \sqrt{1 - \frac{b^2}{a^2}} \qquad \varepsilon = 1 - \frac{r_{min}}{a}$$

*Where b = semi-minor axis

$\varepsilon < 1 \rightarrow ellipse$

$\varepsilon = 1 \rightarrow parabola$

$\varepsilon > 1 \rightarrow hyperbola$

$\varepsilon = 0 \rightarrow circle$

Vis-Viva:

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

Perigee Radius:

$$r_p = a(1-\varepsilon)$$

Perigee Velocity:

$$V_p = \sqrt{\frac{\mu}{a}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)} \qquad V_p = \sqrt{\frac{\mu}{r_p}\frac{2r_a}{(r_a + r_p)}}$$

Apogee Radius:

$$r_a = a(1+\varepsilon)$$

Apogee Velocity:

$$V_a = \sqrt{\frac{\mu}{a}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)} \qquad V_a = \sqrt{\frac{\mu}{r_a}\frac{2r_p}{(r_a + r_p)}} \qquad V_a = \sqrt{\frac{\mu}{a}\frac{r_a}{r_p}}$$

Orbital Period:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Energy (per unit mass):

$$e = -\frac{\mu}{2a}$$

Circular Orbits: $r(\theta) = R = constant,\ e < 0,\ \varepsilon = 0$

Velocity:

$$V = \sqrt{\frac{\mu}{R}}$$

Period:

$$T = 2\pi\sqrt{\frac{R^3}{\mu}}$$

Escape from Circular Orbit:

$$\Delta V_1 \geq (\sqrt{2} - 1)\sqrt{\frac{\mu}{R}}$$

Parabolic Trajectories: (not orbit) $e = 0,\ a = \infty,\ \varepsilon = 1$

$$V_\infty = 0$$

Escape Velocity:

$$V_{esc} = \sqrt{\frac{2\mu}{r}}$$

Escape from planet surface:

$$V_{esc} = \sqrt{\frac{2\mu}{R_{planet}}}$$

Hyperbolic Trajectories: (not orbit) $e > 0$

Excess Velocity:

$$V_\infty = \sqrt{2e} \qquad V_\infty = \sqrt{\frac{\mu}{-a}}$$

Asymptote Angle:

$$\theta_\infty = \pi - \cos^{-1}\left(\frac{1}{\varepsilon}\right)$$

Turning Angle:

$$\delta = 2\sin^{-1}\left(\frac{1}{\varepsilon}\right)$$

Miss Distance:

$$\Delta = -a\sqrt{\varepsilon^2 - 1}$$

Orbital Maneuvers

Circularization Burn:

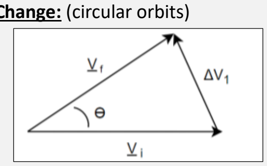
Circularize at r_a:

$$\Delta V_1 = \sqrt{\frac{\mu}{r_a}} \cdot \left[1 - \sqrt{\frac{2r_p}{r_a + r_p}}\right]$$

Circularize at r_p:

$$\Delta V_1 = \sqrt{\frac{\mu}{r_p}} \cdot \left[1 - \sqrt{\frac{2r_a}{r_a + r_p}}\right]$$

Inclination Change: (circular orbits)



*Both other angles are: $\frac{1}{2}\pi - \theta$

$$\Delta V_1 = 2\sqrt{\frac{\mu}{R}}\sin\left(\frac{\theta}{2}\right)$$

Orbital Rendezvous:

Lead Angle:

$$\alpha_L = \frac{\pi}{2}\sqrt{\frac{(R_1 + R_2)^3}{2(R_2)^3}}$$

Transfer Time:

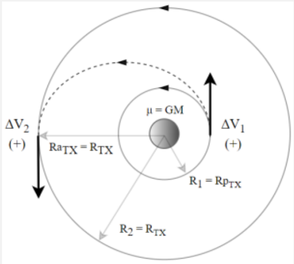
$$T_{TX} = T_{Hohmann} = \frac{\pi}{2}\sqrt{\frac{(R_1 + R_2)^3}{2\mu}}$$

Intercept Opportunity Times:

$$\frac{1}{T} = \frac{1}{(T_C)_1} - \frac{1}{(T_C)_2} \qquad where\ (T_C)_i = 2\pi\sqrt{\frac{R_i}{\mu}}$$

Orbital Transfers

Hohmann Transfer:



Circular Orbit Velocities:

$$(V_C)_1 = \sqrt{\frac{\mu}{R_1}} \qquad (V_C)_2 = \sqrt{\frac{\mu}{R_2}}$$

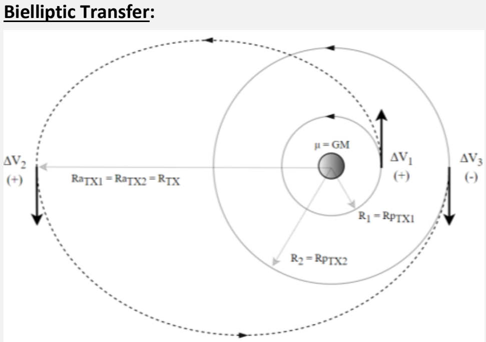
Transfer Orbit Parameters: (Trans time, see Rendezvous)

$$r_p = R_1$$
$$r_a = R_2$$
$$V_p = \sqrt{\frac{\mu}{R_1}\frac{2R_2}{R_1 + R_2}}$$
$$V_a = \sqrt{\frac{\mu}{R_2}\frac{2R_1}{R_1 + R_2}}$$

Velocity Increments:

$$\Delta V_1 = \sqrt{\frac{\mu}{R_1}}\left[\sqrt{\frac{2R_2}{R_1 + R_2}} - 1\right]$$
$$\Delta V_2 = \sqrt{\frac{\mu}{R_2}}\left[1 - \sqrt{\frac{2R_1}{R_1 + R_2}}\right]$$

Bielliptic Transfer:



Circular Orbit Velocities:

$$(V_C)_1 = \sqrt{\frac{\mu}{R_1}} \qquad (V_C)_2 = \sqrt{\frac{\mu}{R_2}}$$

Transfer Orbit Parameters: (R_{TX} can be chosen)

$$R_1 = R_{p_{TX1}}$$
$$R_{TX} = R_{a_{TX1}}$$
$$R_2 = R_{p_{TX2}}$$
$$R_{TX} = R_{a_{TX2}}$$
$$a_{TX1} = \frac{1}{2}(R_1 + R_{TX})$$
$$a_{TX2} = \frac{1}{2}(R_2 + R_{TX})$$

Transit Time:

$$T_{TX} = \pi\frac{\sqrt{(a_{TX1})^3} + \sqrt{(a_{TX2})^3}}{\sqrt{\mu}}$$

Velocity Increments:

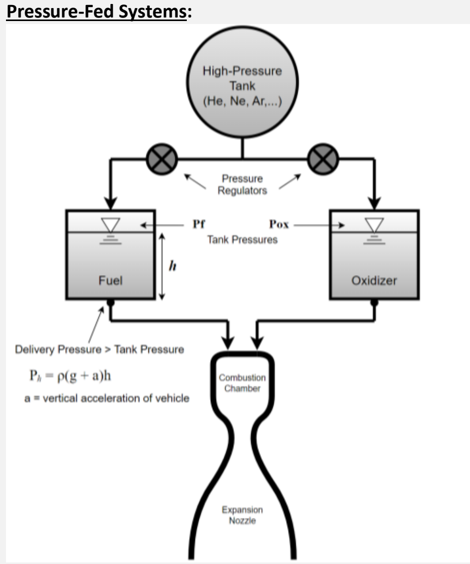
$$\Delta V_1 = \sqrt{\mu\left(\frac{2}{R_1} - \frac{1}{a_{TX1}}\right)} - \sqrt{\frac{\mu}{R_1}}$$
$$\Delta V_2 = \sqrt{\mu\left(\frac{2}{R_{TX}} - \frac{1}{a_{TX2}}\right)} - \sqrt{\mu\left(\frac{2}{R_{TX}} - \frac{1}{a_{TX1}}\right)}$$
$$\Delta V_3 = \sqrt{\frac{\mu}{R_2}} - \sqrt{\mu\left(\frac{2}{R_2} - \frac{1}{a_{TX2}}\right)}$$

Standard Gravitation Parameters

Body	$\mu = GM'\left[\frac{m^3}{s^2}\right]$
Sun	1.327×10^{20}
Mercury	2.203×10^{13}
Venus	3.249×10^{14}
Earth	3.986×10^{14}
(Moon)	4.903×10^{12}
Mars	4.283×10^{13}
Jupiter	1.267×10^{17}
Saturn	3.793×10^{16}
Uranus	5.794×10^{15}
Neptune	6.837×10^{15}
Pluto	1.108×10^{12}

Propellant Feed Systems

Pressure-Fed Systems:

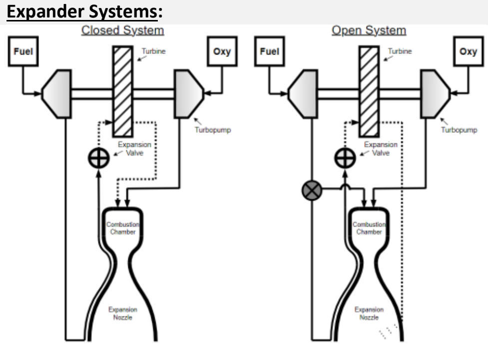


Deliver Pressure > Tank Pressure,

$$P_h = \rho(g + a)h$$

	$\rho\ [kg/m^3]$	$C_V\ [J/kg \cdot K]$	$T\ [K]$
LO2	1141	1669	90
LH2	70.8	9668	20
LCH4	422	4216	111
RP-1	810	2188	300

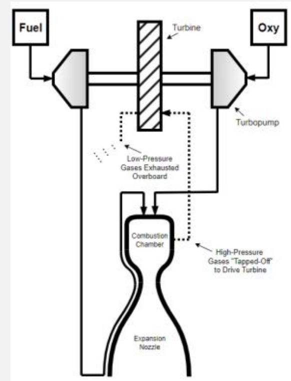
Expander Systems:



*Fuel is expanded to gaseous form to drive turbine that powers pumps which then push propellant into CC.

- Low/moderate thrust applications
- Often used for upper stage engines

Combustion Tap-Off Systems:



*Uses combustion product gas from combustion chamber to drive gas turbine that then drives pumps.

Open Systems:

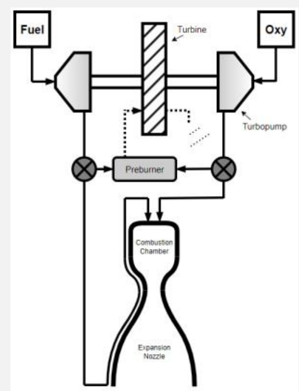
- Turbine exit gas pressure low, dumped overboard

Closed Systems:

- Turbine exit gas dumped into nozzle at low pressure location to add $\dot{m}_p$ at nozzle exit

*Moderate/high thrust (Saturn I-B/Blue Origin)

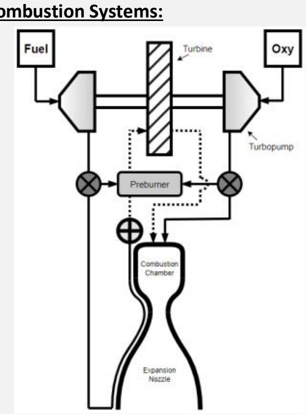
Gas Generator Systems:



*Part of fuel and oxidizer are burned in preburner to produce high-pressure gas that drives turbopumps.

- Preburner operated very fuel-rich to keep temperature sufficiently low
- Fuel-rich exhaust dumped overboard (open) or dumped into nozzle (closed)
- Power controlled by adjusting fuel/ox rates into preburner
- Can achieve higher CC pressure than expander sys.

Staged Combustion Systems:



*All propellant is burned in two stages (nothing wasted). First in fuel-rich preburner, then in main CC.

- Allows very high CC pressures (high thrust)
- Can be used with any liquid propellants
- More efficient than gas generator (all fuel burned)
- Requires very high-pressure turbopumps

Propellants and Aerothermochemistry

Propellant Types:

Liquid Propellants:

Fuels:

a) Cryogenic
- LH_2 – liquid hydrogen (20 K) "deep cryogen"
- LCH_4 – liquid methane (111 K)

b) Non-cryogenic
- RP-1 – kerosene $[C_1H_{1.96}]$
- ethyl alcohol – $[C_2H_5OH]$

c) Hypergolic ($E_a \leq 0$) (reacts on contact with oxi.)
- hydrazine $[N_2H_4]$
- monomethyl hydrazine (MMH)
- unsymmetrical dimethyl hydrazine (UDMH)

Oxidizers:

a) Cryogenic
- LOX – liquid oxygen
- LF_2 – liquid fluorine
- FLOX – fluorine-enhanced LOX
- N_2O_4 – dinitrogen tetroxide

b) Non-cryogenic
- N_2O – nitrous oxide
- N_2O_4 – dinitrogen tetroxide
- ClF_5 – chlorine pentafluoride

Monopropellants:

* Both fuel and oxidizer are contained in the same molecule; contact with a catalyst bed initiates reaction (breaks oxidizer and fuel components apart).

* Usually used for RCS and in-space propulsion at low-mid thrust applications.

- hydrazine (N_2H_4) + granular Al w/iridium coating
- hydrogen peroxide (H_2O_2) + many possible catalysts
- nitromethane (CH_3NO_2)
- ethylene oxide (C_2H_4O)
- nitrous oxide (N_2O)
- hydroxylammonium nitrate (HAN) ($H_4N_2O_4$)

Solid Propellants:

***** Both fuel and oxidizer are initially in solid form.

Homogenous Propellants:

* Fuel and oxidizer are contained in the same molecule or in a homogeneous mixture.
  - nitroglycerine + nitrocellulose
    $C_3H_5(NO_2)_3$ + $C_6H_7O_2(NO_2)_3$
  - Single/Double/Triple base propellants
    Metal powders often added to increase heat of combustion.

Heterogeneous (composite) Propellants:

* Heterogeneous mixture of oxidizing crystals held in an organic plastic-like fuel "binder"

a) Fuel Component (common binders)
- hydroxyl-terminated polybutadiene (HTPB)
- rubber/asphalt

b) Oxidizer Component (ground crystals of one or more of the following)
- ammonium perchlorate (AP)
- ammonium nitrate (AN)
- nitronium perchlorate (NP)
- potassium perchlorate (KP)
- potassium nitrate (PN)
- cyclotrimethylene trinitomine (RDX)
- cyclotetramethylene tetranitramine (HMX)

* The combination of fuel/ox specifies the propellant (e.g. AP-HTPB, KP-HTPB, …)
  Many possible combinations.

Hydrocarbon Combustion:

Overall Stoichiometric Reaction:

$$1\, C_nH_m + \left(n + \frac{m}{4}\right) O_2 \;\rightarrow\; nCO_2 + \frac{m}{2} H_2O$$

$$n = carbon\ atoms, \quad m = hydrogen\ atoms$$

Mixture Ratio:

$$r = \left(\frac{M_{O_2}}{M_f}\right)$$

Stoichiometric Mixture Ratio:

$$r_s = \frac{32n + 8m}{12n + m}$$

Equivalence Ratio:

$$\varphi = \frac{r_s}{r}$$

$$\varphi = 1 \rightarrow stoichiometric\ combustion$$
$$\varphi > 1 \rightarrow fuel-rich\ combustion$$
$$\varphi < 1 \rightarrow fuel-lean\ combustion$$

Molar Enthalpy of Combustion:

$$\Delta\tilde{h}_C = \left[\sum_i Bond\ Energy\right]_{react} - \left[\sum_i Bond\ Energy\right]_{prod}$$

$$\Delta\tilde{h}_C < 0 \rightarrow exothermic$$
$$\Delta\tilde{h}_C > 0 \rightarrow endothermic$$

Convert to mass-specific enthalpy:

$$\Delta h_C = \frac{\Delta\tilde{h}_C}{MW_{HC}}$$

$$MW_{HC} = 12n + 1m \left[\frac{kcal}{g}\right]$$

Convert to $[kJ/kg]$:

$$\Delta h_C\left[\frac{kcal}{g}\right] \cdot \left(\frac{1000\ cal}{kcal}\right)\left(\frac{1000g}{kg}\right)\left(\frac{4.1814\ J}{cal}\right)\left(\frac{1\ kJ}{1000J}\right)$$

Turbopumps and Turbines

Turbopumps:

Shaft Power from Turbine:

$$\dot{W} = \tau \cdot \dot{\Omega}$$

Ideal Pump Work:

$$|w|_{ideal} = \frac{1}{\rho}(p_{t2} - p_{t1})$$

Ideal Pump Power:

$$|\dot{w}|_{ideal} = \dot{m}|w|_{ideal} = \dot{m}\frac{1}{\rho}(p_{t2} - p_{t1})$$

Pump Efficiency:

$$\eta_P = \frac{|w|_{ideal}}{|w|_{actual}} = \frac{|\dot{w}|_{ideal}}{|\dot{w}|_{actual}}$$

Non-Ideal Pump Work:

$$|w|_{actual} = \frac{1}{\eta_P}\frac{1}{\rho}(p_{t2} - p_{t1})$$

Non-Ideal Pump Power:

$$|\dot{w}|_{actual} = \dot{m}|w|_{actual} = \dot{m}\frac{1}{\eta_P}\frac{1}{\rho}(p_{t2} - p_{t1})$$

$$*from\ the\ above$$

$$\Delta p_t = (p_{t2} - p_{t1}) = w_P\, \eta_P\, \frac{\rho}{\dot{m}}$$

1st Law for Liquid Flow Through Pump:

$$c_V = (T_2 - T_1) + \frac{1}{\rho}(p_2 - p_1) + \frac{1}{2}(V_2^2 - V_1^2) = |w|$$

$$c_V = (T_2 - T_1) + \frac{1}{\rho}(p_{t2} - p_{t1}) = |w|$$

Temperature Change Across Pump:

$$(T_2 - T_1) = \left(\frac{1 - \eta_P}{\eta_P}\right)\frac{(p_{t2} - p_{t1})}{\rho \cdot c_V}$$

Turbines:

Application of 1st Law:

$$(\Delta h_t)_{1,2} = q_{1,2} - w_{1,2} = 0$$
$$\rightarrow h_{t2} - h_{t1} = 0$$
$$\rightarrow C_p(T_{t2} - T_{t1}) = 0$$
$$\therefore\ T_{t2} = T_{t1}$$

Turbine Work:

$$|W|_{nonideal} = \eta_T \frac{1}{\rho}\Delta p_t$$

Temperature Change Across Turbine:

$$\Delta T = (1 - \eta_T)\frac{\Delta p_t}{\rho \cdot c_V}$$

Turbine Work: (per unit mass)

$$|w_T| = |C_P(T_{te} - T_{ti})|$$

Turbine Power:

$$\dot{W}_T = \dot{m}|w_T|$$

Turbine Stage Efficiency:

$$\eta_S = \frac{|w|}{|w|_S}$$

Total Turbine Efficiency:

$$\eta_T = \frac{1 - \left(\frac{T_{t3}}{T_{t1}}\right)}{1 - \left(\frac{p_{t3}}{p_{t1}}\right)^{\frac{\gamma-1}{\gamma}}}$$

Turbine Total Temperature Ratio:

$$\frac{T_{t3}}{T_{t1}} = 1 - \eta_T\left[1 - \left(\frac{p_{t3}}{p_{t1}}\right)^{\frac{\gamma-1}{\gamma}}\right]$$

Turbine Total Pressure Ratio:

$$\frac{p_{t3}}{p_{t1}} = \left[1 - \frac{1}{\eta_T}\left(1 - \frac{T_{t3}}{T_{t1}}\right)\right]^{\frac{\gamma}{\gamma-1}}$$

Total-to-Static Temperature Relation:

$$T_3 = T_{t3}\left[1 + \frac{\gamma - 1}{2}M_3^2\right]^{-1}$$

Total-to-Static Pressure Relation:

$$p_3 = p_{t3}\left[1 + \frac{\gamma - 1}{2}M_3^2\right]^{-\frac{\gamma}{\gamma-1}}$$

Entropy Change Across Turbine:

$$\Delta s = s_3 - s_1 = C_p \ln\left(\frac{T_{t3}}{T_{t1}}\right) - R\ln\left(\frac{p_{t3}}{p_{t1}}\right)$$

$$R = C_p\left(\frac{\gamma - 1}{\gamma}\right)$$

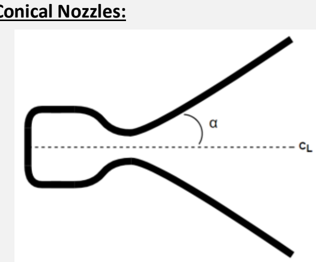
$$R = C_v(\gamma - 1)$$

Temperature-Velocity Relation:

$$(T_t - T) = \frac{V^2}{2 \cdot C_p}$$

Nozzle Design

Simple Conical Nozzles:



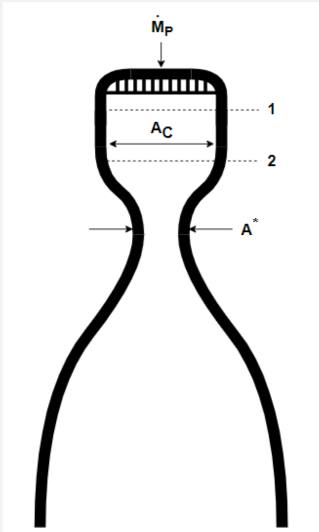
Axial Thrust:

$$T_{axial} = \frac{1 + \cos(\alpha)}{2}\dot{m}_p V_e + (p_e - p_\infty)A_e$$

*See notes for Rao and TOP nozzle stuff

Combustion Chamber Performance

Combustion Chambers:



Thrust Coefficient:

$$C_T = \frac{T}{p_{t2} \cdot A^*}$$

Thrust:

$$T = \dot{m}_p \left(\frac{p_{t2} \cdot A^*}{\dot{m}_p}\right) C_T$$

C* "Characteristic Velocity":

$$C^* = \frac{p_{t2} \cdot A^*}{\dot{m}_p}$$

$$C^*_{ideal} = \frac{p_{t2}}{pt_1}\sqrt{RT_{t2}}\left[\gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]^{\frac{1}{2}}$$

C* Efficiency:

$$\eta_{C^*} = \frac{C^*_{actual}}{C^*_{ideal}}$$

Specific Impulse:

$$I_{SP} = C^* C_T \frac{1}{g}$$

Pressure Ratio:

$$\frac{p_2}{p_1} = \frac{(1+\gamma M_1^2)}{(1+\gamma M_2^2)}$$

Temperature Ratio:

$$\frac{T_2}{T_1} = \left[\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right] \cdot \left(\frac{M_2}{M_1}\right)^2$$

Total Temperature Ratio:

$$\frac{T_{t2}}{T_{t1}} = \left[\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right] \cdot \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2$$

$$\frac{T_{t2}}{T_{t1}} = \frac{q_{1,2}}{C_p T_{t1}} + 1$$

Total Pressure Ratio:

$$\frac{p_{t2}}{p_{t1}} = \left[\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}} \cdot \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)$$

Combustion Chamber-to-Throat Area Ratio:

$$\frac{A_C}{A^*} = \frac{1}{M_2}\left\{\left(\frac{2}{\gamma+1}\right)\left[1+\frac{\gamma-1}{2}M_2^2\right]\right\}^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}$$

Thermal Choking Criterion:

$$\frac{T_{t2}}{T_{t1}} \geq \frac{1}{2(1+\gamma)} \frac{(1+\gamma M_1^2)^2}{\left[1+\frac{\gamma-1}{2}M_1^2\right]M_1^2}$$

*subtract 1 from result to get max allowable T_{t2}

DOF and Molecular Energy

Monatomic Molecules:

$$(DOF)_{trans} = 3$$
$$(DOF)_{vib} = 0$$
$$(DOF)_{rot} = 0$$
$$\rightarrow (DOF)_{avail} = 3$$

Diatomic Molecules:

$$(DOF)_{trans} = 3$$
$$(DOF)_{vib} = 2$$
$$(DOF)_{rot} = 2$$
$$\rightarrow (DOF)_{avail} = 7$$

Polyatomic Molecules:

$$(DOF)_{trans} = 3$$

$$(DOF)_{vib} = \begin{cases} 2 \; if \; linear \; (CO_2) \\ 3 \; if \; nonlinear \; (all \; others) \end{cases}$$

$$(DOF)_{rot} = \begin{cases} 2(3n-5) \; if \; linear \\ 2(3n-6) \; if \; nonlinear \end{cases}$$

$$\rightarrow (DOF)_{avail} = \begin{cases} 5 + 2(3n-5) \; if \; linear \\ 6 + 2(3n-6) \; if \; nonlinear \end{cases}$$

Energy Stored in One Molecule:

$$e' = \frac{1}{2}kT(DOF)_{active}$$

$$k = \frac{\tilde{R}}{A_v} = \frac{8.314\, J/mol \cdot K}{6.022 \times 10^{23} \frac{part}{mol}} \quad (Boltzmann\ Constant)$$

$$\tilde{e} = A_v \cdot e' = \frac{1}{2}\tilde{R}T \cdot (DOF)_{active} \quad (per\ mole)$$

$$e = \tilde{e}/MW = \frac{1}{2}RT \cdot (DOF)_{active} \quad (per\ mass)$$

*$(DOF)_{active}$ depends on temperature

- Low T = only trans. active
- Med T = trans. and rot. active
- High T = trans., rot., and vib. are active

Specific Heat and Gas Constant Relations:

Specific Heat - Constant Volume:

$$\tilde{C}_V = \tilde{R}\frac{1}{2}(DOF)_{active}$$

$$C_V = R\frac{1}{2}(DOF)_{active}$$

Specific Heat - Constant Pressure:

$$\tilde{C}_P = \tilde{R}\left[1 + \frac{1}{2}(DOF)_{active}\right]$$

$$C_P = R\left[1 + \frac{1}{2}(DOF)_{active}\right]$$

Gas Constant Relations:

$$R = C_P - C_V$$
$$\tilde{R} = \tilde{C}_P - \tilde{C}_V$$

*given C_P and γ

$$R = C_P\left(1 - \frac{C_V}{C_P}\right) = C_P\left(1 - \frac{1}{\gamma}\right) = C_P\left(\frac{\gamma-1}{\gamma}\right)$$

*given C_V and γ

$$R = C_P - C_v = C_V\left(\frac{C_P}{C_V} - 1\right) = C_V(\gamma - 1)$$

Specific Heat Ratio:

$$\gamma = 1 + \frac{2}{(DOF)_{active}}$$

Monatomic:

$(DOF)_{active} = 3$ (independent of temp.)

$$\rightarrow \gamma = 1 + \frac{2}{(3)} = 1.667$$

Diatomic:

- Medium temp

$$(DOF)_{active} = 5 \quad \rightarrow \gamma = 1 + \frac{2}{(5)} = 1.4$$

- High temp

$$(DOF)_{active} = 7 \quad \rightarrow \gamma = 1 + \frac{2}{(7)} = 1.286$$

Non-Isentropic Nozzle Flow

Application of 1st Law:

$$(\Delta h_t)_{2,e} = q_{2,e} - w_{2,e} = 0$$
$$\rightarrow h_{t2} - h_{te} = 0$$
$$\rightarrow C_p(T_{t2} - T_{te}) = 0$$
$$\therefore \; T_{t2} = T_{te}$$

Temperature-Velocity Relation:

$$(T_t - T) = \frac{V^2}{2 \cdot C_P}$$

Total Pressure Ratio:

$$\frac{p_{te}}{p_{t2}} = e^{-\frac{\Delta s}{R}}$$

*where, $p_{te} < p_{t2}$ and $\Delta s > 0$

Total-to-Static Pressure Ratio:

$$\frac{p_e}{p_{t2}} = \left\{1 - \frac{1}{\eta_N}\left[\frac{(\gamma-1)M_e^2}{2+(\gamma-1)M_e^2}\right]\right\}^{\frac{\gamma}{\gamma-1}}$$

Nozzle Efficiency:

$$\eta_N = \frac{1 - \left[1 + \frac{\gamma-1}{2}M_e^2\right]^{-1}}{1 - \left(\frac{p_e}{p_{t2}}\right)^{\frac{\gamma-1}{\gamma}}}$$

Area-Mach Relation:

$$\frac{A_e}{A^*} =$$

$$\frac{1}{M_e}\left\{\left(\frac{2}{\gamma+1}\right)\left[1+\frac{\gamma-1}{2}M_e^2\right]\right\}^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \cdot \left\{1 + \left(1 - \frac{1}{\eta_N}\right)\frac{\gamma-1}{2}M_e^2\right\}^{-\frac{\gamma}{\gamma-1}}$$

*from resulting M_e, get:

$\rightarrow p_e$ from total-to-static (above)

$$\rightarrow T_e = T_{te}\left[1 + \frac{\gamma-1}{2}M_e^2\right]^{-1}$$

$$\rightarrow a_e = \sqrt{\gamma R T_e} \quad where, R = C_p\left(\frac{\gamma-1}{\gamma}\right)$$

$$\rightarrow V_e = M_e \cdot a_e$$

$$\rightarrow T = m_p V_e + (p_e - p_\infty)A_e$$

Non-Isentropic Thrust Coefficient:

$$C_T = \gamma\left\{\eta_n\left(\frac{2}{\gamma-1}\right)\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1 - \left(\frac{p_e}{p_{t2}}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{\frac{1}{2}} + \frac{(p_e - p_\infty)}{p_{t2}}\frac{A_e}{A^*}$$

Injectors and Droplet Vaporization

Droplet Characterization:

Droplet Diameter Probability:

$$P(D) = \frac{Prob\left(D + \frac{1}{2}dD\right)}{dD}$$

*Sauter Mean Diameter (SMD): the droplet diameter with same surface-to-volume ratio as the entire spray.

*Weber Number: ratio of inertia (mass) to surface tension

*Ohnesorge Number: ratio of friction to surface tension

$$\frac{SMD}{\ell} = c \cdot We^p \cdot Oh^q$$

$$(SMD)_{CC} = (SMD)_{lab} \cdot \left(\frac{\ell_{CC}}{\ell_{lab}}\right)\left(\frac{We_{CC}}{We_{lab}}\right)^p\left(\frac{Oh_{CC}}{Oh_{lab}}\right)$$

Droplet Vaporization:

Heat of Vaporization: (heat flux into droplet)

$$\dot{m}_V \cdot \Delta h_V = \dot{Q}_{in} \qquad where,$$

$$\dot{Q}_{in} = -k\left.\frac{dT}{dr}\right|_{R(t)} \cdot 4\pi R^2$$

$$\dot{m}_V = -4\pi\rho_L R^2(t)\frac{dR}{dt}$$

Droplet Diameter: (as function of time)

$$D^2(t) = D_0 - \left(\frac{8k}{\rho_L C_P}\right)\ln(1+B) \cdot t$$

$$B \equiv \frac{C_P(T_\infty - T_V)}{\Delta h_V}$$

Vaporization time:

$$t_V = D_0^2 \frac{\rho_L C_P}{8k\ln(1+B)} = \frac{D_0^2}{k_V}$$